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The potential distribution around growing fractal clusters

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THE process of diffusion-limited aggregation (DLA) is a common means by which clusters grow from their constituent particles, as exemplified by the formation of soot and the aggregation of colloids in solution. DLA growth is a probabilistic process which results in the formation of fractal (self-similar) clusters. It is controlled by the harmonic measure (the gradient of the electrostatic potential) around the cluster's boundary. Here we show that interactive computer graphics can provide new insight into this potential distribution. We find that points of highest and lowest growth probability can lie unexpectedly close together, and that the lowest growth probabilities may lie very far from the initial seed. Our illustrations also reveal the prevalence of 'fjords' in which the pattern of equipotential lines involves a 'mainstream' with almost parallel walls. We suggest that an understanding of the low values of the harmonic measure will provide new understanding of the growth mechanism itself.

The DLA model of Witten and Sander[1] captures the essential fractal aspects of a wide range of physical phenomena[2-7], such as particle aggregation, dielectric breakdown, viscous fingering and electrochemical deposition. In DLA growth, an 'atom' executes Brownian motion until it hits, and becomes attached to, the curve that bounds a given 'target' or 'seed'. Another atom is then launched, and the whole process repeats. Initially, the growth rule and the pattern are very simple, but both become extremely complex in time. The distribution of hitting points of the atoms on the boundary of a fixed pattern becomes increasingly irregular and complex with repeated application of the rules. Simultaneously, the pattern becomes more complex (Fig. 1). Overwhelming evidence from computer simulations[1-7] indicates that these patterns are near self-similar fractals[8], meaning that their complication is about the same at all scales of observation that are sufficiently above the scale of the atom.

The distribution of the hitting points on the target's boundary is the harmonic measure, μ (ref. 9). This measure is obtained by solving the Laplace equation for the (electrostatic) potential, the boundary of the cluster being put at zero potential and a 'circle at infinity' being put at unit potential. The gradient of the potential at a point P of the boundary defines the harmonic measure $\mu(P)$ at P. It is approximated by the potential at the lattice point nearest to P (ref. 10), and is normalized to add up to one. The study of the laplacian potentials and their harmonic measures has been enriched by the example of DLA, because the process creates its own boundary conditions. The difficulties encountered in analytical approaches have made DLA a challenge in theoretical physics[6,7,11-14] and in pure mathematics.

Figures 1, 2 and 3 show half of one of the many clusters we have grown on a cylindrical lattice, the initial target being the cylinder base. The many clusters drawn on a lattice in the original circle geometry[1,10], in which the initial target is a cell in a square lattice, look similar (see cover figure). The cylindrical cluster shown in Figs 1-3 was grown to a height equal to its base $L=512$, and contains ~30,000 atoms. Once the cluster is obtained, the custom has been to solve the discrete Laplace equation[10] on the original lattice on which DLA has been grown. But in the resulting illustrations, the potential cannot be seen with sufficient resolution in the many narrow 'fjords'. We therefore solved the Laplace equation on a lattice that was twice as fine. This equation was solved iteratively and the relative computation error was ~0.01.

Figure 1 and and the cover figure illustrate our first finding. As is usual in this field[4], each atom is coloured to indicate the time it joined the cluster. In the background, rendered in black, the potential is above a certain threshold. Elsewhere, the 'whiteness' of the background increases with −log (potential). The low-potential zones look like ghostly 'cobwebs' extending to spatial scales larger than the atoms. Many occur in the bottom of fjords that may be short, but are narrow. Their location is very sample-dependent, and in many clusters they appear

