Easy and Natural Generation of Multifractals: Multiplying Harmonics of Periodic Functions

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Abstract. Simple multifractal measures are constructed by multiplying a periodically extended function with copies of itself. The frequencies of the copies form a geometric series, $(1, b, b^2, \ldots, b^n, \ldots)$, where b is a *real* number larger than 1. This deterministic construction leads to measures that are similar to random multifractal measures, yet are easier to build. At the same time, they do not have the unphysical disadvantages of other deterministic multifractals, such as the multinomial measures. The effect of random phase shifts is also considered.

1 Introduction

Multifractals are encountered in many fields [22]. Examples are the energy dissipation in a turbulent fluid [14, 15, 24, 25, 22], the growth rate along a DLA-cluster [23, 4, 6], the reaction rate along a fractal catalyst surface [8, 2], the current distribution in a percolation cluster [28], and time in a model for price variation [21]. This variety of multifractal measures calls for a better, more intuitive understanding of how multifractals originate, and for alternative simple constructions. Some constructions, such as the popular multinomial multiplicative cascade, are recursive and subdivide space into boxes, using an integer base b. In general, b is nonphysical, so that it would be desirable to minimize its role. The new method presented in this paper does just that.

A self-similar fractal set is identical at all scales; in a (multiplicative) multifractal measure m, the distribution of a quantity to be called mass is similar at different resolutions. The space over which mass is distributed is divided into fractal subsets, each with the same value for m. The fractal dimensions of these subsets can be plotted as a function of the Hölder exponent α defined by $m \sim \delta^\alpha$; when the resolution δ approaches zero, the plot yields the multifractal spectrum $f(\alpha)$. While this formalism may not be the most general one (see, *e.g.*, [16, 19, 20]), it suffices for most applications. For a general introduction to multifractals, see [17, 18, 5] and the recently published book [22].

Much work on multifractals, especially after their popularization by Frisch and Parisi [7] and Halsey *et al.* [10], assumes implicitly that a multifractal measure is close to being multinomial or even simply binomial. The construction of

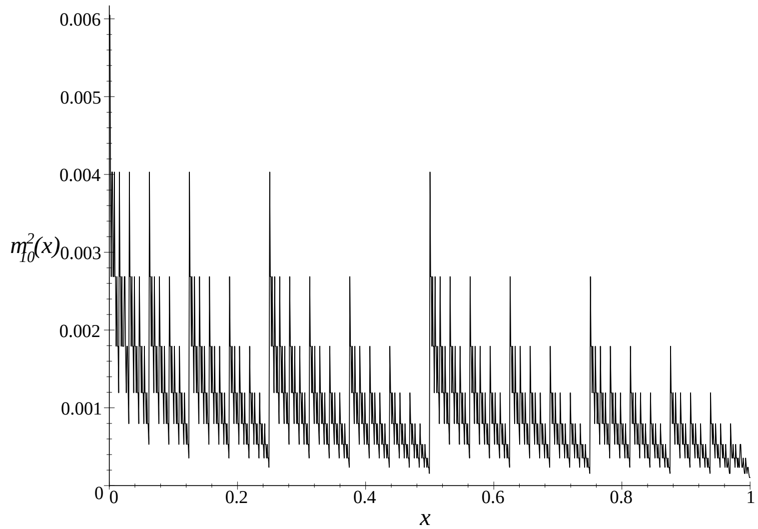

Fig. 1. Binomial measure, after 10 generations, with $p_1 = 0.6$ and $p_2 = 0.4$.

