POLYLOGARITHMS IN ARITHMETIC AND GEOMETRY

A.B.GONCHAROV

The classical polylogarithms were invented in correspondence of Leibniz with Joh.Bernoulli in 1696 ([Lei]). They are defined by the series

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \qquad |z| < 1$$

and continued analytically to a covering of $\mathbb{C}P^1 \backslash \{0, 1, \infty\}$:

$$Li_n(z) := \int_0^z Li_{n-1}(t)\frac{dt}{t}, \qquad Li_1(z) = -\log(1-z)$$

1. The dilogarithm. It was studied by Spence, Abel, Kummer, Lobachevsky, ..., Rogers,Ramanujan, ([L]). The main discovery was that the dilogarithm satisfies many functional equations. For example Rogers' version of the dilogarithm $L_2(x) := Li_2(x) + \frac{1}{2}\log(x)\log(1-x) - \frac{\pi^2}{6}$ for $1 > x > y > 0$ satisfies the relation

$$L_2(x) - L_2(y) + L_2(y/x) - L_2(\frac{1-x^{-1}}{1-y^{-1}}) + L_2(\frac{1-x}{1-y}) = 0 \qquad (1)$$

After a century of neglect the dilogarithm appeared twenty years ago in works of Gabrielov-Gelfand-Losik [GGL] on a combinatorial formula for the first Pontryagin class, Bloch on K-theory and regulators [Bl1] and Wigner on Lie groups.

The dilogarithm has a single-valued cousin : the Bloch - Wigner function

$$\mathcal{L}_2(z) := \mathrm{Im} Li_2(z) + \arg(1-z)\log|z|.$$

Let $r(x_1, ..., x_4)$ be the cross-ratio of 4 distinct points on $\mathbb{C}P^1$. Then

$$\sum_{i=0}^{4}(-1)^i \mathcal{L}_2(r(z_0, ..., \hat{z}_i, ..., z_4)) = 0 \qquad z_i \in \mathbb{C}P^1 \qquad (2)$$

If $(z_1, ..., z_5) = (\infty, 0, 1, x, y)$ the arguments here are the same as in (1).

Choose $x \in \mathbb{C}P^1$. Then (2) just means that the function $c_3(g_0, ..., g_3) := \mathcal{L}_2(r(g_0x, ..., g_3x))$, where $g_i \in GL_2(\mathbb{C})$ and $g_ix \neq g_jx$, is a *measurable* 3-cocycle on $GL_2(\mathbb{C})$. (Wigner).

The function $\log|x|$ is characterized by its functional equation $\log|xy| = \log|x| + \log|y|$. The 5-term equation plays a similar role for the dilogarithm: any measurable function $f(z), z \in \mathbb{C}$ satisfying (2) is proportional to $\mathcal{L}_2(z)$ ([Bl1]). Moreover, any functional equation for $\mathcal{L}_2(z)$ is a formal consequence of (2).

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For a set X denote by $\mathbb{Z}[X]$ the free abelian group generated by symbols $\{x\}$, $x \in X$. Let F be a field. Consider the homomorphism ([Bl1])

$$\tilde{\delta}_2 : \mathbb{Z}[F^*\backslash 1] \longrightarrow \Lambda^2 F^*, \quad \{x\} \longmapsto (1-x) \wedge x$$

By Matsumoto theorem Coker$\tilde{\delta}_2 = K_2(F)$.

Let $R_2(F)$ be the subgroup of $\mathbb{Z}[F^*\backslash 1]$ generated by the elements $\sum (-1)^i \{r(z_0, ..., \hat{z}_i, ..., z_4)\}$ where $z_i \neq z_j \in P^1_F$. One can check that $\tilde{\delta}_2(R_2(F)) = 0$. So setting $B_2(F) := \mathbb{Z}[F^*\backslash 1]/R_2(F)$ we get the Bloch complex ([DS], [S1])

$$B_2(F) \stackrel{\delta_2}{\longrightarrow} \Lambda^2 F^*, \quad \{x\} \mapsto (1-x) \wedge x \tag{3}$$

For an abelian group A put $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. Suslin proved that Ker$\delta_2 \otimes \mathbb{Q} = K_3^{ind}(F)_{\mathbb{Q}}$ ([S1]). Here $K_3^{ind}(F)$ is the cokernel of the multiplication $K_1(F)^{\otimes 3} \to K_3(F)$.

If $F = \mathbb{C}$ any real-valued function $f(z)$ defines a homomorphism $\tilde{f} : \mathbb{Z}[\mathbb{C}] \longrightarrow \mathbb{R}, \quad \{z\} \longmapsto f(z)$. Thanks to (2) we have a homomorphism $\tilde{\mathcal{L}}_2 : B_2(\mathbb{C}) \to \mathbb{R}$. Combined with the above homomorphism $K_3(\mathbb{C}) \to$ Kerδ_2 it gives an explicit formula for the Borel regulator $K_3(\mathbb{C}) \to \mathbb{R}$ and hence ([Bo2]) a formula for $\zeta_F(2)$ for any number field F (see s.5 below).

Let $\mathcal{H}^3$ be the 3-dimensional hyperbolic space. Denote by $I(z_0, ..., z_3)$ the ideal geodesic symplex with vertices at points $z_0, ..., z_3$ of the absolute $\partial\mathcal{H}^3 = \mathbb{C}P^1$. Then $volI(z_0, ..., z_3) = 3/2\mathcal{L}_2(r(z_0, ..., z_3))$ (Lobachevsky).

Any complete hyperbolic 3-fold of finite volume V^3 can be represented as a formal sum of ideal geodesic simplices. So $volV^3 = 3/2 \sum \mathcal{L}_2(x_i)$. It turns out the condition "V^3 is a manifold" implies $\delta_2 \sum \{x_i\} = 0$. (Thurston, [DS], [NZ]).

At first glance many features of this picture seem special for the dilogarithm. For example the classical n-logarithms are functions of just 1 variable, but for $n > 2$ GL_n does not act on P^1 , $\partial\mathcal{H}^n$ is no longer a complex manifold and so on. In this lecture I will explain how most of these facts about the dilogarithm are generalized to the trilogarithm and outline what should happen in general.

