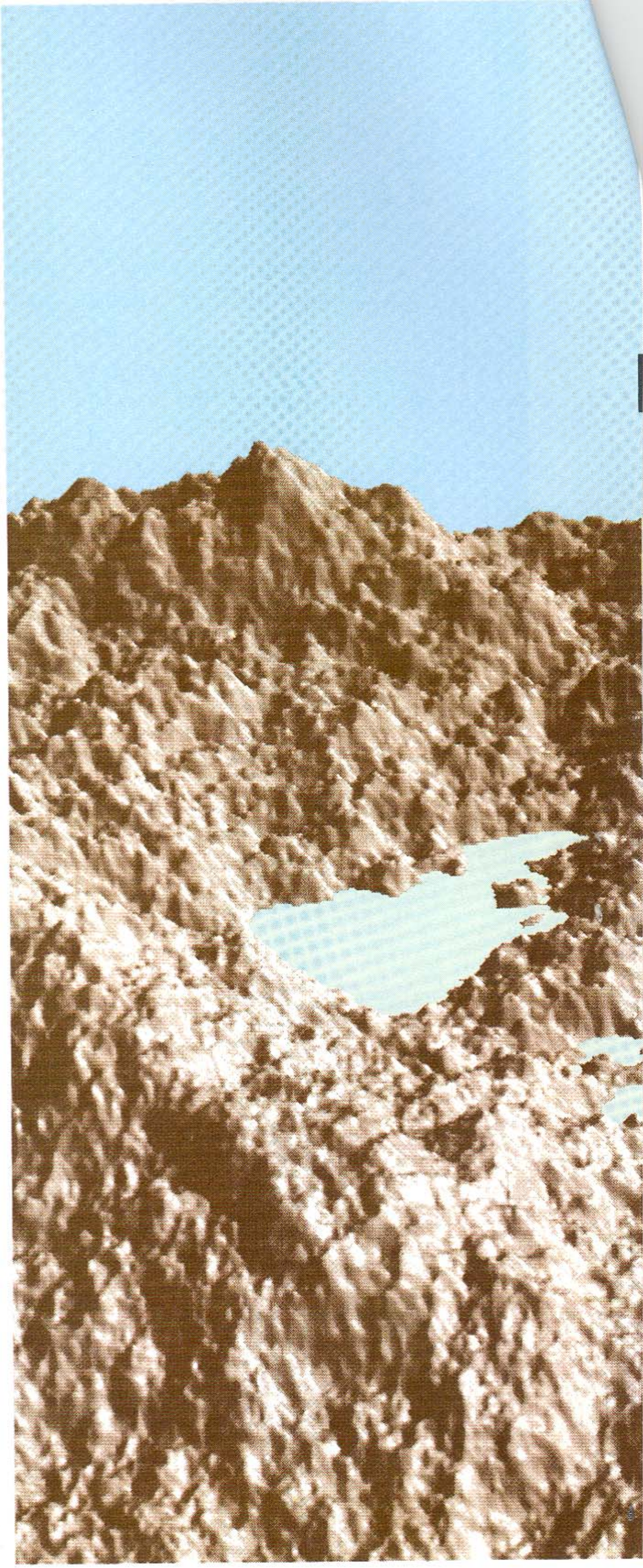
Fractals and the Geometry of Nature

by Benoit B. Mandelbrot

Guided by the mathematics underlying a recently revived family of "monstrous" geometric shapes, computer drawing machines are producing realistic representations of some familiar but grossly irregular patterns in nature.

Before beginning to understand what fractals are, one should know what they look like. The reader is therefore asked to begin this article with a careful examination of its illustrations and to read the captions only after the introduction below.

Now pay special attention to figure 1, at right. It does not represent what it may seem to. It is neither a photograph of a landscape on the Earth, the Moon, or any other planet, nor is it a painting by a science fiction artist. None of the illustrations in this article represents any actual facet of nature, and none is what is ordinarily called a work of art. All are guaranteed 100% geometric fakes. They are computer-generated and computer-plotted representations of selected members of the family of purely geometric shapes called fractals.



