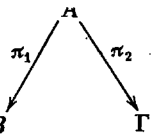
INTEGRAL GEOMETRY ON FAMILIES OF k-DIMENSIONAL SUBMANIFOLDS

A. B. Goncharov

UDC 517.43

1. Definitions. By a double bundle we mean a diagram of manifolds (cf. [17])



(1)

for which $\pi_1 \times \pi_2$: $A \subset B \times \Gamma$ is an imbedding. For $x \in B$ and $\xi \in \Gamma$ we set $B_\xi = \pi_1 \circ \pi_2^{-1}\xi$; $\Gamma_x = \pi_2 \circ \pi_1^{-1}x$.

In order to define a double bundle (1) it suffices to give a family of submanifolds B_ξ of B parametrized by the manifold Γ or a family of submanifolds Γ_x of Γ ($x \in B$). The latter family is called dual to the original.

On the submanifolds B_ξ we give densities μ_ξ and we define the operation of integration I: $f(x) \to \int_{B_\xi} f(x)\mu_\xi$. Then I is an operator from $C_0^\infty(B)$ to $C^\infty(\Gamma)$ whose Schwartz kernel has the form $\mu(x, \xi)\delta(Z)db$, where $\delta(A)$ is the δ-function of the submanifold A, db is the volume element on B, and $\mu(x, \xi)$ is a function on A.

By a local inversion formula is meant an inverse operator J: $C^\infty(\Gamma) \to C^\infty(B)$ whose Schwartz kernel has the form $L\cdot\delta(A)d\gamma$, where L is a differential operator on $B \times \Gamma$ and $d\gamma$ is the volume form on Γ.

In other words, one can reconstruct the value of the function f on B at any point x, knowing the integrals of f over submanifolds of the family passing through an infinitesimal neighborhood of the point x. Clearly, in this case, $\dim B \le \dim \Gamma$.

Definition 0.1. Let $\dim B = \dim \Gamma$. The double bundle (1) is called admissible if there exists a collection of densities μ_ξ such that the integral transformation I has a local inversion formula.

Although Definition 0.1 also makes sense for $\dim B < \dim \Gamma$, it is necessary to define admissibility of a double bundle differently in this case. We do this a little later (cf. Definition 2.1).

In what follows, all manifolds will be complex algebraic although we will integrate smooth functions as before. The fact is that the study of the integral transformation I in the complex case is considerably simpler than in the real one. For example, one can show that if $\dim B_\xi$ is odd, then there do not exist local inversion formulas. Although the majority of results on complex admissible double bundles generalize to the case of families of even-dimensional real manifolds, their formulations and proofs are more complicated.

Let $\dim B = \dim \Gamma$. A double bundle in the category of complex manifolds is called admissible if there exist (0, n)-forms μ_ξ on B_ξ, such that for the integral transformation

$$I\colon C_0^\infty(B) \to C^\infty(\Gamma);\quad I\colon f(x) \mapsto \int_{B_\xi} f(x)\mu_\xi\bar{\mu}_\xi$$

there exists an inverse operator J whose Schwartz kernel has the form $L\bar{L}\delta(A)d\gamma\bar{d}\gamma$, where L is a differential operator on $B \times \Gamma$.

Scientific Council of the Academy of Sciences of the USSR on the Complex Problem of "Cybernetics." Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 23, No. 3, pp. 11-23, July-September, 1989. Original article submitted May 24, 1988.

 0016-2663/89/2303-0178$12.50 © 1990 Plenum Publishing Corporation

2. Basic Results. With each double bundle there is associated a diagram

$$\begin{array}{ccc} & T_A^*(B\times\Gamma) & \\ {}^{\rho_B}\swarrow & & \searrow^{\rho_\Gamma} \\ T^*B & & T^*\Gamma \end{array} \tag{2}$$

where $T_Y^* X \subset T^*X$ is the conormal bundle to Y in X and ρ_B and ρ_Γ are the restrictions to $T_A^*(B \times \Gamma)$ of the projections $T^*(B \times \Gamma) = T^*B \times T^*\Gamma$ onto the factors.

The construction of this diagram goes back to Sophus Lie, who used contact geometry systematically in problems about complex curves in R^3 (rectilinear—spherical correspondence, etc., cf., [9]).

When $\dim B = \dim \Gamma$, all three manifolds in the diagram (2) have the same dimension. We denote by $d(\Gamma)$ the degree of the map ρ_Γ.

BASIC THEOREM. If the double bundle (1) is admissible, then $d(\Gamma) = 1$.

We prove it in Sec. 3. In Sec. 4, we show that the rationality of the manifolds B_ξ and the existence of a canonical rational structure on the projectivization $PT_{B_\xi}^*B$ follow from the condition $d(\Gamma) = 1$. Conjectures about when the necessary condition $d(\Gamma) = 1$ is also sufficient ($\dim B = \dim \Gamma$) are formulated in Sec. 2. Results corroborating these conjectures will be published in [26].

The basic results of this paper were announced in [21].

I sincerely thank I. M. Gel'fand, S. G. Gindikin, and M. I. Graev for many helpful discussions and interest in the paper.

Section 1. Admissible Families of Curves [1-7]

They were written up in two stages. Firstly, in [1-3], Gel'fand, Gindikin, Graev, and Shapiro found a necessary and sufficient condition for admissibility of a family of curves when $\dim B = \dim \Gamma$. We note that in this case, for a generic point $x \in B$, Γ_x is also a curve.

THEOREM 1.1 [1-3]. Let us assume that $\dim B = \dim \Gamma$ and $\dim B_\xi = 1$. Let $\xi_1, \ldots, \xi_n$ be a coordinate system on Γ. Then the double bundle (1) is admissible if and only if the curves Γ_x are graphs of solutions of a system of differential equations

$$\begin{cases} \dfrac{d\xi_i}{d\xi_1} = u_i(\xi)\dfrac{d\xi_2}{d\xi_1} + v_i(\xi), \quad i = 3, 4, \ldots, n, & \text{(3A)} \\ \dfrac{d^2\xi_2}{d\xi_1^2} = p_0(\xi) + p_1(\xi)\dfrac{d\xi_2}{d\xi_1} + p_2(\xi)\left(\dfrac{d\xi_2}{d\xi_1}\right)^2 + p_3(\xi)\left(\dfrac{d\xi_3}{d\xi_1}\right)^3 & \text{(3B)} \end{cases}$$

and $\pi_1(A)$ is dense in B.

The system (3A) means that at a generic point $\xi \in \Gamma$ there exists a 2-dimensional subspace $\Pi_\xi \subset T_\xi\Gamma$ such that at a generic point $x \in B_\xi$ the tangent line $T_\xi\Gamma_x$ lies in Π_ξ.

The form of these equations is independent of the choice of coordinate system. $d^2\xi_i/d\xi_1^2$ are 3rd degree polynomials in the first derivatives for $i \geq 2$.