2. The trilogarithm and $\zeta_F(3)$ ([G2]). A single-valued version of $Li_3(z)$ is

$$\mathcal{L}_3(z) := Re\Big(Li_3(z) - Li_2(z) \cdot \log|z| + \frac{1}{3} Li_1(z) \cdot \log^2|z|\Big)$$

Denote by $\{x\}_2$ the image of $\{x\}$ in $B_2(F)$. Set

$$\mathbb{Z}[F^*] \stackrel{\delta_3}{\longrightarrow} B_2(F) \otimes F^*, \qquad \delta_3 : \{x\} \mapsto \{x\}_2 \otimes x, \quad \{1\} \mapsto 0 \tag{4}$$

Let F be a number field with r_1 real and r_2 complex places, $\{\sigma_j\}$ be the set of all possible embeddings $F \hookrightarrow \mathbb{C}$ numbered so that $\sigma_{r_1+k} = \overline{\sigma_{r_1+r_2+k}}$ and d_F be the discriminant of F. For $x \in \mathbb{Z}[F^*]$ one get numbers $\tilde{\mathcal{L}}_3(\sigma_j(x))$ defined via the composition $\mathbb{Z}[F^*] \stackrel{\sigma_j}{\hookrightarrow} \mathbb{Z}[\mathbb{C}^*] \stackrel{\tilde{\mathcal{L}}_3}{\longrightarrow} \mathbb{R}$.

Theorem 0.1. *For any number field F there exist elements $y_1, \dots y_{r_1+r_2} \in$ Ker$\delta_3 \otimes \mathbb{Q} \subset \mathbb{Q}[F^*]$ such that*

$$\zeta_F(3) = \pi^{3r_2} d_F^{-\frac{1}{2}} \det|\tilde{\mathcal{L}}_3(\sigma_j(y_i))| \,, \quad (1 \le i,j \le r_1+r_2) \,. \tag{5}$$

It was conjectured by Zagier, who gave many numerical examples ([Z1]). Here is one them:

$$\zeta_{\mathbb{Q}(\sqrt{5})}(3) = \frac{24}{25\sqrt{5}} \cdot \mathcal{L}_3(1) \cdot \Big(\mathcal{L}_3(\frac{1+\sqrt{5}}{2}) - \mathcal{L}_3(\frac{1-\sqrt{5}}{2})\Big)$$

If $\alpha := \frac{1+\sqrt{5}}{2}$ and $\bar{\alpha} := \frac{1-\sqrt{5}}{2}$ then $\alpha \cdot \bar{\alpha} = -1, \alpha + \bar{\alpha} = 1$, so $\{\alpha\}_2 \otimes \alpha - \{\bar{\alpha}\}_2 \otimes \bar{\alpha} = (\{\alpha\}_2 + \{1-\alpha\}_2) \otimes \alpha = 0$ modulo torsion because $6 \cdot (\{x\}_2 + \{1-x\}_2) \in R_2(F)$.

Let $\Delta : G \to G \times G$ be the diagonal map. An element $x \in H_n(G)$ is called primitive if $\Delta_*(x) = x \otimes 1 + 1 \otimes x$. For any field F one can define $K_n(F)_{\mathbb{Q}}$ as the subspace of primitive elements in $H_n(GL(F), \mathbb{Q})$.

Let $H_c^*(G, \mathbb{R})$ be continous cohomology of a Lie group G. It is known that $H_c^*(GL(\mathbb{C}), \mathbb{R}) = \Lambda_{\mathbb{R}}^*(c_1, c_3, ...)$ where $c_{2n-1} \in H_c^{2n-1}(GL(\mathbb{C}), \mathbb{R})$ are certain classes.. For example $c_1(g_1, g_2) = \log|detg_1^{-1}g_2|$. Considered as a functional on homology c_{2n-1} induces a map $r_{\mathbb{C}}(n) : K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \to \mathbb{R}$. It is called the Borel regulator [Bo]. Let F be a number field. Then the image of the composition

$$r(n) : K_{2n-1}(F) \longrightarrow \oplus_{Hom(F,\mathbb{C})} K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \stackrel{r_{\mathbb{C}}(n) \otimes \mathbb{R}(n-1)}{\longrightarrow} \mathbb{Z}^{Hom(F,\mathbb{C})} \otimes \mathbb{R}(n-1)$$

is invariant under the complex conjugation. So we get a regulator map $r(n) : K_{2n-1}(F) \longrightarrow \mathbb{R}(n-1)^{d_n}$. Here $d_n = r_1 + r_2$ for odd n and r_2 for even. We will use notation $a \sim b$ if $a/b \in \mathbb{Q}^*$. Borel proved that $r(n)(K_{2n-1}(F))$ is a lattice with covolume $\sim d_F^{1/2} \zeta_F(n)(\pi i)^{-nd_{n-1}}$.

The proof of our theorem is based on an explicit description of the regulator $K_5(\mathbb{C}) \to \mathbb{R}$ by means of the trilogarithm $\mathcal{L}_3$. The key step is a formula for a measurable 5-cocycle of $GL(\mathbb{C})$ representing the class c_5. For $GL_3(\mathbb{C})$ it looks as follows.

Let V^3 be a 3-dimensional vector space over F. Choose a volume form $\omega \in \wedge^3(V^3)^*$. For 6 vectors $l_1, \dots, l_6$ in generic position in V^3 set $\Delta(l_i, l_j, l_k) := \langle \omega, l_i \wedge l_j \wedge l_k \rangle \in F^*$. Let $\text{Alt}_6 f(l_1, \dots, l_6) := \sum_{\sigma \in S_6} (-1)^{|\sigma|} f(l_{\sigma(1)}, \dots, f_{\sigma(6)})$. Set

$$r_3(l_1, \dots, l_6) := \text{Alt}_6 \left\{ \frac{\Delta(l_1, l_2, l_4)\Delta(l_2, l_3, l_5)\Delta(l_3, l_1, l_6)}{\Delta(l_1, l_2, l_5)\Delta(l_2, l_3, l_6)\Delta(l_3, l_1, l_4)} \right\} \in \mathbb{Z}[F^*] \qquad (6)$$

$r_3(l_1, \dots, l_6)$ clearly does not depend on the lengths of vectors l_i and so is a generalized cross-ratio of 6 points on the projective plane.

Theorem 0.2. *a) For any 7 points $(m_1, \dots, m_7)$ in generic position in $\mathbb{C}P^2$*

$$\sum_{i=1}^{7} (-1)^i \tilde{\mathcal{L}}_3(r_3(m_1, ..., \hat{m}_i, ..., m_7)) = 0$$

b) Choose $x \in \mathbb{C}P^2$. Then the function $c_5(g_0, ..., g_5) := \tilde{\mathcal{L}}_3(r_3(g_0x, ..., g_5x))$ defined for $g_i \in GL_3(\mathbb{C})$ such that $(g_0x, ..., g_5x)$ is in general position, is a measurable 5-cocycle representing a nontrivial cohomology class of the group $GL_3(\mathbb{C})$.

3. Trilogarithm and algebraic K-theory. Let $R_3(F)$ be the subgroup of $\mathbb{Z}[F^*]$ generated by $\{x\} + \{x^{-1}\}$ and $\sum_{i=1}^{7} (-1)^i r_3(m_1, ..., \hat{m}_i, ..., m_7)$ where $(m_1, ..., m_7)$ run through all generic configurations of 7 points in P_F^2. Then $\delta_3 R_3(F) =$

0. Let $B_3(F)$ be the quotient of $\mathbb{Z}[F^*]$ by $R_3(F)$. We get a complex $B_F(3)$

$$B_3(F) \xrightarrow{\delta_3} B_2(F) \otimes F^* \xrightarrow{\delta_2 \wedge id} \Lambda^3 F^*$$

placed in degrees [1,3]. (δ_3,δ_2 were defined in (3), (4)).

