Exercise I

MATH 305b/ ENAS 514b - Spring Semester 2017

Due Monday, 01/30/2015, 1:00 PM

1. Let $\mathscr{J}$ be a countable set, and let $a_j \in [0,\infty]$, $j \in \mathscr{J}$, be nonnegative extended reals indexed by it. Define the following notion of summation over countable index sets:

$$\sum_{j\in\mathscr{J}} a_j \triangleq \sup\left\{\sum_{j\in S} a_j : S \subseteq \mathscr{J} \text{ is finite}\right\}.$$

   Prove the following results:

   (a) This notion agrees with the standard notion of sum of a series: $\sum_{n\in\mathbb{N}} a_n = \sum_{n=1}^{\infty} a_n$

   (b) For double-indexed $a_{n,k} \in [0,\infty]$ we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{n,k} = \sum_{(n,k)\in\mathbb{N}^2} a_{n,k} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_{n,k}\ ,$$

   where $\mathbb{N}^2 = \mathbb{N}\times\mathbb{N}$.

2. Using the result for closed bounded intervals, prove that for *every* interval $I \subseteq \mathbb{R}$ (not just closed bounded ones), $m^*(I) = |I|$.

3. Show that for any two sets $A, B \subseteq [0,1]$ with $A \cup B = [0,1]$, their outer measures satisfy $m^*(A) \geq 1 - m^*(B)$.

4. Prove that any open set $A \subseteq \mathbb{R}$ can be written as a countable union $A = \cup_{n=1}^{\infty} I_n$ of disjoint open intervals (i.e., $I_n = (a_n, b_n)$ for some real $a_n \leq b_n$, and $I_j \cap I_k = \emptyset$ when $j \neq k$). Notice that some (or all) of these intervals can be chosen to be empty.

5. Prove that if $A \subseteq \mathbb{R}$ is a union of disjoint open intervals $I_1, I_2, \ldots$, then its outer measure is $m^*(A) = \sum_{n=1}^{\infty} |I_n|$.

6. Prove that for any set $B \subseteq \mathbb{R}$, its outer measure is $m^*(B) = \inf\{m^*(A) : B \subseteq A \subseteq \mathbb{R}, A \text{ is open}\}$.

7. Recall the Cantor set (e.g., from Beals, Sec. 10B, Example 3) has zero outer measure while also being uncountable, closed, and nowhere dense. Show that it is possible to construct a *fat* Cantor set $C \subset [0,1]$, which is closed and nowhere dense, with positive (nonzero) outer measure. Furthermore, show that for every $\varepsilon > 0$ such a set can be constructed to have outer measure $m^*(C) > 1 - \varepsilon$.

8. For each rational number $x \in \mathbb{Q}\cap[0,1]$, written in lowest terms as $x = \frac{p}{q}$ for $p, q \in \mathbb{N}$, let $J_x = (x - \frac{1}{4q^3}, x + \frac{1}{4q^3})$. Prove that $[0,1] \not\subseteq \bigcup_{x\in\mathbb{Q}\cap[0,1]} J_x$, and that furthermore,

$$m^*\left([0,1] \setminus \bigcup_{x\in\mathbb{Q}\cap[0,1]} J_x\right) > 0.$$ *Hint:* notice that $\sum_{q=1}^{\infty} q^{-2} < 2$.