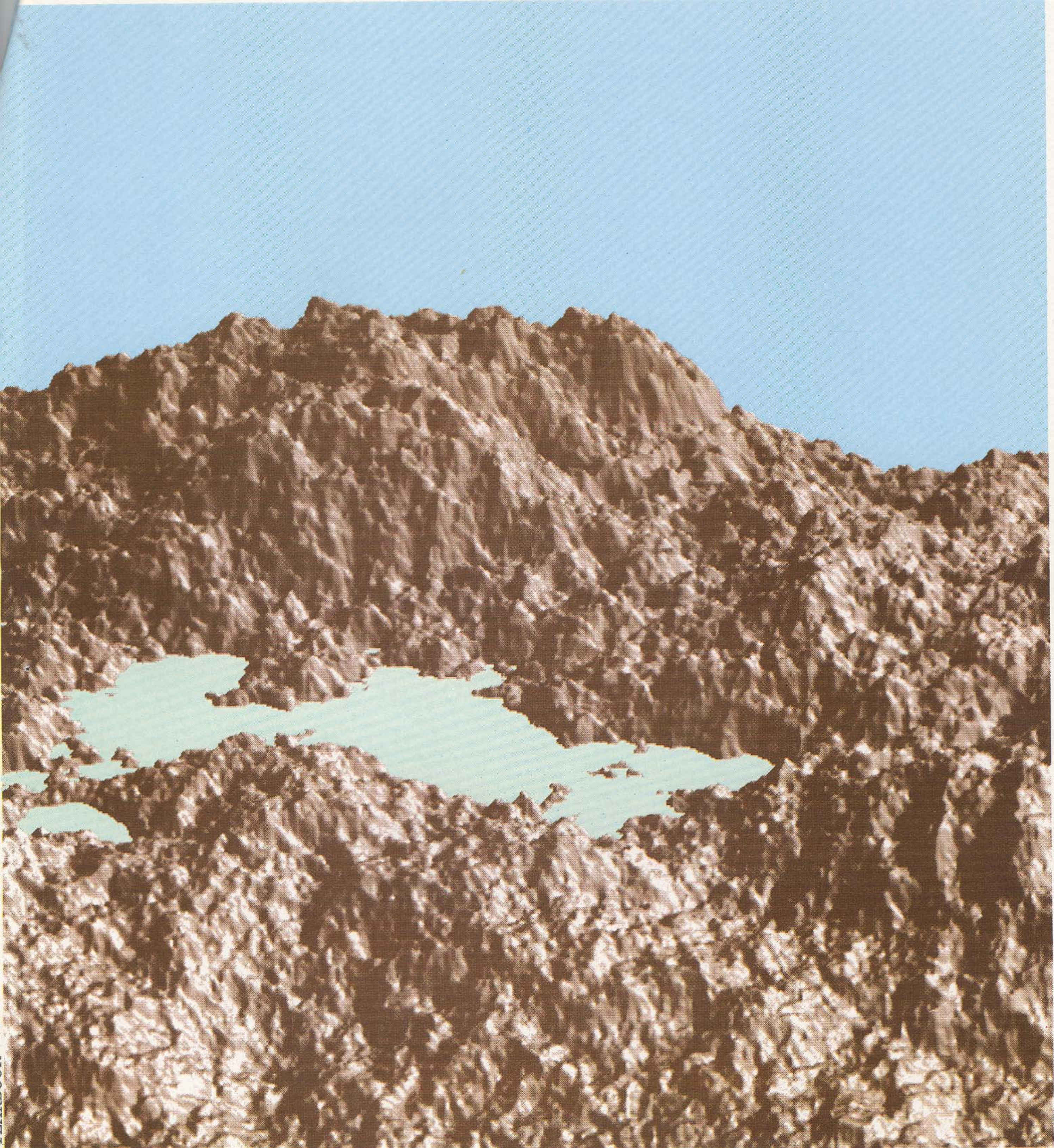


Figure 1 (overleaf): a fractal landscape that never was.

Fractal geometry can imitate nature

The illustrated fractal shapes are really very simple in the sense that every one of their details has been deduced unambiguously from a few lines of instruction given to the computers that drew them. These shapes are extremely involved, however, and are strikingly unlike anything in the familiar discipline of classical geometry, or "Euclid." The new fractal geometry that they exemplify is very different from Euclid. Especially conspicuous is the fact that the number of dimensions, or dimensionality, of a fractal may be a fraction. This idea is by no means "geometry fiction" but part of a chapter of mathematics that is classical but was obscure until recently for lack of widely interesting applications.

The applicability of fractal geometry in describing some grossly irregular and fragmented facets of nature is so strikingly evident from the illustrations that it is reasonable to wonder why it had not been heard of before 1975, when this author's first comprehensive publication on fractals introduced the term and marked the founding of the discipline. For, had it been founded earlier, though doubtlessly under a different name, it would have filled an obvious need for describing some of many conspicuous natural patterns—including the shapes of mountains, coastlines, and clouds—at which the straight lines, circles, ellipses, squares, and other components of classical geometry are almost completely inept.

The answer is a hard-to-believe tale of extreme self-delusion on the part of many great minds over a full century. "Fractional dimension" and several other basic components later to be fitted into the system of fractal geometry had been known to mathematicians and to a few scientists and philosophers since the period 1875–1925 but were knowingly left to remain as unrelated odds and ends of specialized consequence. No one favored them with careful attention because they were believed to deserve none; hence no one even felt the need of a word to denote them.

Some unexpectedly simple shapes

Before touching again on history, it will be helpful to contrast some fractal shapes with those in Euclid. The circle, the square, the sphere, and the cube are the very simplest Euclidean shapes because, position aside, one needs only one parameter to describe any of them; say, a diameter or a diagonal length. Any alternative parameter is merely a fixed multiple of the one chosen as a base. A circle is simpler than a square because it involves fewer position parameters. A rectangle, an ellipse, or several circles strung shish-kebab style involve only two parameters. In addition, it is obvious that each of the parameters required by these examples is a scale of length.

This last feature leads to a strong temptation to identify the notions of scale and of parameter and to conclude that a geometric shape which is simple to describe must also involve few distinct scales of length. However, fractal illustrations suffice to demonstrate that this temptation has no merit at all. To take an example, the fake mountain view in figure 1 includes identifiable hills and hillocks of every conceivable scale, between barely perceptible ones and ones that nearly fill the picture. If the picture were more finely grained, an even wider range of hill sizes would be seen. On the

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Illustrations implemented on computer by Richard F. Voss (figures 1 and 5), Mark R. Laff (figures 2, 3, 6, and 7), and Sigmund W. Handelman (figures 3 and 4) of the IBM Thomas J. Watson Research Center.