According to [S2] $H_n(GL_n(F), \mathbb{Q}) = H_n(GL(F), \mathbb{Q})$. Let

$$K_n^{(i)}(F)_{\mathbb{Q}} := K_n(F)_{\mathbb{Q}} \cap Im\Big(H_n(GL_{n-i}(F), \mathbb{Q}) \to H_n(GL_n(F), \mathbb{Q})\Big)$$

be the rank filtration. Denote by $K_n^{[i]}(F)_{\mathbb{Q}}$ its graded quotients.

Theorem 0.3. *There are canonical maps* $K_{6-i}^{[3-i]}(F)_{\mathbb{Q}} \longrightarrow H^i(B_F(3) \otimes \mathbb{Q})$

They should be isomorphisms. This is known for $i = 3$ ([S2]).

4. Classical polylogarithms and motivic complexes. The following single-valued version of $Li_n(z)$ was invented by Zagier [Z1], see also [BD].

$$\mathcal{L}_n(z) \quad := \quad \begin{matrix} \text{Re} & (n: \text{ odd}) \\ \text{Im} & (n: \text{ even}) \end{matrix} \left(\sum_{k=0}^{n-1} \beta_k \log^k |z| \cdot Li_{n-k}(z) \right) , \quad n \geq 2$$

It is continuos on $\mathbb{C}P^1$. Here $\frac{2x}{e^{2x}-1} = \sum_{k=0}^{\infty} \beta_k x^k$.

Let us define inductively for each $n \geq 1$ a subgroup $\mathcal{R}_n(F) \subset \mathbb{Z}[P_F^1]$, which for $F = \mathbb{C}$ will be the subgroup of *all* functional equations for $\mathcal{L}_n(z)$.

Put $\mathcal{B}_n(F) := \mathbb{Z}[P_F^1]/\mathcal{R}_n(F)$. Set $\mathcal{R}_1(F) := (\{x\}+\{y\}-\{xy\}; \{0\}; \{\infty\})$.Then $\mathcal{B}_1(F) = F^*$. Let $\{x\}_n$ be the image of $\{x\}$ in $\mathcal{B}_n(F)$. Consider homomorphisms

$$\mathbb{Z}[P_F^1] \xrightarrow{\delta_n} \begin{cases} \mathcal{B}_{n-1}(F) \otimes F^* & : \quad n \geq 3 \\ \Lambda^2 F^* & : \quad n = 2 \end{cases} \tag{7}$$

$$\delta_n : \{x\} \mapsto \begin{cases} \{x\}_{n-1} \otimes x & : \quad n \geq 3 \\ (1-x) \wedge x & : \quad n = 2 \end{cases} \qquad \delta_n : \{\infty\}, \{0\}, \{1\} \mapsto 0 \tag{8}$$

Set $\mathcal{A}_n(F) :=$ Ker δ_n . Any element $\alpha(t) = \Sigma n_i \{f_i(t)\} \in \mathbb{Z}[P_{F(t)}^1]$ has a specialization $\alpha(t_0) := \Sigma n_i \{f_i(t_0)\} \in \mathbb{Z}[P_F^1]$ at each point $t_0 \in P_F^1$.

Definition 0.4. $\mathcal{R}_n(F)$ *is generated by elements* $\{\infty\}, \{0\}$ *and* $\alpha(0) - \alpha(1)$ *where* $\alpha(t)$ *runs through all elements of* $\mathcal{A}_n(F(t))$.

One can show that $\delta_n \mathcal{R}_n(F) = 0$ ([G1], 1.16). So we get homomorphisms

$$\delta_n : \mathcal{B}_n(F) \longrightarrow \mathcal{B}_{n-1}(F) \otimes F^*, \quad n \geq 3; \quad \delta_2 : \mathcal{B}_2(F) \longrightarrow \Lambda^2 F^*$$

and finally the following complex $\Gamma(F, n)$:

$$\mathcal{B}_n \xrightarrow{\delta} \mathcal{B}_{n-1} \otimes F^* \xrightarrow{\delta} \mathcal{B}_{n-2} \otimes \Lambda^2 F^* \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{B}_2 \otimes \Lambda^{n-2} F^* \xrightarrow{\delta} \Lambda^n F^*$$

where $\delta : \{x\}_p \otimes \bigwedge_{i=1}^{n-p} y_i \to \delta_p(\{x\}_p) \wedge \bigwedge_{i=1}^{n-p} y_i$ has degree +1 and $\mathcal{B}_n \equiv \mathcal{B}_n(F)$ placed in degree 1. One can prove that $\tilde{\mathcal{L}}_n\Big(\mathcal{R}_n(\mathbb{C})\Big)$ (see [G2] theorem 1.13).

Conjecture 0.5. *Let* $f(z)$ *be a measurable function such that* $\tilde{f}(\mathcal{R}_n(\mathbb{C})) = 0$. *Then* $f(z) = \lambda_0 \mathcal{L}_n(z) + \lambda_1 \mathcal{L}_{n-1}(z) \log|z| + ... + \lambda_{n-2} \mathcal{L}_2(z) \log|z|^{n-2}$, $\lambda_i \in \mathbb{C}$.

This is true for $n=2$ ([Bl]) and $n=3$ (unpublished).

Let γ be the Adams filtration on $K_n(F)_{\mathbb{Q}}$. Hypothetically it is opposite to the rank filtration. For number fields $gr^{\gamma}_n K_m(F)_{\mathbb{Q}} = 0$ if $m \neq 2n-1$.

Conjecture A *a)For any field F* $H^i\Gamma(F,n)\otimes\mathbb{Q} = gr^{\gamma}_n K_{2n-i}(F)\otimes\mathbb{Q}$.

b) The composition $gr^{\gamma}_n K_{2n-1}(\mathbb{C})_{\mathbb{Q}} \to H^1\Gamma(\mathbb{C},n)_{\mathbb{Q}} \to \mathbb{R}$ *is a nonzero rational multiple of the Borel regulator.*

For number fields the isomorphism $K_{2n-1}(F)_{\mathbb{Q}} = Ker_{\delta_n}$ was conjectured (slightly differently, without the complexes $\Gamma(F,n)$) by Zagier [[Z1]).

So we get a hypothetical description of Quillen's K-groups by symbols that generalizes Milnor's approach to K-theory ($H^n\Gamma(F,n) = K^M(F)$ by definition):

$$K_m(F)_{\mathbb{Q}} \stackrel{?}{=} \oplus_n H^{2n-m}(\Gamma(F,n)\otimes\mathbb{Q}) \tag{9}$$

This suggests that $\Gamma(F,n)\otimes\mathbb{Q}$ should be the weight n motivic complex conjectured by Beilinson and Lichtenbaum ([B1], [Li]). Another approach see in [Bl2].