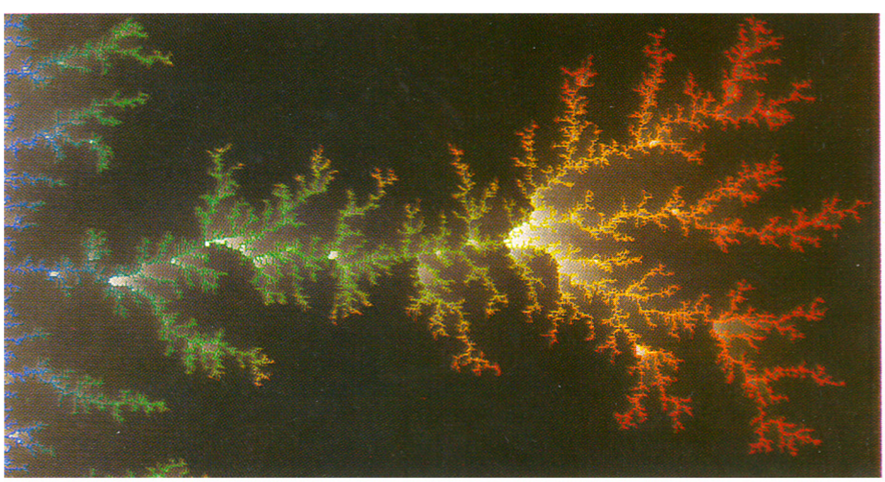

FIG. 1 'Cobwebs' of very low potentials near a cylindrical cluster of diffusion-limited aggregation. The vertical cylinder base is to the left.

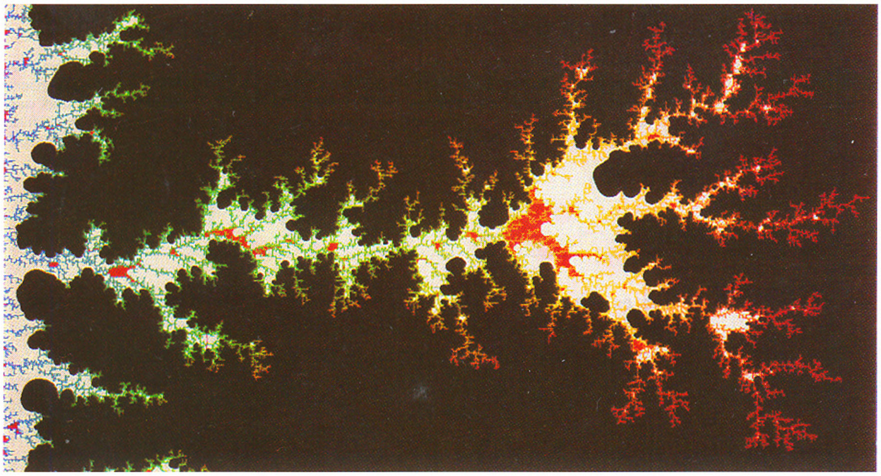

FIG. 2 Low-potential regions around part of a cluster. The smallest value of the growth probability occurs at the lowest point of the largest red domain.

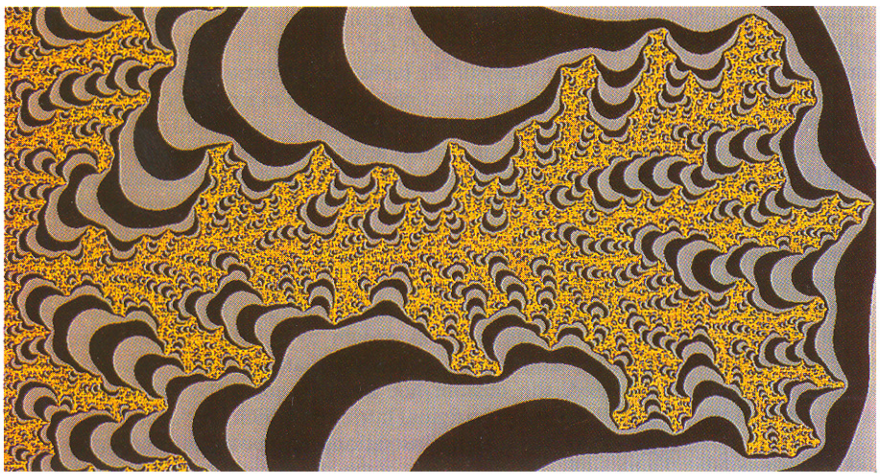

FIG. 3 Equipotential lines around cluster. Each combination of one black and one white zebra stripe marks a fall in the potential by a factor of ten.

between young growing branches positioned near the top of very large trees. The very bright and localized spots seen in Fig. 1 seem irrelevant to the understanding of the structure of DLA above the atomic level, and our conclusions do not depend on their presence or absence.