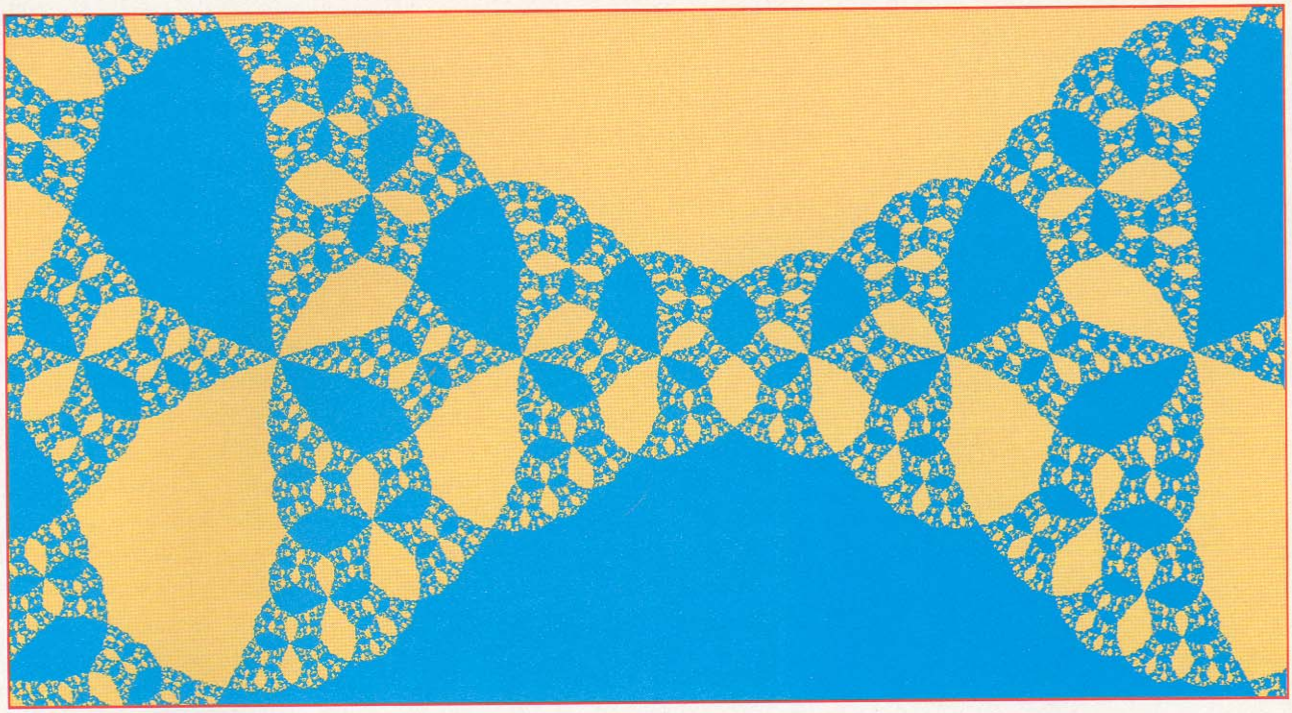


Figure 2: a fractal attractor.

other hand, only three parameters suffice to identify this mountain scene within a certain colossal "portfolio" of alternative scenes to which it belongs. (Not counted are the parameters that control the angles of lighting and of observation.) Furthermore, only one of the three parameters is related to length: it doubles when altitudes of the landscape double.

The second parameter, denoted *D*, is the most interesting and the most important one. In the case of surfaces it lies between two and three. Its most obvious role in the construction of these landscapes is to control the relative numbers of large and small hills: when *D* is close to two, the scene has a huge hill with tiny pimples, while when *D* is close to three, the scene contains many middling hills with barely a trace of a large one. The term denoted by *D* is "fractal dimension," which demands some explanation. *D* differs from the standard view of dimension as the number of distinct coordinates needed to specify a point in space. For instance, on a straight line one needs but a single coordinate to identify a point; on a plane or a landscape one needs two coordinates. The notion of dimension, however, has more than one meaning, and fractals are characterized by the fact that different definitions of dimension yield distinct numerical values. (This issue is discussed further in the box on pp. 176–177.)

The third and last parameter, called "random seed" or "chance," is best thought of as a scene's page number in the portfolio of alternatives mentioned above. The notion of chance in this sense is a subtle one. Ordinarily a game of, say, 1,000 coin tosses is viewed as a sequence of 1,000 independent chance events, each the outcome of one coin toss. But one can also imagine that there exists somewhere a big book of 2^{1000} pages (a number greater than 1 followed by 300 zeros), in which the progress of each possible

Figure 3: the Koch snowflake and a snowflake sweep.

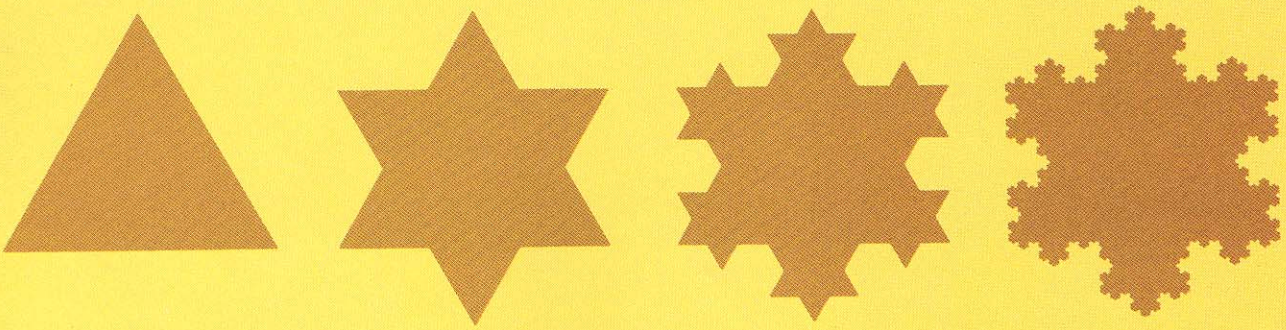
outcome of 1,000 coin tosses is recorded on a separate page. Thus, any game of 1,000 tosses can be specified by selecting a page in this book. The parameter of chance is simply the page number so selected. Landscapes are a more complicated story, but again the computer program responsible for such fractals as figure 1 has a finite number of conceivable outcomes. The specific figure that one obtains depends on a number that one gives in advance, called "seed," which can be viewed as a page number in a virtual portfolio of different landscapes.