For a compact smooth i-dimensional variety X over $\mathbb{Q}$ Beilinson defined a regulator map to Deligne cohomology ([B2]) $r_{Be} : gr^{\gamma}_n K_{2n-i}(X) \longrightarrow H^i_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$

A regular model $X_{\mathbb{Z}}$ of X over $\mathbb{Z}$ defines a subgroup $gr^{\gamma}_n K_{2n-i}(X_{\mathbb{Z}}) \subset gr^{\gamma}_n K_{2n-i}(X)$. For $n > i+1$ they should coincide. Beilinson conjectured that $r_{Be}(gr^{\gamma}_n K_{2n-i}(X_{\mathbb{Z}}))$ is a lattice whose covolume with respect to the natural $\mathbb{Q}$- structure provided by $H^i_{\mathcal{D}}(X/\mathbb{R}, \mathbb{Q}(n))$ up to a standard factor coincides with the value of L-function $L(h^i(X), s)$ at $s=i$. Unfortunately the definition of the regulator is rather implicit.

Conjecture A together with Beilinson's conjecture should give explicit formulas for special values of the L-functions of varieties over number fields in terms of *classical* polylogarithms. Below two examples are discussed: ζ-functions of number fields and L-functions of elliptic curves.

5. Zagier's conjecture. Conjecture A b) and Borel's theorem [Bo2] lead to

Conjecture 0.6. *For any number field F there exists elements* $y_1, \dots y_{d_n} \in \mathrm{Ker}\delta_n \otimes \mathbb{Q} \subset \mathcal{B}_n(F)_{\mathbb{Q}}$ *such that*

$$\zeta_F(n) = \pi^{d_{n-1}\cdot n} d_F^{-\frac{1}{2}} \det|\tilde{\mathcal{L}}_n(\sigma_j(y_i))| \ , \quad (1 \le i,j \le d_n) \ , \tag{10}$$

It was stated in [Z1]. The case $n=2$, essentially proved in [Z2], follows from the Borel theorem and the results of Bloch [Bl1] and Suslin [S2] (see s.1); a simpler proof see in s.2 of [G1]. It is not proved for $n>3$.

Theorem 0.7. *For any number field F there is a map* $l_n : Ker\delta_n \otimes \mathbb{Q} \to K_{2n-1}(F)_{\mathbb{Q}}$ *such that for any* $\sigma : F \hookrightarrow \mathbb{C}$ *one has* $r_{\mathbb{C}}(n)(\sigma \circ l_n(y)) = \tilde{\mathcal{L}}_n(\sigma(y))$.

This was proved by Beilinson-Deligne [BD] and later de Jeu [J]. It can be deduced from the existence of the triangulated category of mixed Tate motives over $Spec(F)$ constructed by Levine [L] and Voevodsky [V].

6. Motivic complexes for curves. Let K be a field with a discrete valuation v, the residue field k_v and the group of units U. Let $u \to \bar{u}$ be the projection $U \to k_v^*$. Choose a uniformizer π. There is a homomorphism $\theta : \Lambda^n F^* \longrightarrow \Lambda^{n-1} F_v^*$ uniquely defined by the following properties ($u_i \in U$):

$$\theta\,(\pi \wedge u_1 \wedge \dots \wedge u_{n-1}) = \bar{u}_1 \wedge \dots \wedge \bar{u}_{n-1}; \qquad \theta\,(u_1 \wedge \dots \wedge u_n) = 0$$

It is clearly independent of π. Let us define a homomorphism $s_v : \mathbb{Z}[P^1_K] \longrightarrow \mathbb{Z}[P^1_{k_v}]$ setting $s_v\{x\} = \{\bar{x}\}$ if x is a unit and 0 otherwise. It induces a homomorphism $s_v : \mathcal{B}_m(K) \longrightarrow \mathcal{B}_m(k_v)$. Put

$$\partial_v := s_v \otimes \theta : \mathcal{B}_m(K) \otimes \Lambda^{n-m} K^* \longrightarrow \mathcal{B}_m(k_v) \otimes \Lambda^{n-m-1} k_v^* \ .$$

It defines a morphism of complexes $\partial_v : \Gamma(K, n) \longrightarrow \Gamma(k_v, n-1)[-1]$. Let X be a regular curve over a field F and F_x be the residue field of a point $x \in X$. Let us define the motivic complex $\Gamma(X, n)$ as follows ($\mathcal{B}_n(F(X))$ is in degree 1):

$$\begin{array}{ccccccc} \mathcal{B}_n(F(X)) & \xrightarrow{\delta} & \mathcal{B}_{n-1}(F(X)) \otimes F(X)^* & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \Lambda^n F(X)^* \\ & & \downarrow \coprod_x \partial_x & & & & \downarrow \coprod_x \partial_x \\ & & \coprod_{x \in X^1} \mathcal{B}_{n-1}(F_x) & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \coprod_{x \in X^1} \Lambda^{n-1} F_x^* \end{array} \qquad (11)$$

Conjecture 0.8. *For a regular curve X one has $H^i(\Gamma(X,n) \otimes \mathbb{Q}) = gr^{\gamma}_n K_{2n-i}(X)_{\mathbb{Q}}$.*

7. Explicit formulas for regulators in the case of curves ([G6]). Let me recall that for a curve X over $\mathbb{R}$ and $n > 1$ $H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n)) = H^2(X, \mathbb{R}(n-1))^+$ where "+" means invariants of the complex conjugation acting both on $X(\mathbb{C})$ and $\mathbb{R}(n-1)$. Beilinson's regulator for curves over $\mathbb{Q}$ is a homomorphism

$$r_{Be}(n) : K_{2n-2}(X)_{\mathbb{Q}} \longrightarrow H^2_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$$

Cup product with $\omega \in \Omega^1(\bar{X})$ identifies $H^1(\bar{X}, \mathbb{R}(n-1))$ with $H^0(\bar{X}, \Omega^1)^{\vee}$. So we will view elements of $H^2_{\mathcal{D}}(\bar{X}/\mathbb{R}, \mathbb{R}(n))$ as functionals on $H^0(\bar{X}, \Omega^1)^{\vee}$.

Set $\alpha(f, g) := \log|f| d\log|g| - \log|g| d\log|f|$.