The fact that the right side of (3B) is a single-valued function of $d\xi_2/d\xi_1$ is equivalent to the fact that for a generic direction in the plane Π_ξ there exists exactly one curve of the dual family which is tangent to it. Hence the map $x \in B_\xi \to T_\xi\Gamma_x$ identifies B_ξ with a domain in $P\Pi_\xi \simeq CP^1$. Thus, the curves B_ξ are rational.

By tradition we will call the family of submanifolds B_ξ in the case when $\dim B = \dim \Gamma$ a complex.

The geometric structure of admissible families of curves was determined by Bernshtein and Gindikin [3-7]. We formulate only part of their results: the description of admissible complexes in general position.

THEOREM 1.2 [3-7]. Let Γ' be a complete family of smooth compact rational curves B_ξ on the manifold B_ξ, i.e., $\dim \Gamma' = \dim H^0(B_\xi, N_{B_\xi}B)$. Then an admissible complex in general position consists of curves tangent to r_1 hypersurfaces and intersecting r_2 submanifolds of codimension 2 in B, where $r_1 + r_2 = \dim\Gamma' - \dim B$.

For admissible complexes of lines Theorem 1.3 was already proved by Gel'fand and Graev in 1968 [8].

Section 2. Conjectures

1. In point 2 of Sec. 4 we show that the degree $d(\Gamma)$ has the following geometric interpretation. Let ξ be a generic point in Γ and H be a hyperplane (containing 0) in general position in $T_\xi\Gamma$. Then $d(\Gamma)$ is equal to the number of points $x \in B_\xi$, such that $T_x B_\xi$ lies in H (cf. with the definition of the Crofton number in [14]). Hence when $\dim B = \dim\Gamma$ and $\dim B_\xi = 1$ the projectivization of the cone $\bigcup_{x\in B_\xi} T_\xi\Gamma_x$ is a curve of degree $d(\Gamma)$ in $PT_\xi\Gamma$. Consequently, if $d(\Gamma) = 1$, then this curve is a line and the map $x \in B_\xi \to PT_\xi\Gamma_x$ is an imbedding. It follows from this that the curves Γ_x are graphs of solutions of a system of differential equations

$$\frac{d\xi_i}{d\xi_1} = u_i(\xi)\frac{d\xi_2}{d\xi_1} + v_i(\xi); \quad \frac{d^2\xi_2}{d\xi_1^2} = \Phi\left(\xi; \frac{d\xi_2}{d\xi_1}\right), \tag{4}$$

where Φ is a <u>single-valued</u> analytic function of $d\xi_2/d\xi_1$. However, according to Theorem 1.1, Φ is a polynomial of degree 3. Hence the condition $d(\Gamma) = 1$ is a fortiori insufficient for the admissibility of the double bundle.

2. <u>Conjecture A.</u> A complex Γ of k-dimensional planes in $\mathbb{C}P^n$ is admissible if and only if $d(\Gamma) = 1$.

When k = 1 the validity of this conjecture follows from the results of Gel'fand and Graev [8]. For complexes of (n − 2)-dimensional planes in $\mathbb{C}P^n$ and 2-planes in $\mathbb{C}P^5$ it is proved in [26]. For linear complexes of k-planes Conjecture A is examined in [21] with the help of the results of [16].

I have no doubt of its validity. In order to formulate a much more courageous conjecture (and at the same time to explain how one proves the admissibility of certain complexes or others), I need the concept of <u>universal</u> local inversion formula [17; 1].

Let us assume that $\dim B < \dim\Gamma$. Then $\dim B_\xi < \dim\Gamma_x$. Let $\chi_x\colon C^\infty(\Gamma) \to \Omega^k(\Gamma_x)$ be a differential operator of order $k = \dim B_\xi$, such that $d\chi_x(If) = 0$ for any function f from $C_0^\infty(B)$.

If γ is a k-dimensional cycle in Γ_x, then

$$\int_\gamma \varkappa_x(If) = c(\gamma) f(x), \tag{5}$$

where $c(\gamma)$ is independent of f. Indeed, since the form $\chi_x(If)$ is closed, the integral (5) is unchanged by deformation of the cycle γ in Γ_x (we recall that $\dim\Gamma_x > \dim\gamma$). Hence if $x' \neq x$, then one can find a cycle $\tilde\gamma$ homologous to γ, such that $x_0 \notin \tilde\gamma$. Hence the integral (5) defines a generalized function on B with support at the point x. With the help of homogeneity considerations it is easy to show that this generalized function is proportional to $\delta(x)$ (here one uses only the fact that the order of the differential operator χ_x coincides with the dimension of B_ξ).

It can happen that $c(\gamma)$ is identically equal to zero. For example, if $\chi_x(If) = d\tau_x \times (If)$, where $\tau_x\colon C^\infty(\Gamma) \to \Omega^{k-1}(\Gamma_x)$, then $c(\gamma) \equiv 0$.

If for any point x there exists an operator χ_x and a cycle γ in Γ_x such that $c(\gamma) \neq 0$, then one says that the integral transformation I has a universal local inversion formula. Different inversion formulas are obtained from one another by suitable choice of cycles $\tilde\gamma$ homologous to γ.

As usual, in the complex case, we assume that the integral (5) has the form $\int_\gamma \chi_x \wedge \overline{\chi}_x(If)$.

Universal local inversion formulas for the family of all planes in $\mathbb{C}^n$ were discovered by Gel'fand, Graev, and Shapiro more than 20 years ago [17]. Cf. [18] for another example.

Definition 2.1. When $\dim B < \dim \Gamma$, the double bundle (1) is called admissible if there exist measures of the form μ_ξ, $\bar{\mu}_\xi$ on B_ξ, such that the corresponding integral transformation has a universal local inversion formula.

The following question arises: how can we define the functional $c(\gamma)$ on $H_{2k}(\Gamma_x)$ geometrically? Let h be a hyperplane in general position in $T_\xi B$. By C_x we denote the homology class of codimension 2k in Γ_x which is defined by the cycle consisting of all those points $\xi \in \Gamma_x$ such that $T_x B_\xi \subset h$ [cf. the geometric interpretation of the number $d(\Gamma)$ at the beginning of Sec. 2!].

PROPOSITION 2.2. $c(\gamma)$ is equal to the intersection index of $\hat{C}_x$ and γ.

CONJECTURE B. Let $\dim B < \dim \Gamma'$ and let the double bundle (1) be admissible. Then the complex $\Gamma \subset \Gamma'$ is admissible if and only if its degree $d(\Gamma)$ is equal to 1.

The validity of this conjecture for curves ($\dim B_\xi = 1$) follows from the results of Bernshtein and Gindikin [4-7].

Since the manifold of all k-planes in $\mathbb{C}P^n$ is admissible [17], Conjecture A is a very special case of Conjecture B.

Conjecture B is proved for "typical" complexes of hypersurfaces in $\mathbb{C}P^n$ (cf. [26]). The case of quadrics in $\mathbb{C}P^n$ is examined in [11].