Some of the preceding assertions may be clarified by examining their counterparts in the case of other fractals—notably of the snowflake curve in figure 3 on p. 173. Whereas a computer that is programmed to draw a circle will perform that single task and then stop, a typical fractal-drawing computer is programmed to loop endlessly. In other words, after it has performed a simple task assigned to it and has finished drawing a curve with a limited amount of detail, it immediately starts again to perform the same task on a smaller scale of length, thus adding more detail—and so on to infinity. The ratio r between the scales involved in successive stages enters into the principal parameter, D, characteristic of such pattern-generation loops. For the snowflake curve this parameter is 1.26; for variants of the snowflake it lies between one and two and is again a fractal dimension. Endless looping is the trick that makes it possible for geometric shapes that involve many scales of lengths to be counted among the simplest in geometry. Thus, the product of an interrupted loop that stops after a finite number of steps is, paradoxically, less simple than that of an endless loop: to be able to know when to stop, the drawing program must include an additional signal; *i.e.*, one more parameter, such as a counter or a smallest scale.

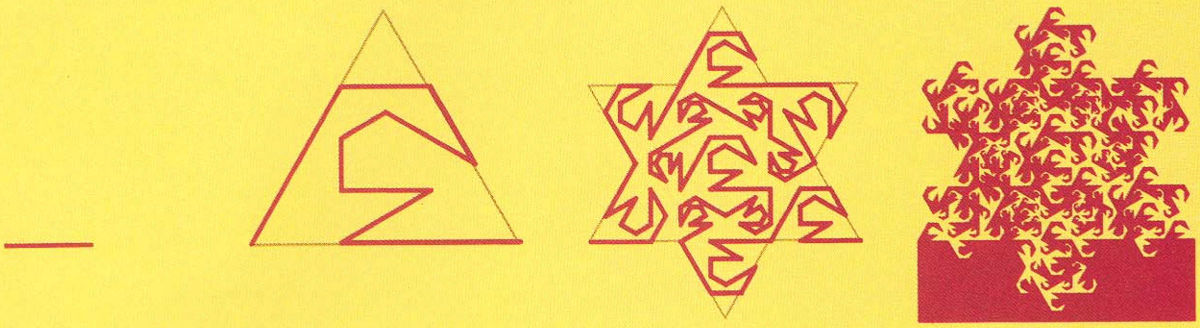
It will be noticed that in loop-generated fractals, successive scales of length fall in a discrete, geometric sequence. This is an undesirable and unnatural feature in most cases in which one seeks a faithful model of nature, but it vanishes in such other fractals as fake mountains, whose scales are continuously governed by the chance parameter. When a loop is absent, a shape is typically extremely smooth. When a purely repetitive loop is present, a shape is typically extremely rough and irregular and in some cases also fragmented into separate islandlike pieces. These extremes happen to be much simpler than intermediate shapes of moderate irregularity.

Taming the mathematical monsters

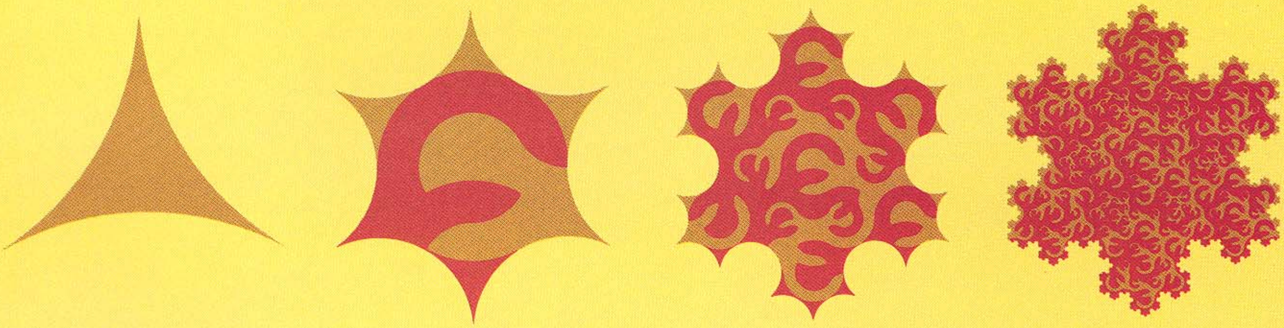
This author's search for a word to denote the fake mountains, loop-generated curves, and their kin eventually led to coining the term fractal. The word is related to the Latin verb *frangere,* which means "to break." In the Roman mind, *frangere* may have evoked the action of breaking a stone, since the adjective derived from it combines the two most obvious properties of broken stones, irregularity and fragmentation. This adjective is *fractus,* which led to fractal. Eventually this author proposed a precise definition of the mathematical term fractal set (see box). The etymological kinship with "fraction" is also significant if one interprets "fraction" as a number that lies between integers. Indeed, a fractal set can be considered as lying between the shapes of Euclid.