a multinomial measure on a line segment [0,1) starts by dividing the segment in b pieces of length $1/b$, and associating a different weight or probability p_i to each of these pieces. Each of the b shorter segments is replaced by a b times smaller copy of this generator, in which mass or probability is redistributed in the same way as in the generator; therefore, the j-th piece in the i-th segment of the first iteration is given the weight $p_i p_j$. The number of segments N with the same weight $p = p_i p_j$ follows the multinomial distribution, and this *multiplicative cascade* continued *ad infinitum* creates a multinomial multifractal measure. For $b = 2$, the measure is binomial. An example with $p_1 = 0.6$ and $p_2 = 0.4$ is shown in Fig. 1. When the resolution $\delta = b^{-n}$ approaches zero in this cascade, sets with the same Hölder exponent $\alpha = \lim_{\delta \to 0} \log p / \log \delta$ are fractal, and they have a fractal similarity dimension $f(\alpha) = -\lim_{\delta \to 0} \log N / \log \delta$. The link between $f(\alpha)$ and the moment generating function involves a Legendre transform (the thermodynamic formalism), and the explicit formulas for α and $f(\alpha)$ are well-known and will not be repeated here (see [10, 16–18]).

The multinomial measures are a fine first example, yet inadequate for most practical purposes. In the first place, why the arbitrary division in segments, based on some integer b ? Why should this multiplicative cascade proceed with the same set of probabilities $\{p_1, \ldots, p_b\}$? Moreover, the measures obtained in turbulence [14, 15] and growth phenomena [23, 4, 6] are definitely *not* binomial or multinomial. Even the best known signature, the $f(\alpha)$ spectrum, need not have the familiar $\bigcap$-shape, unless special conditions are satisfied as Coppens discusses in the practical example of catalysis [2].

The random multiplicative multifractals introduced by Mandelbrot [14, 15, 22] can lead to $f(\alpha)$ spectra that have different shapes, and do not require an arbitrary choice of a base b. Ingenious mathematical examples, such as the Minkowski measure [9, 20], were studied to show how the spectrum can become left-sided (*i.e.*, the maximum value of $f(\alpha)$ is obtained for $\alpha \to \infty$) [27], or how α or $f(\alpha)$ can be negative. Moreover, there can be several maxima in the $f(\alpha)$ curve [26].

The experimental evidence shows that these spectra are not just esoteric creations. But they are complicated and make it very desirable to have some simple method to generate various spectra other than the inadequate multiplicative scheme behind the multinomial measures. The method introduced in this paper responds to this strong desire and proves particularly powerful, despite its simplicity. The method merely involves the repeated multiplication of a periodic function $w(t)$ (the *generator*) with rescaled copies $w(b^k x)$ of itself. We will show how this product of harmonics leads to a multifractal measure.

The main advantage of our method is threefold. First of all, a great variety of measures can be constructed through a multiplication of functions that is not much harder than the generation of multinomial measures, but much more general. Secondly, as opposed to the multinomial measure, in which the "base" b is an integer larger than 1, the base is now any *real* number larger than 1. Thirdly, we will prove how the essential characteristics of measures constructed using this completely *deterministic* method are the same as those of Mandelbrot's *random* multifractal measures, when the multipliers are chosen in a way to be discussed in this paper.

This is a preliminary announcement of the method and some of the general features of the *multifractal product of function* (*MPF*). Detailed mathematical proofs will appear soon [3].

2 Methodology of the *MPF*

Define a function $w(x)$ with period 1, *i.e.*:

$$w(x+k) = w(x), \qquad \forall k \in \mathbf{N}. \tag{1}$$

This "generator" and a base $b > 1$ define the functions $w(b^{i-1}x), i = 1, 2, 3, \ldots$ These functions' frequency is b^{i-1} times higher than for $w(x)$. Now, let:

$$m_n^b(x) = \prod_{i=1}^{n} w(b^{i-1}x). \tag{2}$$

Like $w(x)$, the functions $m_n^b(x)$ have period 1, if b is an integer.

When $w(x) = p_j$ for $(j-1)/b \le x < j/b$, with $\sum_j p_j = 1$, the b-nomial measure is recovered. Note the resemblance to a Fourier series, where a *sum* of periodic functions w_i (cosines, sines) with a geometric series of frequencies converges to the limit Weierstrass function. Here, the sum is replaced by a product, and, if a nondegenerate limit $m(x) \equiv m^b_\infty(x)$ exists (in the sense of measures rather than functions), it can be shown to be a multifractal measure that we call a *multifractal product of functions* (*MPF*).