The scheme of the proof of the majority of the results on the classification of admissible complexes $\Gamma \subset \Gamma'$ is the following: first one finds all complexes for which $d(\Gamma) = 1$; afterwards one proves that the restriction of a suitably chosen form $\chi_x \varphi$ to Γ_x can be calculated from $\varphi|_\Gamma$. A priori for this one must know the restriction of φ to the k-th infinitesimal neighborhood of Γ in Γ'. Hence it is quite astonishing that at least in some cases the condition $d(\Gamma) = 1$, a first-order condition, is sufficient.

3. Guillemin proved (cf. [11, 13] that if in the category of C^∞ manifolds the map

$$\rho_\Gamma^0 \colon T_A^*(B \times \Gamma)\setminus 0 \to T^*\Gamma \setminus 0$$

is an injective immersion, $\pi_1\colon A \to B$ is proper, and I^t is the operator of integration with respect to some measure on Γ_x, then $I^t \circ I$ is an elliptic pseudodifferential operator. According to point c) of Theorem 4.2, which we prove in point 2 of Sec. 4, in the complex case under the condition $\operatorname{codim} B_\xi > 1$ there are no such examples.

4. One can verify that the function Φ i (4) is a polynomial of degree 3 if and only if it has no singularities in any coordinate system on Γ (cf. point 2 of Sec. 4). In this case, there exists a family of curves on Γ containing all curves Γ_x such that for any point ξ in each direction which lies in the plane Π_ξ precisely one curve of this family issues.

It would be very interesting to find the analog of this condition at least for complex hypersurfaces in $\mathbb{C}^n$ for $n \geq 3$. The following theorem obtained by Gel'fand and the author shows that the answer will be different from that in the case $n = 2$.

THEOREM 2.3 [20]. Let a family of holomorphic hypersurfaces in the domain $\mathscr{U}$, $\dim \mathscr{U} \geq 3$ satisfy the following condition: at any point $x \in \mathscr{U}$ each hyperplane in $T_x\mathscr{U}$ is tangent to exactly one hypersurface of the family.

Then this family is locally isomorphic to the family of all hyperplanes in $\mathbb{C}P^n$.

This theorem is false in the category of C^∞-manifolds: a counterexample is given at the end of point 2 in [20].

Section 3. Proof of the Basic Theorem

1. THEOREM 3.1. If a double bundle in the category of complex manifolds is admissible ($\dim B = \dim \Gamma$) and $\pi_1\colon A \to B$ is a proper map, then $d(\Gamma) = 1$.

Remark. Let $\mathscr{L}$ and $\mathscr{L}_\Gamma$ be holomorphic line bundles on B and Γ. The proof of Theorem 3.1 given below carries over word for word to the case when the integration operator I acts from $C_0^\infty(B, \mathscr{L}_B \otimes \bar{\mathscr{L}}_B)$ to $C^\infty(\Gamma, \mathscr{L}_\Gamma \otimes \bar{\mathscr{L}}_\Gamma)$. We note that Theorem 3.1 is usually applied precisely in this situation (cf. [26]).

The proof makes essential use of the technique of Fourier integral operators. We recall some definitions [13, Chapter 6]. Let X and Y be n-dimensional manifolds, $\Lambda \subset X \times Y$,

$L = T_A^*(X \times Y)$, $\tau \in I_m(L)$ be a generalized semiform on $X \times Y$ associated with L of order m. If $(a_1, \ldots, a_{2n-k}, z_1, \ldots, z_k)$ are coordinates in $X \times Y$ in which A is defined by the equations $z_1 = \ldots = z_k = 0$, then modulo distributions of lower order

$$\tau = \int b(a, \xi)\, e^{i(z,\xi)} d\xi \cdot \sqrt{da\, dz}, \tag{6}$$

where (a, ξ) are coordinates in L, while the coordinates ξ and z are adjoint and $b(a, \xi)$ is homogeneous of degree $m - (n/2)$ in ξ.

We denote by $\sigma(\tau)$ the symbol of the generalized semiform τ. By definition $\sigma(\tau)$ is a homogeneous semiform of degree $m + (n/2)$ on L. For example, if τ has the form (6), then $\sigma(\tau) = b(a, \xi)\sqrt{da\, d\xi}$. In particular,

$$\tau = \sum_{|s| \leq m} l_s(a) \frac{\partial^s}{\partial z^s} \delta(z) \sqrt{da\, dz},$$

then, $\tau \in I_{m+n/2}(L)$ and $\sigma(\tau) = i^m \sum_{|s|=m} l_s(a)\xi^s \sqrt{da\, d\xi}$.

We denote by I_τ the integral operator defined by the kernel τ:

$$I_\tau\colon C_0^\infty(\Lambda_X^{1/2}) \to C^\infty(\Lambda_Y^{1/2}),$$

$$I_\tau\colon \mu \mapsto \pi_{Y*}\tau\pi_X^*\mu,$$

where $\Lambda_X{}^{1/2}$ is the bundle of semiforms on X; π_X, π_Y are the projections of $X \times Y$ to the factors and π_{Y*} is integration along X of sections of $\pi_X^*\Lambda_X \otimes \pi_Y^*\Lambda_Y{}^{1/2}$.

If $\tau \in I_{m_1+n/2}(L)$, then I_τ is a Fourier integral operator of order m_1. The set of all such operators is denoted by $(F.I.)^{m_1}(L)$.

Let $L' \subset T^*Y \times T^*Z$ be a homogeneous Lagrangian submanifold and $\dim Z = n$. We have the diagram

$$\begin{array}{ccccccc} & & L & & & L' & \\ & \swarrow^{\rho_X} & & \searrow^{\rho_Y} & \swarrow^{\rho'_Y} & & \searrow^{\rho_Z} \\ T^*X & & & T^*Y & & & T^*Z \end{array} \tag{7}$$

Here ρ_X is the projection of $L \subset T^*X \times T^*Y$ onto the first factor, etc. Let us assume that all maps in the diagram (6) have finite degree. We choose open domains $\tilde{L}$ and $\tilde{L}'$ in L and L' on which these maps are unbranched.

Let ω_Y be the canonical 2-form defining the symplectic structure in T^*Y. We represent $\sigma(\tau)|_{\tilde{L}}$ in the form $\tilde{\sigma}(\tau)\cdot\rho_Y^*\sqrt{\omega_Y^n}$.

Let $J \in (F.I.)^{m_2}(L')$,

$$J\colon C^\infty(\Lambda_Y^{1/2}) \to C^\infty(\Lambda_Z^{1/2}).$$

We denote by $\tilde{L} \circ \tilde{L}'$ the composition of the correspondences $\tilde{L}$ and $\tilde{L}'$. By definition

$$\tilde{L} \circ \tilde{L}' = \{(\xi_X, \xi_Z) \mid \exists \xi_Y \in T^*Y\colon (\xi_X, \xi_Y) \in \tilde{L}, (\xi_Y, \xi_Z) \in \tilde{L}'\}.$$

If τ is the Schwartz kernel of the operator I_τ, then we will write $\sigma(I)$, instead of $\tilde{\sigma}(\tau)$.