3a: four stages in the construction of the Koch snowflake



3b: four stages in the construction of a snowflake sweep



3c: rounded approximations of 3a and 3b, superposed

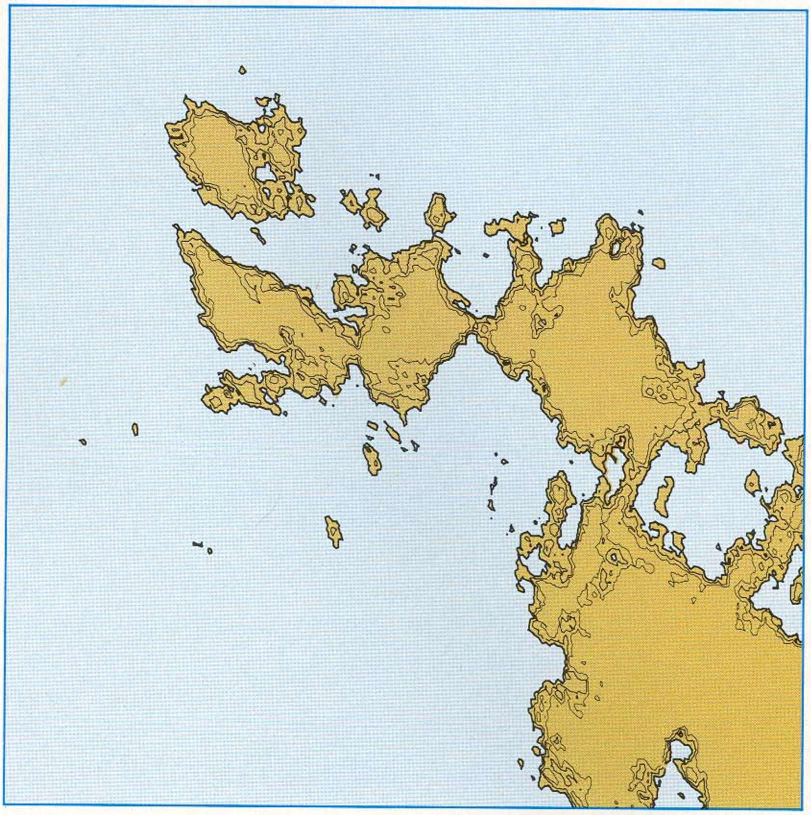


Figure 4: map of a country that never was.

The reason for coining the term fractal and founding fractal geometry was well stated by Freeman J. Dyson in the journal *Science*:

> "Fractal" is a word invented by Mandelbrot to bring together under one heading a large class of objects that have [played an] . . . historical role . . . in the development of pure mathematics. A great revolution of ideas separates the classical mathematics of the 19th century from the modern mathematics of the 20th. Classical mathematics had its roots in the regular geometric structures of Euclid and the continuously evolving dynamics of Newton. Modern mathematics began with Cantor's set theory and Peano's space-filling curve. Historically, the revolution was forced by the discovery of mathematical structures that did not fit the patterns of Euclid and Newton. These new structures were regarded by contemporary mathematicians as "pathological." They were described as a "gallery of monsters," kin to the cubist painting and atonal music that were upsetting established standards of taste in the arts at about the same time. The mathematicians who created the monsters regarded them as important in showing that the world of pure mathematics contains a richness of possibilities going far beyond the simple structures that they saw in nature. Twentieth-century mathematics flowered in the belief that it had transcended completely the limitations imposed by its natural origins.
>
> Now, as Mandelbrot points out [in his book, *Fractals*] . . . nature has played a joke on the mathematicians. The 19th-century mathematicians may have been lacking in imagination, but nature was not. The same pathological structures that the mathematicians invented to break loose from 19th-century naturalism turn out to be inherent in familiar objects all around us in nature.

Thus, the theory of fractals is not properly an application of 20th-century mathematics, but the sudden revival and belated blooming of odds and ends not intended to become a theory. In the 20th century more so than in preceding ones, mathematics is influenced and often dominated by the search for generality for its own sake. The results that this search achieves (for example, the properties true of all curves) are typically of little use in science. Science had exhausted the old curves of Euclid and was in dire need of new ones, but it needed curves that are sufficiently special to have interesting properties subject to comparison with natural phenomena. Mathematics of intermediate generality created around 1900 involved a cache of curves and other shapes that the "mainstream" had leapfrogged much too hastily, and fractal geometry is the new discipline that is being built around this cache.

The fact that mathematics, viewed by its own creators as "absolutely pure," should respond so well to the needs of science is striking and surprising but follows a well-worn pattern. That pattern was first set when Johannes Kepler concluded that, to model the path of Mars around the Sun, one must resort to an intellectual plaything of the Greeks—the ellipse. Soon after, Galileo concluded that, to model the fall of bodies toward the Earth, one needs a different curve—a parabola. And he proclaimed that "the great book [of nature] . . . is written in mathematical language and the characters are triangles, circles and other geometric figures . . . without which one wanders in vain through a dark labyrinth." In the pithy words of Scottish biologist D'Arcy Thompson: "God always geometrizes." With the advent of fractal geometry, the meanings of further geometric "characters" have been revealed, and a few more pages of the great book of nature have become understandable.

