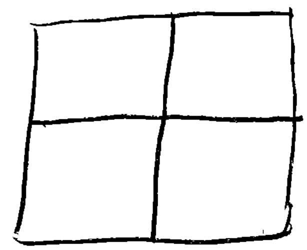
Multifractals, again

If the multifractal is generated by transformations T_1, T_2, T_3, T_4 (any number can be used) all scaled by the same factor r



$N = 4$
$r = 1/2$

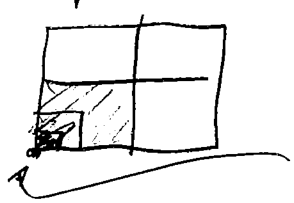
Suppose p_1, p_2, p_3, and p_4 are the probabilities of applying T_1, T_2, T_3, T_4

Then $\max(\alpha) = \dfrac{\log(\min(p_i))}{\log(r)}$

$\min(\alpha) = \dfrac{\log(\max(p_i))}{\log(r)}$

$f(\min(\alpha))$ = dim of the part of the multifractal generated by $\max(p_i)$

If $\max(p_i)$ occurs for a single p_i, say p_1, then



$\max(p_i)$ occurs at a point,
so $f(\min(\alpha)) = \dim(\text{point}) = 0$

If $\max(p_i)$ occurs on two probabilities, say $p_1 = p_2 > p_3 > p_4$, $\max(p_i)$ occurs on a line,
so $f(\min(\alpha)) = \dim(\text{line}) = 1$

If $\max(p_i)$ occurs on three probabilities, say $p_1 = p_2 = p_3 > p_4$, $\max(p_i)$ occurs on a gasket
so $f(\min(\alpha)) = \dim(\text{gasket}) = \dfrac{\log 3}{\log 2}$

f(max(α)) = dim of the part of the multifractal generated by min(p_i)

The max(f(α)) is the dimension generated by the IFS.

N = 16 transformations: each with $r = 1/4$

p_2	p_2	p_3	p_3
p_2	p_2	p_2	p_2
p_1	p_1	p_1	p_1
p_1	p_1	p_1	p_1

$p_1 > p_2 > p_3$

min(α), f(min(α))

max(α) f(max(α))

max(f(α))

$$\min(\alpha) = \frac{\log(p_1)}{\log(1/4)} \qquad f(\min(\alpha)) = \frac{\log 8}{\log 4}$$

N=8 boxes have p_1, $r = 1/4$

$$\max(\alpha) = \frac{\log(p_3)}{\log(1/4)} \qquad f(\max(\alpha)) = \frac{\log 2}{\log 4}$$

$N = 2$

$r = 1/4$

$$\max(f(\alpha)) = \frac{\log(16)}{\log(1/(1/4))} = \frac{\log 16}{\log 4} = \frac{\log(4^2)}{\log(4)} = \frac{2\,\cancel{\log(4)}}{\cancel{\log(4)}} = 2$$

Fractals are homogeneous - they look the same everywhere. Multifractals, unlike fractals which are characterized by a single dimension, are described by many dimensions; they are made of a <u>mul</u>titude of <u>fractals.</u>

Randomized Moran equation:

Construct a Cantor set with two pieces, the left is scaled by $r_1 = \frac{1}{3}$, the right is scaled by $r_2 = 1/3$ with prob 1/2, and by $r_2 = 1/9$ with prob 1/2

The Moran equation becomes

$$\underbrace{1 = 1\cdot\left(\frac{1}{3}\right)^d}_{\text{left side}} + \underbrace{\frac{1}{2}\left(\frac{1}{3}\right)^d + \frac{1}{2}\left(\frac{1}{9}\right)^d}_{\text{right side}}$$

Take $x = \left(\frac{1}{3}\right)^d$, then $\left(\frac{1}{9}\right)^d = \left(\left(\frac{1}{3}\right)^2\right)^d = \left(\left(\frac{1}{3}\right)^d\right)^2 = x^2$

$$1 = x + \frac{1}{2}x + \frac{1}{2}x^2$$

$$2 = 2x + x + x^2$$

$$x^2 + 3x - 2 = 0 \qquad x = \frac{-3 \pm \sqrt{3^2 - 4\cdot 1\cdot(-2)}}{2\cdot 1}$$

$$= \frac{-3 \pm \sqrt{17}}{2}$$

pos value of x is $\frac{-3+\sqrt{17}}{2}$

$$\left(\frac{1}{3}\right)^d = \frac{-3+\sqrt{17}}{2}$$

$$\log\left(\left(\frac{1}{3}\right)^d\right) = \log\left(\frac{-3+\sqrt{17}}{2}\right)$$

$$d = \frac{\log\left(\frac{-3+\sqrt{17}}{2}\right)}{\log(1/3)}$$