We define a function $\tilde{\sigma}(J) \circ \tilde{\sigma}(I)$ on $\tilde{L} \circ \tilde{L}'$ by

$$\tilde{\sigma}(J) \circ \tilde{\sigma}(I)(\xi_X, \xi_Z) = \sum_{\xi_Y} \tilde{\sigma}(I)(\xi_X \xi_Y)\tilde{\sigma}(J)(\xi_Y, \xi_Z),$$

where the summation is over all $\xi_Y \in T^*Y$, such that $(\xi_X, \xi_Y) \in \tilde{L}$ and $(\xi_Y, \xi_Z) \in \tilde{L}'$.

Let $P\tilde{L}$ be the projectivization of $\tilde{L}$. We say that $I \in (F.I.)^{m_1}(\tilde{L})$ is properly concentrated if the projections of $P\tilde{L}$ to PT^*X and PT^*Y are proper maps.

The following lemma generalizes Proposition 6.2 of Chapter VI of [13] and is proved analogously.

LEMMA 3.2. If $I \in (F.I.)^{m_1}(\tilde{L})$ and $J \in (F.I.)^{m_2}(\tilde{L}')$ are properly concentrated, then $J \circ I \in (F.I.)^{m_1+m_2}(\tilde{L} \circ \tilde{L}')$ and $\sigma(J \circ I) = \tilde{\sigma}(I)$.

Now we start directly on the proof of the theorem.

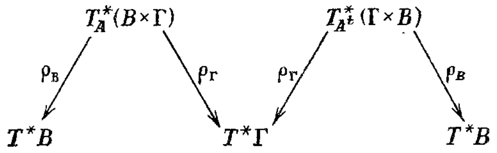
Let ψ_B and ψ_Γ be fixed semidensities on B and Γ which have the form $\psi(z)\sqrt{dz d\bar{z}}$ where $\psi(z) > 0$. We define operators $\tilde{I}$ and $\tilde{J}$ as follows:

$$\tilde{I}:\ C_0^\infty(\Lambda_B^{1/2}) \to C^\infty(\Lambda_\Gamma^{1/2});\quad \tilde{I}:\ f\psi_B \mapsto I(f)\psi_\Gamma,$$

$$\tilde{J}:\ C^\infty(\Lambda_\Gamma^{1/2}) \to C^\infty(\Lambda_B^{1/2});\quad \tilde{J}:\ \varphi\psi_\Gamma \mapsto J(\varphi)\psi_B.$$

In what follows, we will omit the tildes in the notation for the operators $\tilde{I}$ and $\tilde{J}$.

We consider the diagram



Let Δ be the diagonal in B × B.

PROPOSITION 3.3. If the double bundle (1) is admissible, then ρ_B and ρ_Γ are maps of finite degree.

Proof. First, we prove the following lemma.

LEMMA 3.4. If (in the category of C^∞-manifolds) there exists an inverse operator J for the integral operator I, the wave front of whose Schwartz kernel is contained in $T_A^* \times (B \times \Gamma)$, then the image of the map ρ_B is dense in T^*B.

Proof of Lemma 3.4. Let K be the Schwartz kernel of the operator $J \circ I$. This is a generalized semiform on B × B. We denote by WF(K) its wave front. Since $J \circ I$ is the identity operator in $C^\infty(\Lambda_B{}^{1/2})$, one has $WF(K) = T_\Delta^*(B \times B)$. As is known, WF(K) is contained in the closure of $T_A^*(B \times \Gamma) \circ T_{A^t}^*(\Gamma \times B)$. Thus:

$$T_\Delta^*(B\times B) \subset \overline{T_A^*(B\times\Gamma) \circ T_{A^t}^*(\Gamma\times B)}.$$

It follows from this that the image of the map ρ_B is dense in T^*B■.

Since $T_A^*(B \times \Gamma)$ and T^*B are manifolds of the same dimension, the differential of the map ρ_B at a point in general position is an isomorphism.

LEMMA 3.5 [12]. Let V and W be symplectic vector spaces, $L \subset V \oplus W$ be a Lagrangian subspace, π_1 and π_2 be the projections of L to the first and second summands. Then $\pi_2 \times (\operatorname{Ker}\pi_1) = \pi_2(L)^\perp$■.

Applying this lemma to the case when $V \oplus W$ and L are the tangent spaces to $T^*B \times T^*\Gamma$ and $T_A^*(B \times \Gamma)$ at a generic point η, we get that

$$d_\eta\rho_\Gamma:\ T_\eta T_A^*(B\times\Gamma) \to T_{\rho_\Gamma(\eta)}T^*\Gamma$$

is an isomorphism. Hence in the category of complex algebraic manifolds the map ρ_Γ has finite degree ■ ■.

Let $\widetilde{T_A^*(B \times \Gamma)}$ be an open domain in $T_A^*(B \times \Gamma)$ on which the maps ρ_B and ρ_Γ are unbranched. We denote by L the closure of the set $\widetilde{T_A^*(B \times \Gamma)} \circ \widetilde{T_{A^t}^*(\Gamma \times B)}$.

LEMMA 3.6. $\deg\rho_\Gamma = 1$ if and only if $L = T_\Delta^*(B \times B)$.

Remark. If $*$ is the operation of composition of Lagrangian cycles, then $T_\Delta^*(B \times B)$ occurs in L with multiplicity $\deg\rho_B$.

It is very important to keep the following familiar lemma in mind.

LEMMA 3.7. Each homogeneous irreducible algebraic Lagrangian submanifold of T^*X has the form T_Y^*X, where Y is a submanifold of X ■.

Let us assume that $\deg\rho_\Gamma > 1$. Then there exists an irreducible submanifold $C \subset B \times B$ different from Δ, such that $T_C^*(B \times B) \subset L$. To prove that $J \circ I \neq \mathrm{id}_B$, it suffices to prove that $T_C^*(B \times B) \subset WF(K)$. A difficulty which must be overcome is that although $WF \times (K(J \circ I))$ is contained in L, it does not necessarily coincide with L in general. We give a typical example of such a situation.

a) Let $B = R^2$, Γ be the manifold of all nontrivial lines $y = ax + b$ on the plane. We set

$$I_1 f(a, b) = \int_{-\infty}^{+\infty} f(x, ax + b)\, dx,$$

$$J_1 \varphi(x_0 y_0) = \int_{-\infty}^{+\infty} \varphi_b'(a, y_0 - a x_0)\, da.$$

Then $J_1 \circ I_1 \equiv 0$. We note that the complex analogs of these formulas lead to an inversion formula for the Radon transform in C^2. In the following example the manifold Γ_x will be compact.

b) $I_2 f(\xi_1, \xi_2; p) = \int f(x_1, x_2)\, \delta(x_1\xi_1 + x_2\xi_2 - p)\, d^2x,$

$$J_2 \varphi(x_1, x_2) = \int_K \frac{\partial \varphi}{\partial p}(\xi; p)\big|_{p=(\xi, x)} (\xi_1\, d\xi_2 - \xi_2\, d\xi_1).$$

The integral is taken over the closed curve K in the (ξ_1, ξ_2) plane going around zero. Since the integrand changes sign under the substitution $\xi \to -\xi$, this integral is equal to zero.