Figure 2 is another representation of the data in Fig. 1. The smallest potentials from 10^{-25} to 10^{-17} are in red, and the potentials between 10^{-10} and 10^{-17} are in white. The domains of lowest potential are conspicuously absent near the base of the tree.

The rendering used in Fig. 3 is borrowed from the 'zebra stripe' illustrations of the potentials around Julia or Mandelbrot sets[15]. Here, the logarithm of the potential is evaluated in base $b=\sqrt{10}$, and odd and even values of the integer part of $-\log_b$ (potential) are plotted in black and white. Given two points not too close to the boundary, one measures the change in $-\log_{10}$ (potential) by counting 'doublets' made of one black and one white stripe.

There is a stark difference in the structure of the trees formed by the fjords and those formed by the clusters. The most striking feature is the abundance of fjords through which the potential field has near constant width, despite the fractal shape of the walls. A stack of near uniformly spaced binary stripes provides a smoothed inner approximation of such a fjord. Rectangular fjords with parallel walls have been considered (for example, in ref. 16), but only because they are easy to treat analytically. Our figures suggest that one could almost characterize DLA clusters as branching just enough to ensure fjords in which the potential exhibits a 'mainstream' with nearly parallel walls, including fjords that are narrow and surprisingly long. Our observations based on zebra striping fit nicely within the mathematical advances of Carleson and Jones[17] concerning Laplacian potentials around fractal boundaries.

Although the location of the smallest growth probabilities in DLA does not seem to have been discussed explicitly in the literature, in our experience there is a widespread expectation that the lowest potentials should be concentrated near the initial seed. It is well known, and follows from a theorem by Carleson and Jones[17], that if DLA were exactly self similar, the harmonic measure would be smallest at the bottoms of the deepest fjords. The effects we report therefore express deviations from exact self-similarity. They do not originate from branching hierarchies, but from the ubiquitous geometric fluctuations, which result in narrow fjords when branches fail to fan out and instead run in parallel or tend to close in on each other.

The reasoning behind the expectation concerning the smallest probabilities is that the initial seed becomes increasingly 'screened' as the cluster grows: at each new stage of screening, the harmonic measure μ is multiplied by some random factor <1, adding some positive term to $-\log\mu$. This initial step is the very basis of the original multiplicative multifractals[18]. Because the points near the initial seed are subject to the most stages of screening, it seems natural to expect them to correspond in all cases to the largest values of $-\log\mu$. But this assumes that the increment of $-\log\mu$ contributed by each stage of screening has a finite expectation, which is not a safe assumption. On the contrary, our observation that a few stages of screening can be as effective as many (because there are minima in short fjords), combined with extensive numerical work to be reported elsewhere, points toward an infinite expectation value for the increments of $-\log\mu$ in models of DLA. It follows[16,19-21] that the so called 'thermodynamic' distribution function $f(\alpha)$ is left-sided. Here α is the Holder exponent, $\alpha = -\log \mu / \log L$. We therefore believe that left-sidedness is not a pathology related to 'old wood' near the initial seed, but a genuine aspect of fractal growth in DLA and hence physically significant. 'Thermodynamic' implies that $f(\alpha)$ is an asymptotic concept. When $f(\alpha)$ is left-sided, however, the convergence to this asymptote is not only slow but 'singular'[19,21], meaning that even if a cluster is very large, its 'finite size' $f(\alpha)$ differs in shape from the asymptote. This issue will be discussed elsewhere.