Theorem 0.9. *Let X be a curve over $\mathbb{Q}$. Then for each element $\gamma_{2n-2} \in K_{2n-2}(X)$, $n = 3, 4$, there are rational functions $f_i, g_i \in \mathbb{Q}(X)^*$ such that $\sum_i \{f_i\}_{n-1} \otimes g_i$ is a 2-cocycle in (11) and for any $\omega \in \Omega^1(X)$ one has ($a_n, b_n \in \mathbb{Q}^*$):*

$$\int_{X(\mathbb{C})} r_{Be}(n)(\gamma_{2n-2}) \wedge \omega = a_n \cdot \sum_i \int_{X(\mathbb{C})} \mathcal{L}_{n-1}(f_i) d\log|g_i| \wedge \omega = \qquad (12)$$

$$b_n \cdot \sum_i \int_{X(\mathbb{C})} \log|g_i| \log^{n-3}|f_i| \alpha(1-f_i, f_i) \wedge \omega$$

For $n = 2$ this is the famous symbole modéré of Beilinson and Deligne. Hypothetically (12) should be true for all n.

Example. For $n = 3$ the condition "$\sum_i \{f_i\}_2 \otimes g_i$ is a 2-cocycle in (11)" means that $\sum_i (1-f_i) \wedge f_i \wedge f_i = 0$ in $\Lambda^3 \mathbb{Q}(X)^*$ and $\sum_i v_x(g_i)\{f_i(x)\}_2 = 0$ in $\mathcal{B}_2(\bar{\mathbb{Q}})$ for any $x \in X(\bar{\mathbb{Q}})$. Here v_x is the valuation defined by a point x.

8. Special values of L-functions of elliptic curves ([G6]). Let E be an elliptic curve $/\mathbb{Q}$ and $\Gamma := H_1(E(\mathbb{C}), \mathbb{Z})$. A holomorphic 1-form ω defines an embedding $\Gamma \hookrightarrow \mathbb{C}$ together with an isomorphism $E(\mathbb{C}) = \mathbb{C}/\Gamma = \Gamma \otimes \mathbb{R}/\Gamma$. The intersection pairing $\Gamma \times \Gamma \to \mathbb{Z}(1)$ provides a pairing $(\cdot, \cdot) : E(\mathbb{C}) \times \Gamma \longrightarrow U(1) \subset \mathbb{C}^*$. If $\Gamma = \mathbb{Z}u + \mathbb{Z}v \subset \mathbb{C}$ with $Im(u/v) > 0$ then $(z, \gamma) = \exp A(\Gamma)^{-1}(z\bar{\gamma} - \bar{z}\gamma)$ where $A(\Gamma) = \frac{1}{2\pi i}(\bar{u}v - u\bar{v})$. Consider the generalized Eisenstein-Kronecker series ($\gamma_i \in \Gamma$)

$$K_n(x, y, z) := \sum_{\gamma_1 + \ldots + \gamma_n = 0}{}' \frac{(x, \gamma_1)(y, \gamma_2 + \ldots + \gamma_{n-1})(z, \gamma_n)(\bar{\gamma}_n - \bar{\gamma}_{n-1})}{|\gamma_1|^2 |\gamma_2|^2 \ldots |\gamma_n|^2}, \quad n \geq 3$$

They are invariant under the shift $(x,y,z) \to (x+t, y+t, z+t)$ and so live actually on $E(\mathbb{C}) \times E(\mathbb{C})$. For n = 2 put $K_2(x,y,z) := \sum'_\gamma \frac{(x-z,\gamma)}{|\gamma|^2\gamma}$.

Let $\omega \in H^0(E, \Omega^1_{E/\mathbb{Q}})$ and $\Omega = \int_{E(\mathbb{R})} \omega$ be the real period of E.

Conjecture 0.10. *a) Let E be an elliptic curve / $\mathbb{Q}$ and $n \geq 3$. Then there exist functions $f_i, g_i \in \mathbb{Q}(E)^*$ such that $\sum_i \{f_i\}_{n-1} \otimes g_i$ is a 2-cocycle in (11) and*

$$q \cdot L(E,n) = \Big(\frac{2\pi A(\Gamma)}{f_E}\Big)^{n-1} \Omega \cdot \sum_i K_n(x_i, y_i, z_i) \tag{13}$$

where x_i, y_i, z_i are divisors of $f_i, g_i, 1-f_i$ and $q \in \mathbb{Q}^*$.

b) For any $f_i, g_i \in \mathbb{Q}(E)^*$ as above formula (13) holds with (possibly 0) $q \in \mathbb{Q}$.

For n=2 (13) is Bloch's conjecture [Bl1] and for n=3 it was conjectured (slightly differently, using Massey products) by Deninger [Den]. A conjecture for any elliptic curve over a number field involves determinants with entries $K_n(x,y,z)$.

Theorem 0.11. *Conjecture 0.10 holds for modular elliptic curves over $\mathbb{Q}$ in the cases $n = 3$ and $n = 4$.*

The proof uses theorem 0.3, a similar result in weight 4, theorem 0.9 and weak Beilinson's conjecture for modular curves proved in [B3]. For example for $n = 3$ we get the formula

$$L(E,3) \sim \Big(\frac{2\pi A(\Gamma)}{f_E}\Big)^2 \Omega \cdot \sum_i \sum'_{\gamma_1+\gamma_2+\gamma_3=0} \frac{(x_i,\gamma_1)(y_i,\gamma_2)(z_i,\gamma_3)}{|\gamma_1|^2|\gamma_2|^2|\gamma_3|^2}$$

In a similar conjecture about $L(S^nE, n+1)$ appears determinants whose entries are the classical Kronecker-Eisenstein series $\sum_{\gamma\in\Gamma} \frac{(x-y,\gamma)}{\gamma^a\bar{\gamma}^b}$ $(a+b = 2n+1)$. Their motivic interpretation was given in [BL]. One should have it also for functions $K_n(x,y,z)$, and, more generally, for functions needed to compute $L(S^nE, m)$.

9. Motivic Lie algebra $L(F)_\bullet$ ([G2]). Beilinson conjectured ([B1], [BD2]) that for any filed F should exist a Tannakian (i.e. abelian, tensor, ...) category $\mathcal{M}_T(F)$ of mixed Tate motives over F . It is generated (as tensor category) by an invertible object $\mathbb{Q}(1)_\mathcal{M}$ (Tate motive). Set $\mathbb{Q}(n)_\mathcal{M} := \mathbb{Q}(1)^{\otimes n}_\mathcal{M}$, $n \in \mathbb{Z}$. The crucial axiom is:

$$\mathrm{Ext}^i_{\mathcal{M}_T(F)}(\mathbb{Q}(0)_\mathcal{M}, \mathbb{Q}(n)_\mathcal{M}) \overset{?}{\cong} gr^n_\gamma K_{2n-i}(F)_\mathbb{Q} \tag{14}$$

Any object M of this category carries canonical increasing weight filtration $W_\bullet M$ such that $gr^W_{2k} M = \oplus \mathbb{Q}(-k)_\mathcal{M}$ and $gr^W_{2k-1} M = 0$. There is canonical fiber functor ω from $\mathcal{M}_T(F)$ to the category of finite dimensional graded $\mathbb{Q}$-vector spaces: $\omega(M) := \oplus Hom(\mathbb{Q}(-k)_\mathcal{M}, gr^W_{2k} M)$. Let $U(F)_\bullet := End\omega$ be the space of all endomorphisms of the functor ω. It is a graded (pro) Hopf algebra over $\mathbb{Q}$.