Below we show that in the complex case there are no such reductions.

LEMMA 3.8. I and J are Fourier integral operators while $\sigma(I)$ and $\sigma(J)\cdot(-1)^{\dim \Gamma_x}$ have the same sign on $T_A^*(B \times \Gamma)$.

Proof. Let $(a_i\, ;\, z_j)$ be a holomorphic coordinate system in a neighborhood of the point $a \in A$, in which A is defined by the equations $z_1 = \ldots = z_k = 0$. The kernel defining the operator I in this neighborhood has the form

$$f(a)\, \overline{f(a)}\, \delta(a) \sqrt{da\, dz\, \overline{da}\, \overline{dz}},$$

where $f(a)$ is a holomorphic function.

Hence $\sigma(I) = |f(a)|^2 \sqrt{da\, d\xi\, \overline{da}\, \overline{d\xi}}$ where (a, ξ) are coordinates in $T_A^*(B \times \Gamma)$, where the coordinates ξ are dual to the coordinates z.

Let ω_X be the canonical holomorphic 2-form defining the symplectic structure on T^*X. We set $\omega_X^{n,n} = \omega_X^n \wedge \bar{\omega}_X^n$. Let ρ: $Z \to T^*X$ be an unbranched holomorphic covering. Then if $\omega' \in \Omega_Z$, then $\omega' \wedge \bar{\omega}' = |h|^2 \rho^* \omega^{n,n}$. Hence $\sigma(I) > 0$.

The Schwartz kernel of the operator J has the form

$$g(a)\, \overline{g(a)}\, L\bar{L}\delta(a) \sqrt{da\, dz\, \overline{da}\, \overline{dz}}.$$

where L is a holomorphic differential operator:

$$L = \sum l_{k,s}(a) \frac{\partial^k}{\partial z^k} \frac{\partial^s}{\partial a^s}.$$

Applying a partition of unity and integrating by parts, one can assume that $l_{k,s}(a) = 0$ for $|s| > 0$.

We show that $\deg L = \dim \Gamma_x$. Indeed, we know that

$$I \in (\mathrm{F.I.})^{\frac{k-n}{2}} (T_A^*(B \times \Gamma)), \qquad J \in (\mathrm{F.I.})^{\deg L + \frac{k-n}{2}} (T_{A^t}^*(\Gamma \times B)).$$

Hence according to Lemma 3.2

$$J \circ I \in (\mathrm{F.I.})^{\deg L + (k-n)} (L).$$

Since $J \circ I = \mathrm{id}_B$, one has $\deg L = n - k = \dim \Gamma_x$.

The symbol of the Schwartz kernel defining the operator J has the form

$$\tilde{\sigma}(J) = \sum_{|k| = \dim \Gamma_x} |g(a)|^2 |l_k(a)|^2 (-1)^{\dim \Gamma_x} \sqrt{da\, d\xi\, \overline{da}\, \overline{d\xi}}.$$

Hence, $(-1)^{\dim \Gamma} x\sigma(J) > 0$.

Thus, $(-1)^{\dim \Gamma} x\sigma(J \circ I)$ is strictly greater than zero on $T_C^*(B \times B)$, where C is a submanifold of $B \times B$, different from the diagonal. Hence $J \circ I \neq id_B$, when $\deg \rho_\Gamma > 1$. Theorem 3.1 is completely proved.

2. In this point we show how to integrate a section of line bundles on B.

Let $\mathscr{L}_B$ be a line bundle on B such that

$$H^0(B_\xi, \mathscr{L}_B^*|_{B_\xi} \otimes \Omega_{B_\xi}) \neq 0.$$

We denote by $\mathscr{L}_\Gamma$ the bundle over Γ whose fibre at each point $\xi \in \Gamma$ is

$$\mathrm{Hom}_{B_\xi}(\mathscr{L}_B|_{B_\xi}, \Omega_{B_\xi})^*. \tag{8}$$

We construct the integral transformation

$$I: C_0^\infty(B, \mathscr{L}_B \otimes \overline{\mathscr{L}}_B) \to C^\infty(\Gamma, \mathscr{L}_\Gamma \otimes \overline{\mathscr{L}}_\Gamma).$$

Let $s \oplus \overline{s} \in C_0^\infty(B, \mathscr{L}_B \otimes \overline{\mathscr{L}}_B)$ and

$$i_\xi \in \mathrm{Hom}_{B_\xi}(\mathscr{L}_B|_{B_\xi}, \Omega_B).$$

We define $I(s \oplus \overline{s})$ as the linear functional on $C^\infty(\Gamma, \mathscr{L}_\Gamma \otimes \overline{\mathscr{L}}_\Gamma)^*$ whose value on the vector $i_\xi \oplus \overline{i}_\xi$ is equal to

$$\int_{B_\xi} i_\xi(s|_{B_\xi})\, \overline{i}_\xi(\overline{s}|_{B_\xi}).$$

If π_2: $A \to \Gamma$ is a submersion, then the fibre of the normal bundle to A, $N_A(B \times \Gamma)$ at the point $(x, \xi) \in A$ is canonically identified with the fibre of $N_{B_\xi}B$ at the point x (cf. Proposition 4.2 below). Moreover, Ω_{B_ξ} is canonically isomorphic to $\Omega_B|_{B_\xi} \oplus \det N_{B_\xi}B$. Hence

$$\mathscr{L}_\Gamma = \pi_{2*}\, \underline{\mathrm{Hom}}_A(\det N_A(B \times \Gamma) \otimes \pi_1^*\Omega_B, \pi_1^*\mathscr{L}_B).$$

Example. Let $\Gamma = Gr_k(\mathbb{C}P^n)$ be the Grassmanian of k-dimensional planes in $\mathbb{C}P^n \equiv B$. Then $\Omega_{B_\xi} \cong \mathcal{O}_{P^k}(-k-1)$. Hence the construction given above lets us integrate sections of line bundles $\mathcal{O}(-k-s-1) \oplus \mathcal{O}(-k-s-1)$, where $s > 0$ (cf. [18, Chapter II, Sec. 3]). In particular, for $s = 0$, $\mathscr{L}_\Gamma$ is the determinant bundle over $Gr_k(\mathbb{C}P^n)$, i.e., its fibre over the point ξ corresponding to the $(k + 1)$-dimensional linear subspace h_ξ in $\mathbb{C}^{n+1}$, is $\det h_\xi$.