The multifractal spectrum $f(\alpha)$ is defined as follows. First, the coarse-grained Hölder exponents of the b^n intervals $[ib^{-n}, (i+1)b^{-n}) = [i\epsilon_n, (i+1)\epsilon_n)$ are evaluated for all $i = 0, \ldots, b^n - 1$:

$$\alpha(i) = -\frac{1}{n} \log_b \int_{ib^{-n}}^{(i+1)b^{-n}} m^b_n(x)dx = \frac{\log \int_{i\epsilon_n}^{(i+1)\epsilon_n} m^b_n(x)dx}{\log \epsilon_n}. \tag{3}$$

The histogram $N(\alpha)$ is then constructed, by distributing the Hölder exponents in bins $[\alpha, \alpha + \Delta\alpha)$. The normalized logarithm of this histogram is the pre-multifractal spectrum:

$$f_n(\alpha) = \frac{1}{n} \log_b N(\alpha) = -\frac{\log N(\alpha)}{\log \epsilon_n}. \tag{4}$$

The limit multifractal spectrum is:

$$f(\alpha) = \lim_{n\to\infty} f_n(\alpha). \tag{5}$$

Another way to derive $f(\alpha)$ is from the cumulative distribution $N(A > \alpha)$. It should be noted that b can be any positive real number larger than one and does not have to be an integer in the *MPF*, as opposed to the multiplicative cascades leading to multinomial multifractals.

The multifractals constructed in this way are entirely deterministic, but can be randomized by introducing a random phase shift with every iteration, *i.e.*, we let

$$m^b_n(x) = \prod_{i=1}^{n} w[b^{i-1}(x + \phi_i)], \tag{6}$$

in which ϕ_i is a random deviate uniformly distributed between 0 and 1.

3 Examples

An unlimited variety of measures is generated through this *MPF* by changing the generator $w(x)$. A few examples are presented below.

- A multinomial measure is constructed when $w(x)$ is constant within each of b intervals of length $1/b$.
- The potential well or “fjord”, $w(x) = 1 - 0.25[1 - \cos(2\pi x)]^2$, shown in Fig. 2 generates the measure shown in Fig. 3 after respectively 5 (above) and 10 (below) generations. This could serve as a crude model for the accessibility measure along a rough surface. Even visually, the measures in this and the previous example are quite different from the binomial measure in Fig. 1.

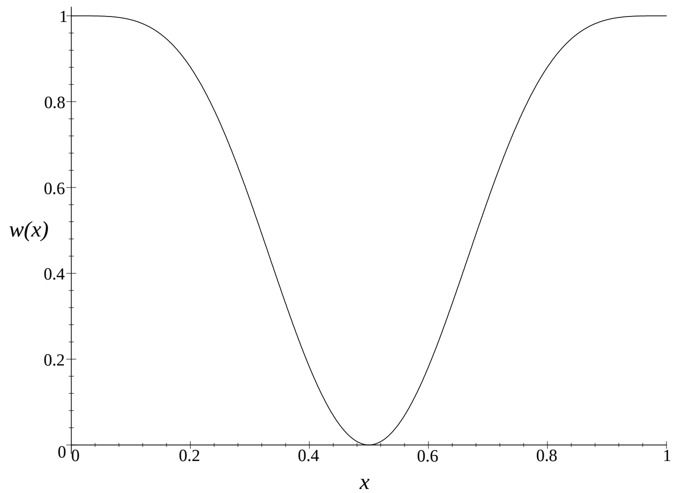

Fig. 2. Potential well or "fjord", $w(x) = 1 - 0.25[1 - \cos(2\pi x)]^2$.