Let $L(F)_\bullet$ be the Lie algebra of all derivations of ω. It is naturally graded: $L(F)_\bullet = \oplus_{n\geq 1} L(F)_{-n}$ and $U(F)_\bullet$ is its universal enveloping algebra. The functor ω is an equivalence of the category $\mathcal{M}_T(F)$ with the category of finite dimensional graded modules over $L(F)_\bullet$.

The degree n part of the cochain complex $(\Lambda^{\bullet}(L(F)^{\vee}_{\bullet}), \partial)$ of the Lie algebra $L(F)_{\bullet}$ forms a subcomplex (here $V^{\vee}$ is dual to V, and $L^{\vee}_{-n}$ is in degree 1):

$$L^{\vee}_{-n} \xrightarrow{\partial} \dots \xrightarrow{\partial} L^{\vee}_{-2} \otimes \Lambda^{n-2} L^{\vee}_{-1} \xrightarrow{\partial} \Lambda^{n} L^{\vee}_{-1} \qquad (15)$$

Its cohomology is predicted by formula (14). Moreover it should be quasiisomorphic to the weight n motivic complex for $Spec(F)$: (14) provides its key property. So conjecture A suggests that it should be quasiisomorphic to our complex $\Gamma(F, n)$.

One should have canonical injective homomorphisms $l_n : \mathcal{B}_n(F) \hookrightarrow L(F)^{\vee}_{-n}$ (see s.12 below). But already for $n = 4$ in degree 2 of (15) appears $\Lambda^2 L^{\vee}_{-2}(F) \stackrel{?}{=} \Lambda^2 \mathcal{B}_2(F)$ which is absent in $\Gamma(F, 4)$. So complex (15) is bigger then $\Gamma(F, n)$

Set $I_{\bullet} := \oplus_{n=2}^{\infty} L(F)_{-n}$ and let $H^1_{(n)}(I(F)_{\bullet})$ be the degree n part of $H^1(I(F)_{\bullet})$. Conjecture A is essentially equivalent to the following one about the structure of the Lie algebra $L(F)_{\bullet}$:

Conjecture B. a) *$I(F)_{\bullet}$ is a free graded pro-Lie algebra.*

b) $H^1_{(n)}(I(F)_{\bullet}) = \mathcal{B}_n(F)_{\mathbb{Q}}$ for $n \geq 2$, i.e. *$I(F)_{\bullet}$ is generated as a graded pro-Lie algebra by the spaces $\mathcal{B}_n(F)^{\vee}$* of degree $-n$.

c) *The action of $L_{\bullet}/I_{\bullet} = F^{*\vee}_{\mathbb{Q}}$ on $H_1^{(n)}(I(F)_{\bullet}) = \mathcal{B}_n(F)^{\vee}_{\mathbb{Q}}$ coming from the extension $0 \to H_1(I_{\bullet}) \to L_{\bullet}/[I_{\bullet}, I_{\bullet}] \to L_{\bullet}/I_{\bullet} \to 0$ is described by the homomorphism dual to $\delta_n : \mathcal{B}_n(F)_{\mathbb{Q}} \to \mathcal{B}_{n-1}(F)_{\mathbb{Q}} \otimes F^*$.*

Assuming conjecture B it is easy to see that the Hochshild-Serre spectral sequence for $H^*_{(n)}(L(F)_{\bullet})$ with respect to the ideal $I_{\bullet}$ reduces exactly to the complex $\Gamma(F, n)$. Indeed thanks to a) and b) we have

$$E_1^{p,q} = C^p(L_{\bullet}/I_{\bullet}, H^q_{(n-p)}(I_{\bullet})) = \begin{cases} \Lambda^p F^*_{\mathbb{Q}} \otimes \mathcal{B}_{n-p}(F)_{\mathbb{Q}} : & q = 1 \\ \Lambda^n F^*_{\mathbb{Q}} & : q = 0, n = p \\ 0 & : \text{otherwise} \end{cases}$$

and the differentials coincide with the ones in $\Gamma(F, n)$ because of c) .

10. Framed mixed Tate motives and $U(F)_{\bullet}$ ([BMS],[BGSV]). A mixed $\mathbb{Q}$ - Hodge structure H is called a Hodge-Tate structure if all the quotients $gr^W_{\bullet} H$ are of Hodge type (p, p). It is an n-framed Hodge-Tate structure if supplied with nonzero vectors $v \in gr^W_{2n} H$ and $f \in (gr^W_0 H)^*$.

Consider the coarsest equivalence relation on the set of all n-framed Hodge-Tate structures for which $H_1 \sim H_2$ if there is a morphism of mixed Hodge structures $H_1 \to H_2$ respecting the frames. Let $\mathcal{H}_n$ be the set of equivalence classes. It has an abelian group structure: $(H; v, f) \oplus (H'; v', f') := (H \oplus H'; (v, v'), f + f')$. Set $\mathcal{H}_0 := \mathbb{Z}$. The tensor product of mixed Hodge structures induces the commutative multiplication $\mu : \mathcal{H}_k \otimes \mathcal{H}_\ell \to \mathcal{H}_{k+\ell}$. A comultiplication $\nu = \bigoplus_k \nu_{k,n-k} : \mathcal{H}_n \to \bigoplus_k \mathcal{H}_k \otimes \mathcal{H}_{n-k}$ is defined as follows. Let $\{e_j\}$ and $\{e^j\}$ be dual bases in $gr^W_{2k} H_{\mathbb{Q}}$ and $gr^W_{2k} H^*_{\mathbb{Q}}$. Set $\nu_{k,n-k}((H; v, f)) := \sum_j (H; v, e^j) \otimes (H; e_j, f)$.

Then $\mathcal{H}_{\bullet} := \oplus \mathcal{H}_n$ is a commutative graded Hopf algebra.

Similary the equivalence classes of n-framed objects in the category $\mathcal{M}_T(F)$ form a commutative graded Hopf algebra $\mathcal{M}_{\bullet}$. It maps to $U(F)^{\vee}_{\bullet}$: the value of the functional defined by $(\omega(M), v, f)$ on $A \in End\omega$ is $< f, Av >$. *This map is an* *isomorphism of Hopf algebras.* In particulary

$$Ker\Big(U(F)^{\vee}_{-n} \stackrel{\Delta}{\longrightarrow} \oplus_k U(F)^{\vee}_{-(n-k)} \otimes U(F)^{\vee}_{-k}\Big) \stackrel{?}{\cong} gr^{\gamma}_{n} K_{2n-1}(F)_{\mathbb{Q}} \qquad (16)$$

It seems that any example of variation of framed mixed Tate motives should be of great interest; the corresponding Hodge periods deserve to be called polylogarithms (don't confuse them with the *classical* polylogarithms!). Below I discuss two such examples where periods are volumes of non-euclidian geodesic simplices and hyperlogarithms. Another example see in [BGSV].