Selected facets of fractals are discussed below in descriptions of the illustrations and in a separate box.

Figure 1: a fractal landscape that never was

In order to determine the degree of validity of a scientific principle, scientists seek formulas that follow from this principle and compare them with empirical formulas derived from actual observations. The correspondence between the fractal model used to generate this mountainous relief and real mountains can be shown to be surprisingly excellent given the model's simplicity. By contrast, other mathematical models of landscape that compete with fractal landscapes lead to drawings remote from reality.

This figure is an example of an unsystematic, or random, fractal. Formally it is a "truncated fractional Brownian surface" of fractal dimension $D = 2.3$. The word truncated simply means that at all points where the mathematical model called for altitude below some threshold value labeled as zero, the altitude was arbitrarily reset to zero. In the illustration these points appear as depressions filled with water. The meaning of the word fractional is technical; it refers to a smoothing operation applied to a markedly rough surface, called a Brownian surface, to make it more in conformity with a natural Earth landscape. The word Brownian calls attention to the fact that each vertical cross section of a Brownian surface is a Brownian function: very nearly a random walk that steps up or down with equal probability independently of its past steps.

Self-similarity and fractal dimension

A straight line segment has a property that is self-evident but deserves to be set out as the basis of a later generalization. Given any integer N, a segment of length L is the sum (union) of N straight segments of length $r = L/N$, each of which can be obtained from the original segment by a similarity of ratio r, with an appropriate focal point. In the same vein a square of side L is the same as the sum of N^2 squares of side $r = L/N$, each of which can be obtained from the original by a similarity of ratio r. The line segment and the square are therefore described as being self-similar entities, and all line segments and squares are simply reduced or enlarged replicas of each other. In Euclid, all self-similar shapes are deducible to the above examples; hence self-similarity is not an especially useful notion. Fractals, however, can be self-similar in a truly overwhelming variety of ways. For example, each third of the snowflake curve in figure 3 is self-similar. It is made of four replicas of itself reduced in the ratio $r = 1/3$.

For the rare self-similar figures in Euclid it is easy to see that the ratio $D = \log N/\log(1/r)$—*i.e.*, the logarithm of N divided by the logarithm of $1/r$—is identical to the figure's dimension, which is one for curves like straight lines and circles and two for planar domains like the interior of a square. The same ratio also deserves to be considered for fractal self-similar shapes. But the values it yields are most surprising. For the snowflake curve, $D = \log 4/\log 3 = 1.26\ldots$, which is a fraction! And for the snowflake sweep in figure 3, a more general formula (slightly more involved than a ratio of logarithms and applicable when the reduction ratios of the parts are not all the same) yields $D = 2$, which reflects the property of the planar domain that the curve fills rather than its curvelike nature.

More generally, the property of a geometric shape that is revealed by D is something that one may informally call "heft." Progressing from shapes

Figure 2: a fractal attractor

The boundary of the blue-colored region in this illustration, meant to contrast with the previous one, is an example of a very systematic (though very complex) fractal curve. The rationale behind it involves dynamic physical systems. A physical system may have a stable attractive point, meaning a point to which it converges in due time and to which it returns if disturbed. For example, a marble tossed into an upright funnel will tend toward a stable point at the funnel's neck. A physical system may also have a stable attractive cycle; say, a circle or an ellipse. The planets and satellites of the solar system, for instance, have established stable, nearly elliptical orbits around their parent bodies. However, dynamic systems whose attractors are points or near circular cycles or other Euclidean shapes are exceptions, and the behavior of most dynamic systems is incomparably more complicated. Viewed in terms of fractal geometry, a remarkable finding by the mathematicians Henri Poincaré (circa 1885) and Pierre Fatou and Gaston Julia (circa 1918) can be expressed by saying that, save for certain simple exceptions, attractors are fractals.

of lower dimension to those of higher dimension, a square is heftier than a line segment, and a cube heftier than a square. From the viewpoint of heft, fractals prove by simple construction that a shape can well lie between standard values of dimension.

But heft does not exhaust all the nuances of dimension. For example, the top third of the snowflake curve has the same kind of connectedness that a straight line segment has: if a few points are deleted from it, it breaks into disconnected pieces. Consequently, mathematicians say that for both the straight line and the snowflake curve the topological dimension is one. It is apparent that this value coincides with the number of coordinates needed to identify a point on either of these curves. The mathematician Georg Cantor, however, demonstrated that identifying dimension with numbers of coordinates is a treacherous notion, and modern mathematics prefers to stress the topological facet of dimension. In a similar manner, instead of noting that a plane or a landscape requires two coordinates in order to identify a point on it, mathematicians stress that both figures possess a topological dimension of two because either can be disconnected by deleting curves from it.

It should be clear that dimension is a notion more delicate than first appearances would indicate. Its dissection into distinct facets is of practical significance, and the author therefore was led to define a fractal set (the entire family of fractal shapes) as being a mathematical set such that D is greater than the topological dimension, D_T. In mathematical symbols, $D > D_T$. To describe fractals requires a number of different parameters, but the topological dimension, D_T, and the fractal dimension calculated as $\log N/\log (^1/r)$, or by more general formulas when needed, are the crucial parameters.

Among the many reasons why this finding remained little known (and interest in this topic waned for half a century) is that the original papers are difficult and devoid of illustrations: fractal attractors are very difficult to draw. The few examples laboriously produced about 1885 keep being reproduced seemingly without recognition that they are incorrect and quite misleading. Drawing fractals is a task for computer graphics, as exemplified by the figure, which is excerpted from a varied and extensive collection developed in the course of the author's most recent research.