- For $w(x) = \sqrt{x(1-x)}$ (a semi-circle), the tenth generation $w_{10}^2(x)$ looks as in Fig. 4. The pre-multifractal spectrum corresponding to this "circle measure" is shown as well. It is clearly different from the familiar symmetrical spectrum of a binomial measure. For higher pre-multifractal generations, similar spectra were found with similar $\alpha_{\min}$ and $\alpha_{\max}$, yet the fluctuations become smaller and occur at other places. They appear to be a result of the typical relatively slow overall convergence and the way in which the spectrum was generated through binning. This could be avoided through a procedure similar to the one described in [1].
- The following class of base functions is particularly interesting, because their shape can be qualitatively changed by modifying the value of the parameters:

$$w(x) = \alpha + \beta x^a + \gamma(1-x)^b, \qquad 0 < x < 1. \tag{7}$$

Depending on the values of the 5 parameters, a wide range of *MPFs* can be constructed. This example will be discussed elsewhere.

4 Link with random multiplicative multifractals

The deterministic measures generated using our new method are interesting by themselves, but also because of their close relation to random canonical multiplicative multifractals (CM^2) [15, 22].

The construction of the random CM^2 measures relies on an integer base b. It starts with a uniform measure of density 1 and begins by dividing $[0, 1]$ into b equal b-adic parts. After the first construction step, the density is constant on each b-adic part and its b values w_i are independent and identically dis-

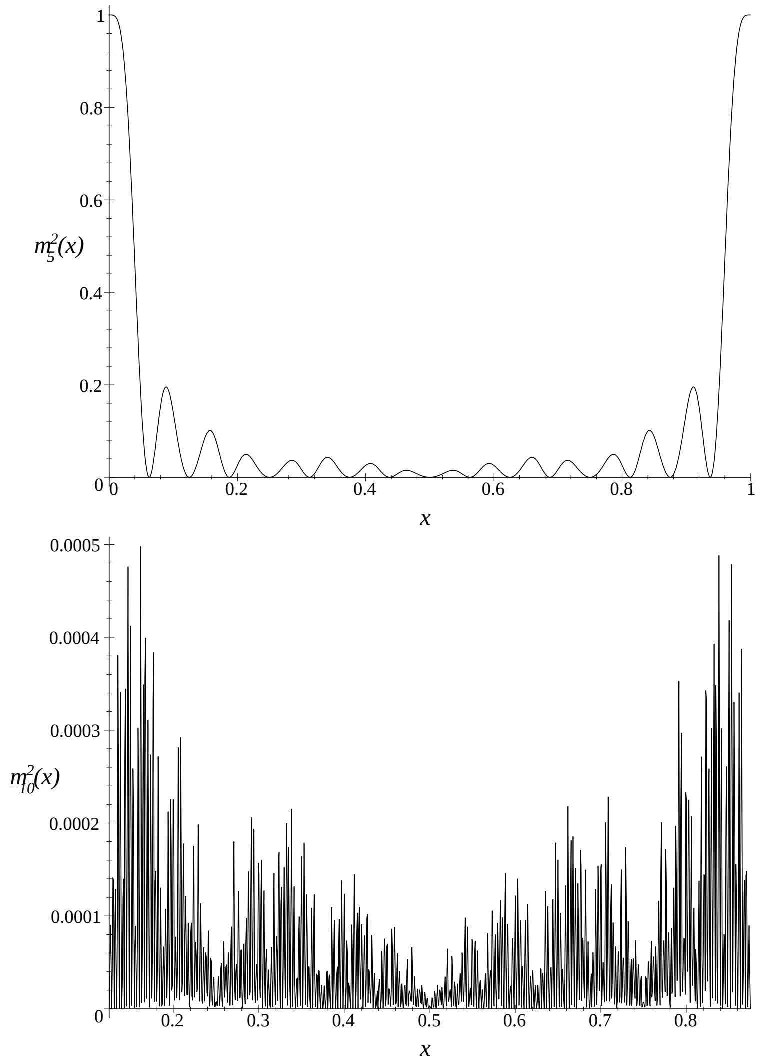

Fig. 3. Fifth (above) and tenth (below) generation of the potential well *MPF* shown in the previous Figure, formed by a binary multiplicative cascade of $w(x) = 1 - 0.25[1 - \cos(2\pi x)]^2$. The interval [1/8, 7/8] is shown for the tenth generation, because a plot of $m_{10}^2(x)$ over the interval [0,1] would be completely dominate by the peaks $m_n^b(0) = m_n^b(1)$. This is typical for a multifractal measure.