10. Hyperbolic geometry ([G4]).

Theorem 0.12. *Let V^5 be a 5-dimensional complete hyperbolic manifold of finite volume. Then there are algebraic numbers $z_i \in \bar{\mathbb{Q}}^*$ such that*

$$\sum_i \{z_i\}_2 \otimes z_i = 0 \ \text{in} \ B_2(\bar{\mathbb{Q}}) \otimes \bar{\mathbb{Q}}^* \quad \text{and} \quad \mathrm{vol}(V^5) = \sum_i \mathcal{L}_3(z_i)$$

Conjecture 0.13. *Let V^{2n-1} be an $(2n-1)$-dimensional complete hyperbolic manifold of finite volume. Then there are algebraic numbers $z_i \in \bar{\mathbb{Q}} \subset \mathbb{C}$ such that $(n \geq 3)$ $\quad \delta_n(\sum_i \{z_i\}_n) = 0$ and $\mathrm{vol}(V^{2n-1}) = \sum_i \mathcal{L}_n(z_i)$.*

A geodesic simplex M in the hyperbolic space $\mathcal{H}^m$ define a mixed Tate motive. Indeed, in the Klein model $\mathcal{H}^m$ is the interior of a ball in $\mathbb{R}^m$ and geodesics are straight lines. So a geodesic simplex is the usual one inside the absolute: sphere Q.

After complexification and compactification we get $\mathbb{CP}^m$ together with a quadric Q (the absolute) and a collection of hyperplanes $M = (M_1, \dots, M_{m+1})$ ($(n-1)$-faces of a geodesic simplex). $H(Q, M) := H^m(\mathbb{C}P^m \backslash Q, M)$ is a Hodge-Tate structure.

Let $m = 2n-1$ and $\tilde{Q}(x) = 0$ be a quadratic equation of Q. Set

$$\omega_Q := \pm \frac{\sqrt{det\tilde{Q}}}{(2\pi i)^n} \frac{\sum_{i=o}^{2n-1} (-1)^i x_i dx_0 \wedge ...\hat{dx_i} \wedge dx_{2n-1}}{\tilde{Q}(x)^n}$$

The sign depends on the choise of a generator in the primitive part of $H^{n-1}(Q, \mathbb{Z})$. It is provided by an orientation of $\mathcal{H}^{2n-1}$. The simplex M defines a chain Δ_M representing a generator in $H_{2n-1}(\mathbb{C}P^{2n-1}, M)$. Then $vol(M) = \int_{\Delta_M} \omega_Q$.

The scissor congruence group $\mathcal{P}(\mathcal{H}^m)$ is an abelian group generated by pairs $[M, \alpha]$ where M is an oriented geodesic simplex and α is an orientation of $\mathcal{H}^m$.The relations are: $[M, \alpha] = [M_1, \alpha] + [M_2, \alpha]$ if $M = M_1 \cup M_2$; $[M, \alpha]$ changes sign if we change orientation of M or α, and $[M, \alpha] = [gM, g\alpha]$ for any $g \in O(m, 1)$. The spherical scissor congruence groups $\mathcal{P}(S^m)$ are defined similary. $\mathcal{P}(\mathcal{S}^{2k}) = 0$.

The volume provides homomorphisms $\mathcal{P}(\mathcal{H}^m) \to \mathbb{R}$ and $\mathcal{P}(S^m) \to \mathbb{R}/\mathbb{Z}$.

We have a vector $[\omega_Q]$ in $H^{2n-1}(\mathbb{C}P^{2n-1} \backslash Q) = gr^W_{2n} H(Q, M)$ and a functional $[\Delta_M]$ on $H^{2n-1}(\mathbb{C}P^{2n-1}, M) = gr^W_0 H(Q, M)$. So we get an n-framed Hodge-Tate structure associated with $[M, \alpha]$. This construction defines a homomorphism of groups $\mathcal{P}(\mathcal{H}^{2n-1}) \to \mathcal{H}_n$ and similary $\mathcal{P}(S^{2n-1}) \to \mathcal{H}_n$.

Let us define the Dehn invariant $\mathcal{P}(\mathcal{H}^{2n-1}) \stackrel{D^h_n}{\longrightarrow} \oplus_k \mathcal{P}(\mathcal{H}^{2k-1}) \otimes \mathcal{P}(S^{2(n-k)-1})$. Each $(2k-1)$-face A of M is a hyperbolic simplex $h(A)$. In the orthogonal plane $A^{\perp}$ M cuts a spherical simplex

$s(A)$. Choose orientations α_A and β_A of A and $A^\perp$ such that $\alpha_A \otimes \beta_B = \alpha$. Then $D_n^h([M,\alpha]) := \sum_A [h(A),\alpha_A] \otimes [s(A),\beta_A]$.

Theorem 0.14. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{P}(\mathcal{H}^{2n-1}) & \xrightarrow{D_n^h} & \oplus_k \mathcal{P}(\mathcal{H}^{2k-1}) \otimes \mathcal{P}(S^{2(n-k)-1}) \\ \downarrow & & \downarrow \\ \mathcal{H}_n & \xrightarrow{\nu} & \oplus_k \mathcal{H}_k \otimes \mathcal{H}_{n-k} \end{array}$$

A similar motivic interpretation has the spherical Dehn invariant $D_n^s : \mathcal{P}(S^{2n-1}) \longrightarrow \oplus_k \mathcal{P}(\mathcal{S}^{2k-1}) \otimes \mathcal{P}(S^{2(n-k)-1})$. So (16) leads to

Conjecture 0.15. *There are canonical injective homomorphisms*

$$KerD_n^h \otimes \mathbb{Q} \hookrightarrow [gr_n^\gamma K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q}(n)]^- \qquad KerD_n^s \otimes \mathbb{Q} \hookrightarrow [gr_n^\gamma K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q}(n)]^+$$

whose composition with Beilinson's regulator coincide with the volume homomorphisms.

If $n = 2$ they exist and are isomorphisms by the results of [D], [DS], [S1].

Each complete hyperbolic $(2n-1)$-manifold can be cuted on geodesic simplices and so produces an element in $\mathcal{P}(\mathcal{H}^{2n-1})$. Its Dehn invariant is equal to zero. So conjecture 0.13 follows from conjectures 0.15 and A.