Remark. A basic point in this construction is the canonical choice of a finite-dimensional space in $\mathrm{Hom}_C(\mathscr{L}_B \otimes \overline{\mathscr{L}}_B|_{B_\xi}, \Omega_{B_\xi} \oplus \overline{\Omega}_{B_\xi})$. If the space (8) is empty, then instead of it, one should take any finite-dimensional space of differential operators from $\mathscr{L}_B|_{B_\xi}$ to Ω_{B_ξ} (imposing suitable conditions on their symbols). Thus, one can, for example, integrate sections of the bundles $\mathcal{O}(-k-s-1) \oplus \mathcal{O}(-k-s-1)$ on P^n for $s < 0$ over k-planes.

One should note that the condition $d(\Gamma) = 1$ follows from the existence of a local inversion formula if and only if $\mathscr{L}_\Gamma$ is a line bundle on Γ.

If the submanifolds B_ξ are the zeros of sections of a holomorphic bundle E on B, then the restriction of $\mathscr{L}_B := \det E \oplus \Omega_B$ to B_ξ is isomorphic to Ω_{B_ξ}, i.e., $\mathscr{L}_\Gamma$ is one-dimensional. Conversely, if $\operatorname{codim}_{B_\xi} B \leq 2$ and there exists a line bundle $\mathscr{L}_B$ on B, such that $\mathscr{L}_B|_{B_\xi} \cong \Omega_{B_\xi}$, then the submanifolds B_ξ are zeros of sections of some bundle on B. (For $\operatorname{codim}_{B_\xi} B = 1$, this is obvious. For $\operatorname{codim}_{B_\xi} B = 2$, it was proved by Serre [27].)

Section 4. Rationality Theorem and Geometric Meaning of the Condition $d(\Gamma) = 1$

1. RATIONALITY THEOREM. If for the double bundle (1) the degree $d(\Gamma)$ is equal to 1, then the manifold B_ξ is rational and on $PT_{B_\xi}^*B$ there is a canonical rational structure.

Remark 4.1. It does not generally follow from the rationality of $PT_{B_\xi}^*B$ that B_ξ, just as the rationality of $PT_{B_\xi}^*B$ does not follow from the rationality of B_ξ.

Proof. Let $\alpha = pdq$ be the canonical 1-form on T^*B. It will be helpful for us to introduce the following notation: $L_\xi := T_{B_\xi}^*B$.

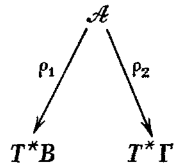
A vector $v \in T_\xi\Gamma$ defines a section $\gamma_v(\lambda)$ of the normal bundle $N_{L_\xi}T^*B$. The formula

$$\lambda \mapsto -\alpha\,(\gamma_v\,(\lambda))$$

defines a map

$$\nu_\xi\colon L_\xi \to T_\xi^*\Gamma.$$

We define the submanifold $\mathcal{A} \subset T^*B \times T^*\Gamma$ as the manifold of pairs $(\lambda \in L_\xi,\ \nu_\xi^*\lambda))$. We have the diagram

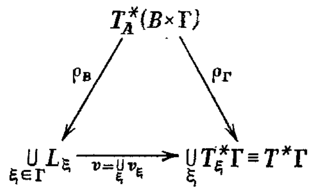


PROPOSITION 4.2. If $\pi_2\colon \mathcal{A} \to \Gamma$ is a submersion, then $\mathcal{A}$ is canonically isomorphic to $T_A^*(B \times \Gamma)$.

Proof. Let $a = (x, \xi) \in A$. Then $T_aA \subset T_xB \oplus T_\xi\Gamma$. By hypothesis $d_a\pi_2\colon T_aA \to T_\xi\Gamma$ is an epimorphism. Hence $\operatorname{Ker} d_a\pi_2$ coincides with $T_xB_\xi \oplus 0$ and $T_a\,(B \times \Gamma)/T_a\,A = T_xB/T_xB_\xi$. We denote by $T_{y,Y}{}^*X$ the fibre of the bundle $T_Y{}^*X$ at the point $y \in Y$. Then the natural map

$$\rho_B\colon T^*_{a,\,A}(B\times\Gamma)\to T^*_{x,\,B_\xi}B \tag{9}$$

is an isomorphism so we need to verify the commutativity of the following diagram:



For this, it suffices to show that if $\lambda \in L_\xi$, then the covector $[\lambda,\ \nu_\xi(\lambda)]$ vanishes on any vector $(v_1,\ v_2) \in T_aA$, i.e., $\langle\nu_\xi(\lambda),\ v_2\rangle = -\langle\lambda,\ v_1\rangle$. But since $\alpha = pdq$, this is the definition of the map ν_ξ ∎.

It follows from this that $d(\Gamma)$ is equal to the degree of the map ν_ξ (if ξ is not a critical value for the map $\pi_2\colon\ A \to \Gamma$). In particular, if $d(\Gamma) = 1$, then the projectivization of the map ν_ξ defines a birational isomorphism $P_{\nu_\xi}\colon\ PT_{B_\xi}{}^*B \to PT_\xi{}^*\Gamma$.

Let $\operatorname{Ann} T_\xi\Gamma_x = \{\ell \in T_\xi{}^*\Gamma \,|\, (\ell,\ v) = 0\ \forall v \in T_\xi\Gamma_x\}$.

LEMMA 4.3. If $\pi_1\colon\ A \to B$ is a submersion, then

$$\rho_B\,(T^*_{a,A}\,(B\times\Gamma)) = \operatorname{Ann}\,T_\xi\Gamma_x\ \ (a=(x,\ \xi)).$$

Proof. If $(\ell_1,\ \ell_2) \in T_{a,}{}_A{}^*(B \times \Gamma) \subset T_x{}^*B \times T_\xi{}^*\Gamma$ and $v \in T_\xi\Gamma_x$, then $0 = \langle(\ell_1,\ \ell_2),\ (0,\ v)\rangle = \langle\ell_2,\ v\rangle$, so that

$$\rho_B\,(T^*_{a,A}\,(B\times\Gamma)) \subset \operatorname{Ann}\,T_\xi\Gamma_x.$$

The opposite inclusion is proved just as in Lemma 4.2 ∎.

PROPOSITION 4.4. The number $d(\Gamma)$ for the double bundle (1) is equal to the number of subspaces of the form $T_\xi\Gamma_x$ (where $x \in B_\xi$) which lie in a hyperplane in general position in $T_\xi\Gamma$.

Proof. The condition $d(\Gamma) = 0$ is equivalent to the fact that $\rho_B(T_{a}\,{}_{,A}{}^*(B \times \Gamma)$ has nonzero codimension in $T_\xi{}^*\Gamma$. According to Lemma 4.3 the latter condition means precisely that in a hyperplane in general position in $T_\xi\Gamma$, there are no subspaces of the form $T_\xi\Gamma_x$, where $x \in B_\xi$.