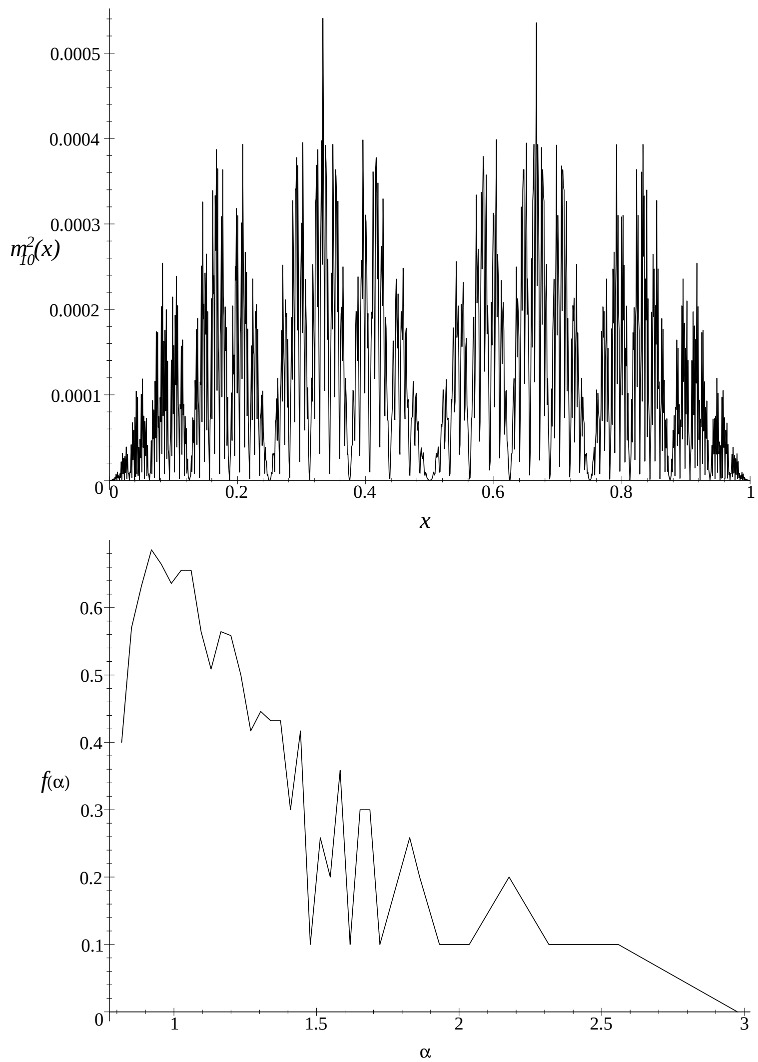

Fig. 4. Above: tenth generation of the "circle measure", formed by a binary multiplicative cascade of a semi-circle. Below: the corresponding pre-multifractal spectrum.

tributed random variables with cumulative distribution function $F(w) =$ Probability $\{W < w\}$ and average $\langle w \rangle = 1$. The notation w is preserved for reasons that will transpire momentarily, but w no longer denotes a periodic and non-random function. The second step of the construction introduces b^2 equal b-adic parts. The corresponding multipliers are independent and identically distributed random variables w_{ij} and the corresponding densities at the end of the second step take the form $w_i w_{ij}$. When this multiplicative procedure is iterated, the random multifractal CM^2 ensues.