Figure 3: the Koch snowflake and a snowflake sweep

While the main display for these figures is shown in 3c, the more sober figures 3a and 3b give a clearer idea of the constructions. In order to construct the snowflake (3a), a shape derived by H. von Koch, one begins with an equilateral triangle with sides one unit long. Next, one attaches to the external middle of each side an equilateral triangle of side $r = {}^1/3$. Then one attaches to the middle of each side a triangle of side r^2, and so on. The "skin" of the snowflake is called a snowflake curve. A striking characteristic is that

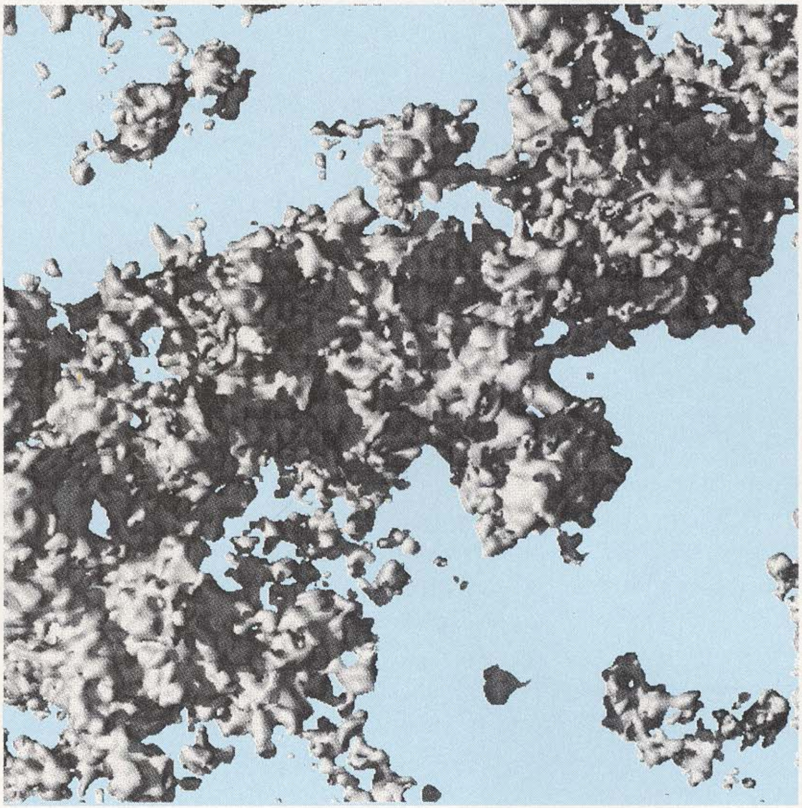


Figure 5: fractal clouds that never were.

its ultimate length is infinite: each time a new series of triangles is attached, the skin's length is multiplied by $^4/_3$. A related fact is that the curve has no tangent at any point. For example, try to draw a tangent at a corner point by drawing the chord which joins this point to another point on the curve that moves increasingly close to the first. It is apparent that this chord oscillates endlessly (within an angle of 30°) without even settling down to a limit one would call a tangent to the snowflake curve.

Is not a curve that has infinite length and is devoid of tangents too bizarre for words or for applications? The theory of fractals advances and defends the opposite view, as is discussed below in figure 4.

In order to construct the snowflake sweep (3b), a curve devised by the author, one starts with a string of length one. Then one stretches and pulls it into a 13-segment shape, called the generator, that fills a regular triangle reasonably uniformly and in any event passes at a distance that is less than $^1/(2\sqrt{3}) = 0.2886\ldots$ from any point in the triangle. Then one stretches and pulls each segment of this generator into a reduced-size version of the whole, thus filling a star hexagon more uniformly than in the preceding stage, by a curve that passes at a distance less than $^1/_{12}$ from any point in this hexagon. And as one repeats the same stretching and pulling, the construction converges to a curve that comes infinitely close to any point within the snowflake (3a) described above.

The fact that a curve can be contrived to fill an area of a plane was first demonstrated by Giuseppe Peano using a different example. Yet, granted

Figure 6: a seven-headed Poincaré fractal curve.

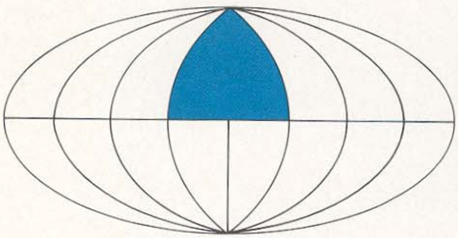
that one can achieve such a monstrous goal in theory, is it not clear that the outcome cannot be anything but extravagant and contrary to all intuition? The theory of fractals shows otherwise. As evidence, figure 3c shows variations of stages of construction of the snowflake curve and snowflake sweep, each being smoothed out by replacing every line segment by an arc, namely $^1/_6$ of a circle. In the variant stages of the snowflake curve, these arcs all bow inward, but in the snowflake sweep stages they bow toward the side along which the next stage of construction will occur. In the resulting pattern one may well find it possible to sense branching trees, licking tongues of fire, and other familiar patterns of nature. In particular, if narrower lines are laid along the middles of the red-colored "fingers," one obtains a branching pattern bearing a strong resemblance to river networks. And this similarity emphasizes the basic requirement of a river network that drains some area effectively: that its cumulative shore should pass within a very small distance of every point of the area to be drained.