11. Hyperlogarithms ([G5]). They where considered by Kummer ([Ku]), Poincare, Lappo-Danilevsky, ... We define them as the following iterated integrals:

$$\Psi_{m_1,...,m_l}(a_1,...,a_l) := \int_0^1 \underbrace{\frac{dt}{t-a_1} \circ \frac{dt}{t} \circ ... \circ \frac{dt}{t}}_{m_1 \text{ times}} \circ ... \circ \underbrace{\frac{dt}{t-a_l} \circ \frac{dt}{t} \circ ... \frac{dt}{t}}_{m_l \text{ times}}$$

This formula means the following. Let $n := m_1 + ... + m_l$ and

$$\Delta := \{(t_1,...,t_n) \subset \mathbb{R}^n | 0 \leq t_1 - a_1 \leq t_2 \leq ... \leq t_{m_1} \leq t_{m_1+1} - a_2 \leq t_{m_1+2} ... \leq t_{m_l}\}$$

Let L be a coordinate simplex in $\mathbb{C}P^n$ related to coordinates $(t_0 : ... : t_n)$ and $\omega_L := \frac{dt_1}{t_1} \wedge ... \wedge \frac{dt_n}{t_n}$. Then $\Psi_{m_1,...,m_l}(a_1,...,a_l) = \int_\Delta \omega_L$.

Let M be collection of all the hyperplanes corresponding to codimension 1 faces of Δ. Then $H(L,M) := H^n(\mathbb{C}P^n \backslash L, M)$ is a Hodge-Tate structure. It has canonical n-framing: $[\omega_L]$ is a vector in $H^n(\mathbb{C}P^n \backslash L) = gr_{2n}^W H(L,M)$ and Δ produces a class $[\Delta] \in H_n(\mathbb{C}P^n, M) = gr_0^W H(L,M)$. So we get an element $\Psi^{\mathcal{H}}_{m_1,...,m_l}(a_1,...,a_l) \in \mathcal{H}_n$. According to the general philosophy *a mixed Hodge structure in the cohomology of a (simplicial) variety is a realisation of a mixed motive.* So we should have an n-framed mixed Tate motive $\Psi^{\mathcal{M}}_{m_1,...,m_l}(a_1,...,a_l)$.

More generally, if F is a field and $a_i \in F^*$ one should also have an n-framed mixed Tate motive $\Psi^{\mathcal{M}}_{m_1,...,m_l}(a_1,...,a_l)$ related to $H^n(P_F^n \backslash L, M)$.

There is a remarkable power series expansion of the hyperlogarithms. Namely, consider *multiple* polylogarithms

$$\Phi_{m_1,...,m_l}(x_1,...,x_l) := \quad (-1)^l \sum_{0<k_1<k_2<...<k_l} \frac{x_1^{k_1} x_2^{k_2} ... x_l^{k_l}}{k_1^{m_1} k_2^{m_2} ... k_l^{m_l}}$$

Theorem 0.16. *([G5]) Suppose* $|a_i/a_{i-1}| < 1$. *Then*

$$\Psi_{m_1,...,m_l}(a_1,...,a_l) = \Phi_{m_1,...,m_l}(\frac{a_2}{a_1}, \frac{a_3}{a_2}, ..., \frac{1}{a_l})$$

In particulary $\zeta(m_1,...,m_l) := \Psi_{m_1,...,m_l}(1,1,...,1)$ are the multiple zeta values of Euler [E], rediscovered and studied by Zagier [Z3], see also [Dr] and [Ko].

Conjecture 0.17. . *a) Any n-framed mixed Tate motive over F is a sum of hyperlogarithmic ones* $\Psi^{\mathcal{M}}_{m_1,...,m_l}(a_1,...,a_l)$*, where* $n = m_1 + ... + m_l$*;* $a_i \in F^*$.

b)Any n-framed mixed Tate motive over Spec$(\mathbb{Z})$ *is a sum of motivic multiple zeta's* $\zeta^{\mathcal{M}}(m_1,...,m_l)$

The first part of the conjecture is motivated by the following

Proposition 0.18. *(***Universality of hyperlogarithms***) Any iterated integral* $F(z) = \int_x^z \omega_1 \circ ... \circ \omega_n$ *of rational 1-forms* ω_i *on a rational variety X is a sum of hyperlogarithms, i.e. there exist* $f_j^{(i)}(z) \in \mathbb{C}(X)^*$ *such that*

$$F(z) = \sum_i \Psi_{m_1^{(i)},...,m_{l^{(i)}}^{(i)}}(f_1^{(i)}(z),...,f_l^{(i)}(z)) \quad + \quad C \qquad (C \text{ is a constant})$$

12. Motivic interpretation of the "weak" part of conjecture A. For any $a \in F^*$ the n-framed mixed Tate motive $\Psi_n^{\mathcal{M}}(a^{-1})$ (corresponding to $Li_n(a)$) provides a homomorphism $\tilde{l}_n : \mathbb{Z}[F^*] \to U(F)^{\vee}_{-n}$. Denote by l_n its composition with the canonical projection $U(F)^{\vee}_{-n} \to L(F)^{\vee}_{-n}$.

One should have $l_n(\mathcal{R}_n(F)) = 0$, so $l_n : \mathcal{B}_n(F) \to L(F)^{\vee}_{-n}$. It turns out that $\partial(l_n\{a\}) = l_{n-1}\{a\} \wedge a$ (we identified $L(F)^{\vee}_{-1}$ with $F^*_{\mathbb{Q}}$), Therefore *homomorphisms* $\{l_i\}$ *provide a canonical homomorphism of the complex* $\Gamma(F,n)$ *to the complex (15).* Using (14) we get canonical maps $H^i(\Gamma(F,n) \otimes \mathbb{Q}) \to gr_n^{\gamma} K_{2n-i}(F)_{\mathbb{Q}}$.

13. The quantum dilogarithm ([FK]). Mixed Tate motives give the best explanation *all* of the different appearances of the dilogarithm discussed above. However recently the dilogarithm appeared in conformal field theory and exactly solvable problems of statistical mechanics. Here is one example.

Let $\Psi(x) := \prod_{n=1}^{\infty}(1 - xq^n)$, $\quad |q| < 1$. Then for $q = exp(\epsilon)$, $Im(\epsilon) < 0$

$$\Psi(x) = \frac{1}{\sqrt{1-x}} exp(Li_2(x)/\epsilon)(1 + O(\epsilon)), \quad \epsilon \to 0$$

Theorem 0.19. *([FK]) Suppose* $\hat{U}$ *and* $\hat{V}$ *satisfies* $\hat{U}\hat{V} = q\hat{V}\hat{U}$. *Then*

$$\Psi(\hat{V})\Psi(\hat{U}) = \Psi(\hat{U})\Psi(-\hat{U}\hat{V})\Psi(\hat{V})$$

and in the classical limit we get the 5-term relation for the Rogers dilogarithm.

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Department of Mathematics, MIT, Cambridge MA 02139 U. S. A.; sashagon@math.mit.edu