Now let $d(\Gamma) > 0$. Then it follows from Proposition 3.3 that $\pi_1\colon\ A \to B$ is a submersion at a generic point. Hence the map $\rho_\Gamma\colon\ T_{a}\,{}_{,A}{}^*(B \times \Gamma) \to T_{\xi,\Gamma_x}\Gamma$ is an isomorphism at a generic point. It remains, just as before, to use Lemma 4.3 ∎.

Now we show that the rationality of the manifolds $d(\Gamma) = 1$ follows from the condition B_ξ. Let $H \subset PT_\xi^*\Gamma$ be a plane in general position of dimension $\dim B_\xi$. According to Proposition 4.4, $d(\Gamma)$ is equal to the degree of the rational map

$$x \in B_\xi \mapsto P(\operatorname{Ann} T_\xi\Gamma_x) \cap H.$$

In conclusion, we note that if $\operatorname{codim} B_\xi = 1$ and $d(\Gamma) = 1$, then on B_ξ there is a canonical rational structure. Namely, the map

$$x \in B_\xi \mapsto T_\xi\Gamma_x \subset T_\xi\Gamma$$

is a birational isomorphism of B_ξ onto the manifold of subspaces of codimension 1 in $T_\xi\Gamma$, i.e., onto CP^{n-1} ∎.

In geometric considerations the following lemma is often helpful.

LEMMA 4.5. Let us assume that $\pi_2\colon\ A < \Gamma$ is a submersion. Let $\eta \in T_x^*B$. Then there exists a natural bijection between $\rho_A^{-1}(x, \eta)$ and $\{\xi \in \Gamma_x |$ the restriction of η to $T_\xi\Gamma_x$ is equal to $0\}$ ∎.

This follows directly from (9).

For example, for a complex of lines in $\mathbf{C}P^3$, $d(B)$ is equal to its degree in the classical sense (cf. [25, Chapter VI]). Hence it is natural to call $d(B)$ the degree and $d(\Gamma)$ the codegree of the double bundle.

2. THEOREM 4.6. Let us assume that for a double bundle in the category of complex manifolds $\dim A > \dim B$ (i.e., $\dim B_\xi \neq 0$). Then

a) if $\dim B = 2$ and the maps $\rho_B^0\colon\ T_A^*(B \times \Gamma)\backslash 0 \to T^*B\backslash 0$ and $\rho_\Gamma^0\colon\ T_A^*(B \times \Gamma)\backslash 0 \to T^*\Gamma\backslash 0$ are isomorphisms, then the family Γ_x is locally-isomorphic to the family of lines in $\mathbf{C}P^2$.

b) if $\dim B = \dim \Gamma \geq 3$, $\pi_1\colon\ A \to B$ is a submersion and the map ρ_Γ^0 is an isomorphism, then the family Γ_x is locally-isomorphic to the family of hyperplanes in $\mathbf{C}P^n$.

c) if $\dim B = \dim \Gamma \geq 3$, $\pi_1\colon\ A \to B$ is a submersion and ρ_Γ^0 is injective, then $\operatorname{codim} B_\xi = 1$.

Interesting results in the C^∞-situation were obtained by Quinto (under considerably stronger hypotheses of the double bundle) (cf. [15; 13, Chapter VI, Sec. 3]).

Proof. First, we prove point c). Since $\pi_1\colon\ a \to B$ is a submersion, the canonical map

$$\rho_\Gamma\colon T^*_{a,A}(B \times \Gamma) \to T^*_{\xi,\Gamma_x}\Gamma$$

is an isomorphism. Hence it follows from the injectivity of the map ρ_Γ^0 that for any x_1, $x_2 \in B_\xi$

$$P\operatorname{Ann} T_\xi\Gamma_{x_1} \cap P\operatorname{Ann} T_\xi\Gamma_{x_2} = \varnothing. \tag{10}$$

Thus, we have a bijective map

$$f_\xi\colon B_\xi \times \mathbf{C}P^{n-k-1} \to \mathbf{C}P^{n-1},$$

where

$$n = \dim B \text{ and } k = \dim \Gamma_x.$$

It defines an imbedding of B_ξ in $Gr_{n-k-1}(\mathbf{C}P^{n-1})$. We denote by $\bar{B}_\xi$ the closure of the image of B_ξ in $Gr_{n-k-1}(\mathbf{C}P^{n-1})$. We get a birational isomorphism

$$\bar{f}_\xi\colon \bar{B}_\xi \times \mathbf{C}P^{n-k-1} \to \mathbf{C}P^n.$$

Let us assume that $k \neq n-1$. We denote by $\mathscr{D}$ the exceptional divisor for $\bar{f}_\xi$. Then $\operatorname{codim} \bar{f}_\xi(\mathscr{D}) \geq 2$. If $y \in \bar{f}_\xi(\mathscr{D})$, then $\bar{f}_\xi^{-1}(y) \geq 1$. It follows from the assumption (10) that there exists a curve $C_y \subset B_\xi$, such that $\bar{f}_\xi(C_y) = y$. Since $C_y \cap (\bar{B}_\xi \backslash B_\xi)$ contains at most one point, in $C_y \cap B_\xi$ one can find two distinct points x_1 and x_2. Then

$$y \in f_\xi(x_1 \times \mathbf{C}P^{n-k-1}) \cap f_\xi(x_2 \times \mathbf{C}P^{n-k-1}),$$

which contradicts (10).

The proof of point b) of the theorem follows from point c) of Lemma 4.5 (in which the places of B and Γ should be interchanged) and Theorem 2.3.

Hence in the proof of point a) one can assume that $\dim B = 2$. According to Lemma 4.5 again the bijectivity of the map $\rho_\Gamma{}^0$ means that through each point $\xi \in \Gamma$ in any direction there passes exactly one curve of the family $\{\Gamma_x\}$. Hence these curves are the graphs of solutions of the differential equation

$$\frac{d^2\xi_2}{d\xi_1^2} = \Phi(\xi_1, \xi_2, \xi_2') \qquad \left(\xi_2' = \frac{d\xi_2}{d\xi_1}\right),$$

where Φ is a polynomial of the third degree in ξ_2'. Indeed, let $\xi(t)$ be a parametrization of the curve Γ_x, $\xi(0) = \xi$. Then, if [,] is a skew-symmetric form in $T_\xi\Gamma$, then $q_\xi(\dot\xi(0) = [\dot\xi(0), \ddot\xi(0))$ is a well-defined homogeneous function of degree 3 on the whole complex plane. Hence this function is a polynomial and hence $\Phi(\xi;\ \xi_2') = q_\xi(\xi_2', 1)$ is a polynomial of degree 3.

An analogous assertion is true for families of curves B_ξ in B. Hence according to the classical theorem of A. Tresse [22] (cf. also [23 or 24, Chapter 1, Sec. 6]) the family of curves Γ_x is locally diffeomorphic to the family of lines in the plane ∎.