The cumulative distribution function $F(w)$ is of course non-decreasing and such that $F(0) = 0$ and $F(\infty) = 1$. If necessary, the graph of $F(w)$ can be processed by filling in each jump by a vertical straight interval. Exchanging the coordinate axes for this filled-in graph yields the filled-in graph of an inverse function $F^{-1}(x)$, which in turn can be made into a left-continuous inverse function. Using integration by parts, $\langle w \rangle = \int_0^1 w dF(w) = \int_0^1 F^{-1}(x)dx$. This result begins to justify using the same letter w for different purposes. Indeed, the same formal condition of conservation on the average applies to CM^2, in the form $\langle w \rangle = 1$, and also to the MPF, in the form $\int_0^1 w(x)dx = 1$. When b is an integer, the inverse $F^{-1}(x)$ of the cumulative probability distribution used to construct the random CM^2 is the function $w(x)$ used to construct the *MPF*.

In the CM^2 construction, only an interval $[0, 1]$ is considered; in the *MPF* construction, the function $w(x)$ is periodically extended beyond this interval. For almost all x, the infinite series of multipliers $w(b^{i-1}x)$ used to construct the *MPF* samples the interval $[0, 1)$ in a uniformly dense way, as is the case for the CM^2, because the β-map:

$$\left(x, \{bx\}, \{b^2x\}, \dots\right), \tag{8}$$

where $\{y\} = y \bmod 1$ is almost surely uniformly dense in $[0, 1)$. The latter was shown by Weyl, Hardy and Littlewood $[29, 11, 12]$ and in a more general case by Kuipers and Niederreiter [13].

To conclude, a left-continuous function $w(x)$ that satisfies conservation on the average can be used in two distinct ways: when b is an integer, to construct a CM^2 measure; for all b, to construct a MPF measure. There is clearly a close relationship between the previously studied multifractal properties of the CM^2 and the simpler to construct *MPF*, which is also multifractal.

This raises the question of whether or not the multifractal formalism that $[15, 22]$ developed to apply to CM^2 also applies to the *MPF*. For integer b, this is already clear from the relation discussed above. The answers provided in [3] for general b are to the affirmative. The criterion of non-degeneracy is:

$$\int_0^1 w(x) \log_b w(x)dx < 1 \tag{9}$$

and the $\tau(q)$ and $f(\alpha)$ functions are those familiar from the study of CM^2.

To be more precise, the preceding affirmative answers apply directly to the randomized *MPF* in which $w(b^{i-1}x)$ is replaced by $w[b^{i-1}(x+\phi_i)]$, with a random phase ϕ_i, as in Eq. (6). For irrational b, the same affirmative answer applies to the non-random product, Eq. (2), for $n \to \infty$, taken over the interval $[X, X+1]$, where X is asymptotically large.

An interesting corrolary of this link between the random measures and the limit *MPF* is that a large number of transformations on $w(x)$ do not change the spectrum either. Such transformations include translations, rotations or reflections of parts of $w(x)$ about vertical axes. However, note in passing that finite generations of the *MPF* do not sample the interval $[0,1)$ uniformly for all x, so that the identity of measures and spectra is not valid for finite generations.

5 Conclusions

This paper introduces a new, simple way to generate a great variety of multifractal measures. We call these measures *Multifractal Products of Functions* (*MPF*). They are constructed by repeatedly multiplying a periodically extended function with copies of itself, each copy having a frequency b times higher than the previous one in the series, where b is a real number larger than 1. The generation is therefore similar to the iterative construction of certain basic fractals, or the cascade leading to a multinomial measure, but is clearly much more general than the latter.

Apart from simplicity, the most important advantage of the *MPF* is the link between them and random multiplicative measures. This opens the way to many applications, because of the ease with which the *MPF* operates. Despite the fact that the *MPF* is either completely *deterministic* or slightly randomized (by using Eq. (6) with random phases), the multifractal spectrum is the same as that of a *random* multiplicative measure.

We sketched the methodology and gave some examples; other papers will refine the mathematical background and discuss the generated multifractal spectra in more detail. The method can be used in various applications in which multiplicative processes are present, such as in the description of turbulence, growth processes and the study of the accessibility distribution over rough interfaces.

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