(The second stage of rounded approximation in figure 3c awaits a "fractalman" to adopt it as his symbol.)

Figure 4: map of a country that never was

This illustration is made up of selected contour lines of a fractional Brownian relief constructed along the same principle as figure 1.

How long is the coast of Britain? This is a deceptively simple question to which the curves in figures 3 and 4 call attention. There is no need to seek



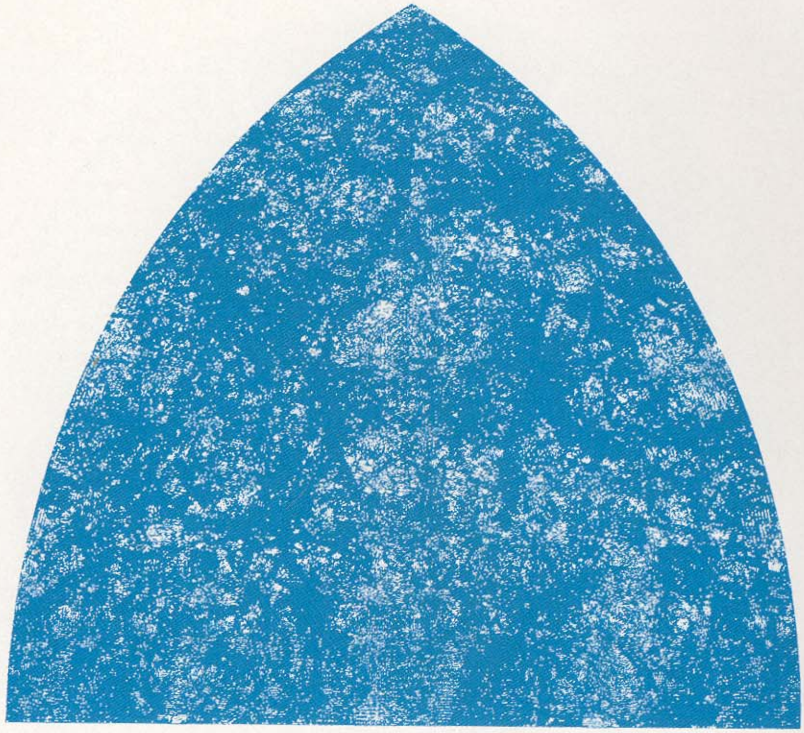


Figure 7: one-eighth of a sky that never was (above right). In the accompanying map (above) the celestial sphere centered on the Earth is projected on a plane. Shaded portion corresponds to the illustrated region of fractal sky.

the answer in an encyclopedia, because a little thought shows that the only sensible answer is "it depends." For example, when the length is measured by walking a compass or dividers along a coastline it is obvious that the result of the measurement depends on how far apart the legs are set: the less broadly they are opened, the better they take account of fine details of coastline and the longer the coastline length they yield.

If a coastline's shape extends to endlessly small detail by self-similarity (*i.e.*, if the irregularity of a coastline segment of any size is the same as that of a segment of any other size), then as the dividers' opening tends to zero, the total measured length will tend to infinity. This behavior exemplifies a surprisingly common phenomenon. When mathematicians concluded about a century ago that the seemingly simple and innocuous notion of "curve" hides profound difficulties, they thought that they were engaging in unreasonable and unrealistic hairsplitting. They had not determined to look out at the real world to analyze it, but to look in at an ideal in the mind. The theory of fractals shows that they had misled themselves.

Figure 5: fractal clouds that never were

The construction of this illustration bears many similarities to that of the fractal landscape in figure 1. The figure is again a fractional Brownian surface that can be thought to represent all points in a three-dimensional space, such as that filled by the Earth's atmosphere, at which the temperature equals 0° C (32° F), the freezing temperature of water. Following this interpretation, the surface bounds a region in the atmosphere that allows the formation and existence of clouds of ice crystals.

Figure 6: a seven-headed Poincaré fractal curve

Much of the caption describing figure 2 applies as well to this curve, which is also a fractal attractor, although of another kind. Its main characteristic is that it is not self-similar, as are the other illustrated figures. Instead it is self-inverse; that is, unchanged by geometric inversion with respect to any of 14 circles.

Figure 7: one-eighth of a sky that never was

Fractal "dusts" of totally disconnected points play a central role in fractal models of the distribution of stars in the Galaxy and of galaxies in the universe. Such dusts are variants of a celebrated mathematical monster called Cantor set. The creators of early models of the sky, not realizing that they were dealing with Cantor sets, largely operated by trial and error. The theory of fractals not only explains the early models but also allows present modeling to proceed systematically in full knowledge of what is actually being done.

In this figure, the distribution of stars is part of a planar projection of the sky, which is a sphere centered on the Earth. The map covers the portion of the Northern Hemisphere between the longitudes − 45° and + 45°. In the illustration it takes the form of an ogive shape colored a dark blue. The "stars" distributed throughout it have been generated according to a model developed by the author; the principal mathematical input of the model consists of several parameters, of which the main one is the estimated fractal dimension of natural stellar distribution in the Galaxy.

FOR ADDITIONAL READING

Benoit B. Mandelbrot, "Getting Snowflakes into Shape," *New Scientist* (June 22, 1978, pp. 808–810).

Benoit B. Mandelbrot, *Fractals: Form, Chance, and Dimension* (W. H. Freeman, 1977).

Martin Gardner, "Mathematical Games," *Scientific American* (November 1976, pp. 132–136, and April 1978, pp. 16–32).