LITERATURE CITED

1. I. M. Gel'fand, S. G. Gindikin, and Z. Ya. Shapiro, "A local problem of integral geometry in a space of curves," Funkts. Anal. Prilozhen., 13, No. 2, 11-31 (1979).
2. I. M. Gel'fand and M. I. Graev, "Admissible n-dimensional complexes of curves in C^n," Funkts. Anal. Prilozhen., 14, No. 4, 36-44 (1980).
3. J. N. Bernstein and S. G. Gindikin, "Admissible families of curves," Seminar on Supermanifolds, Preprint, No. 3, Stockholm University (1986).
4. J. N. Bernstein and S. G. Gindikin, "Geometrical structure of admissible families of curves," Seminar on Supermanifolds, Preprint, No. 3, Stockholm University (1986).
5. J. N. Bernstein, "Description of admissible families of curves by critical points," Seminar on Supermanifolds, Preprint, No. 11, Stockholm University (1986).
6. S. G. Gindikin, "Integral geometry and twistors," Lect. Notes Math., 970, 2-42 (1982).
7. S. G. Gindikin, "Reduction of manifolds of rational curves and related problems of the theory of differential equations," Funkts. Anal. Prilozhen., 18, No. 4, 14-39 (1984).
8. I. M. Gel'fand and M. I. Graev, "Admissible complexes of lines in CP^n," Funkts. Anal. Prilozhen., 2, No. 3, 39-52 (1968).
9. S. Lie, "Über Complex, insbesondere Linien and Kugel-Complex, mit Anwendung auf die Theorie partieller Differentialgleichungen I Ordnung," Math. Ann., 5, 145-256.
10. V. A. Ginzburg, "Symplectic geometry and the theory of representation," Funkts. Anal. Prilozhen., 17, No. 3, 75-76 (1983).
11. A. B. Goncharov, "Integral geometry on surfaces in the space," J. Geom. Physics (1989).
12. V. Guillemin and S. Sternberg, "Some problems in integral geometry and some related problems in microlocal analysis," Am. J. Math., 101, No. 4, 915-955 (1979).
13. V. Guillemin and S. Sternberg, Geometric Asymptotics [Russian translation], Mir, Moscow (1981).
14. I. M. Gel'fand and S. G. Gindikin, "Nonlocal inversion formulas in real integral geometry," Funkts. Anal. Prilozhen., 11, No. 3, 12-19 (1977).
15. E. T. Quinto, "Topological restrictions on double fibrations and Radon transforms," Proc. Am. Math. Soc., 101, No. 4.
16. K. Maius, "Linear complexes of k-planes," Funkts. Anal. Prilozhen., 19, No. 1, 79-80 (1984).
17. I. M. Gel'fand and S. Graev, and Z. Ya. Shapiro, "Integral geometry on k-dimensional planes," Funkts. Anal. Prilozhen., 1, No. 1, 17-31 (1967).
18. I. M. Gel'fand, S. G. Gindikin, and M. I. Graev, "Integral geometry in affine and projective spaces," Itogi Matem. Tekh., Seriya Sovremennye Problemy Matematiki, 16, 53-226 (1980).
19. I. M. Gel'fand and G. S. Shmelev, "Geometric structures of double bundles and their relation to some problems of integral geometry," Funkts. Anal. Prilozhen., 17, No. 2, 7-22 (1983).
20. I. M. Gel'fand and A. B. Goncharov, "A characteriztion of Grassman manifolds," Dokl. Akad. Nauk SSSR, 289, No. 5, 1031-1034 (1986).

21. A. B. Goncharov, "Admissible families of k-dimensional submanifolds," 300, No. 3, 535-539 (1988).
22. A. Tresse, "Sur les invariants differentielles des groupes continus des transformations," Acta Mat., 18, 1-88 (1894).
23. E. Cartan, "Sur les variétés à connexion projectif," in: Oeuvres, III, 1, No. 70, Paris, pp. 825-862.
24. V. I. Arnol'd, Supplementary Chapters of the Theory of Ordinary Differential Equations [in Russian], Nauka, Moscow (1979).
25. P. Griffiths and J. Harris, Principles of Algebraic Geometry [Russian translation], Vol. 2, Mir, Moscow (1982).
26. A. B. Goncharov, "Integral geometry and contact transformations," (to appear).
27. M. Okonek, M. Schneider, and M. Spindler, Vector Bundles over Projective Spaces [Russian translation], Mir, Moscow (1985).

THE ALGEBRA OF INTEGRALS OF MOTION OF TWO-DIMENSIONAL HYDRODYNAMICS IN CLEBSCH VARIABLES

V. E. Zakharov

UDC 517.9

1. We will consider the equations of two-dimensional hydrodynamics of an incompressible fluid

$$\frac{\partial\Omega}{\partial t}+(V\nabla)\Omega=0,\quad \mathbf{V}=(V_x,V_y),\quad \Omega=\frac{\partial V_x}{\partial y}-\frac{\partial V_y}{\partial x},\quad \operatorname{div}\mathbf{V}=0 \tag{1.1}$$

on a torus Γ, i.e., in a square $-L < x < L$, $-L < y < L$ with periodic conditions on the velocity field. Moreover, the mean vorticity equals zero ($\langle\Omega\rangle = \int_\Gamma \Omega dxdy = 0$). We also must equate to zero the mean flow of the fluid

$$\langle V\rangle=\int_\Gamma V\,dx\,dy=0. \tag{1.2}$$

Then one can introduce a periodic function of the current $\psi(V_x = -(\partial\psi/\partial y)$, $V_y = (\partial\psi/\partial x)$ and rewrite (1) in the form

$$\frac{\partial\Omega}{\partial t}+\{\psi,\Omega\}=0,\quad \{A,B\}=AxBy-AyBx. \tag{1.3}$$

Let us remark that $\psi = -(\delta H/\delta\Omega)$, where $H = -1/2 \int_\Gamma \psi\Omega dxdy$ is the kinetic energy of the fluid.

Equation (1.3) is a Hamiltonian system, the phase space of which is the space U of smooth periodic functions $\Omega(x, y)$ with zero mean, the Hamiltonian is the energy H, and the Poisson bracket between the functionals α and β of Ω is determined by the formula

$$[\alpha,\beta]=\int_\Gamma \Omega\left\{\frac{\delta\alpha}{\delta\Omega},\frac{\delta\beta}{\delta\Omega}\right\}dx\,dy. \tag{1.4}$$

Equation (1.3), in which $\psi(x, y, t)$ is an arbitrary given function, has an infinite set of integrals of motion of the form

$$I=\int_\Gamma F(\Omega)\,dx\,dy. \tag{1.5}$$

L. D. Landau Institute of Theoretical Physics. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 23, No. 3, pp. 24-31, July-September, 1989. Original article submitted November 2, 1988.