Much of the research in fractal geometry has been done, and the results discussed, using graphical representations[8]. The ability to graph clusters resulting from DLA and related processes has been crucial to the progress in this field[4]. Nevertheless, we believe that this path deserves to be followed much farther. Potentials have been plotted before (refs 22-25 and P. Meakin, unpublished work) but the plots were, in those cases, intended solely to illustrate already known facts.

Here we have discussed only the observations from a careful interactive study of the graphics, and some implications for the theoretical understanding of DLA. Our quantitative analysis of DLA will be reported elsewhere. Combining the potential field and the cluster has emphasized the interplay between the two, revealing new properties, and, we feel, will lead to a better understanding of fractal growth.

Note added in proof: We have now done the experiments using 'off-lattice' aggregates. The results confirm those reported here. □

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1. Witten T. A. & Sander, L. M. *Phys. Rev. Lett.* **47,** 1400-1403 (1981).
2. Pietronero, L. & Tosatti, E. (eds) *Fractals in Physics* (North-Holland, Amsterdam, 1986).
3. Feder, J. *Fractals* (Plenum, New York, 1988).
4. Meakin, P. in *Phase Transitions and Critical Phenomena* (eds Domb, C. & Lebowitz, J.) Vol. 12, 355-489 (Academic, New York, 1988).
5. Vicsek, T. *Fractal Growth Phenomena* (World Scientific, Singapore, 1989).
6. Aharony, A. & Feder, J. eds *Fractals in Physics* (North-Holland, Amsterdam, 1990).
7. Aharony, A. & Feder, J. *Physica* D**38,** 1-398 (1989).
8. Mandelbrot, B. B. *Fractal Geometry of Nature* (Freeman, New York, 1982).
9. Kakutani, S. *Proc. Imper. Acad. Sci.* (*Tokyo*) **20,** 706-714 (1944).
10. Niemeyer, L., Pietronero, L. & Wiesmann, H. J. *Phys. Rev. Lett.* **52,** 1033-1036 (1984).
11. Pietronero, L., Erzan, A. & Evertsz, C. J. G. *Phys. Rev. Lett.* **61,** 861-864 (1988).
12. Pietronero, L., Erzan, A. & Evertsz, C. S. G. *Physica* A**151,** 207-245 (1988).
13. Tsallis, C. (ed.) *Statistical Physics* (North-Holland, Amsterdam, 1990).
14. Tsallis, C. *Physica* A**163,** 1-428 (1990).
15. Peitgen, H. O. & Richter, P. H. *The Beauty of Fractals* (Springer-Verlag, New York, 1986).
16. Blumenfeld, R. & Aharony, A. *Phys. Rev. Lett.* **62,** 2977-2980 (1989).
17. Carleson, L. & Jones, P. W. *Duke math. J.* (in the press).
18. Mandelbrot, B. B. *J. Fluid Mech.* **62,** 331-358 (1974).
19. Mandelbrot, B. B. *Physica* A**168,** 95-111 (1990).
20. Lee, J. & Stanley, H. E. *Phys. Rev. Lett.* **61,** 2945-2948 (1988).
21. Mandelbrot, B. B., Evertsz, C. J. G. & Hayakawa, Y. *Phys. Rev.* A**42,** 4528-4536 (1990).
22. Pietronero, L., Evertsz, C. J. G. & Siebesma, A. P. in *Stochastic Processes in Physics and Engineering* (ed. Albeverio, S. *et al.*) (Reidel, Dordrecht, 1988).
23. Sander, L. M. *Sci. Am.* **256**(1), 82-88 (1987).
24. Family, F. & Vicsek, T. *Computers in Physics,* **4,** 44-49 (1990).
25. Evertsz, C. J. G., Siebesma, A. P. & Oeinck, F. *Laplacian Fractal Growth,* video (Solid State Physics Laboratory, University of Groningen, 1988).

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