Fractals in Physics (Trieste, 1985).
Edited by Luciano Pietronero
& Erio Tosatti, Amsterdam: North-Holland, 1986, pp.21–28

H24

Diagonally self-affine fractal cartoons. Part 3: "anomalous" Hausdorff dimension and multifractal "localization"

• ⋆***Illustrated long chapter foreword.*** An "anomaly" can be dismissed, or welcomed as a challenge to be faced. Moreover, the familiar mantra applies. When facing a challenge, the recommended first step is to become acquainted with its nature as intimately as possible. It is best to create suitable pictures and very literally to "see" and ponder them. This chapter provides a marvelous fresh example.

The original was written at Harvard in 1985, with no computer access. Belatedly, this reprint created the need for illustrations and preparing them brought fresh understanding of special examples, turned out to be highly educational, and motivated this lengthy illustrated foreword.

In the continuing search for a good definition of self-affine functions, this chapter reveals fresh complexity. It adds a wrinkle to the contrast between generic pure multifractality and the presence of a multifractal mark on the unifractality which is this book's main topic.

I have not followed closely the purely mathematical development of the anomalous Hausdorff dimension, but rumor has it that many questions raised by McMullen 1984 remain open despite considerable progress on many fronts and a large literature (including Lalley & Gatsouras 1992, 1994, Peres 1994, and Peres & Kenyon 1996).

Two constructions that are less closely related than they might seem. For reasons described shortly, Figure 1 will be called "nonlocalized," and Figures 2 and 3 will both be called "localized." In a first approximation, all three – and also Figures 1 and 2 of Chapter H22 – use the same up, down, up and up generator. In a second approximation, the arrows along the sticks of this common generator vary from figure to figure.

Consider the final functions $f(t)$ that those generators yield by context-free recursion (as defined in Chapter ⋆H2, Section 2).

First question: Do changes in the arrows' directions suffice to affect $f(t)$? The answer from pure mathematics is "yes." To wide surprise, McMullen 1984 found that the localized graph in Figure 2 has the "anomalous" Hausdorff–Besicovitch dimension $D_{HB} \sim 1.45$. The non-

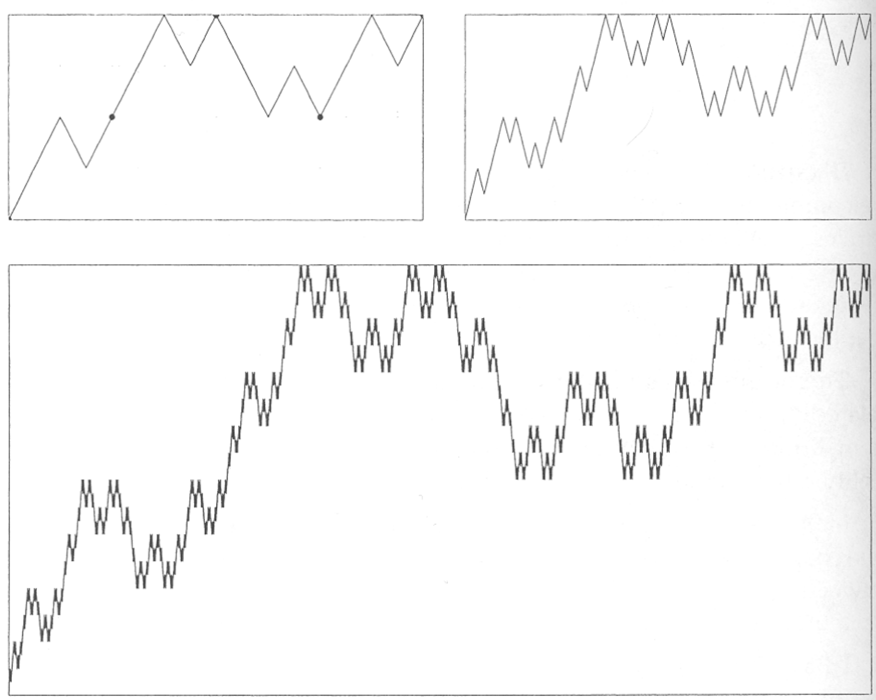

FIGURE C24-1. In this chapter, the figures illustrate the Foreword. Figures 1, 2, and 3 describe the constructions of three self-affine "cartoons" whose generators are seen on the first three sticks on the upper left panels. They are the same except that the directions of some sticks are reversed, as seen on the upper left panels. $H = 1/2$ in all three cases but other visual and numerical aspects are different, showing that seemingly small changes in the algorithm can have spectacular effects. For this graph, $D_{HB} = 1.5 = 2 - H$, as expected. As befits a "cartoon," the up and down oscillation is highly reminiscent of WBM (Chapter ⋆H3) or "Weierstrass–Mandelbrot" functions for the exponent $H = 1/2$ (Chapter ⋆H4).

localized graph in Figure 1 has $D_{\mathrm{HB}} = 1.5 = 2 - H$, as expected. The Hölder exponent is $H = 1/2$ and the box dimension is $1.5 = 2 - H$ in both cases.

The quantification of roughness being a key goal of fractal geometry (recently pushed to the front, as seen in Section 1 of the Overview), it would be wonderful if one could, in each case, improve the measurement of roughness by quoting both dimensions.

A *second question* arises: Are the differences between the two panels visible to the naked eye? One glance at Figures 1, 2, and 3 shows that the answer is an emphatic "yes," again. Clearly, all those graphs are not as "innocuous" as promised by their definitions. The search for telling visual "symptoms" will examine two properties.

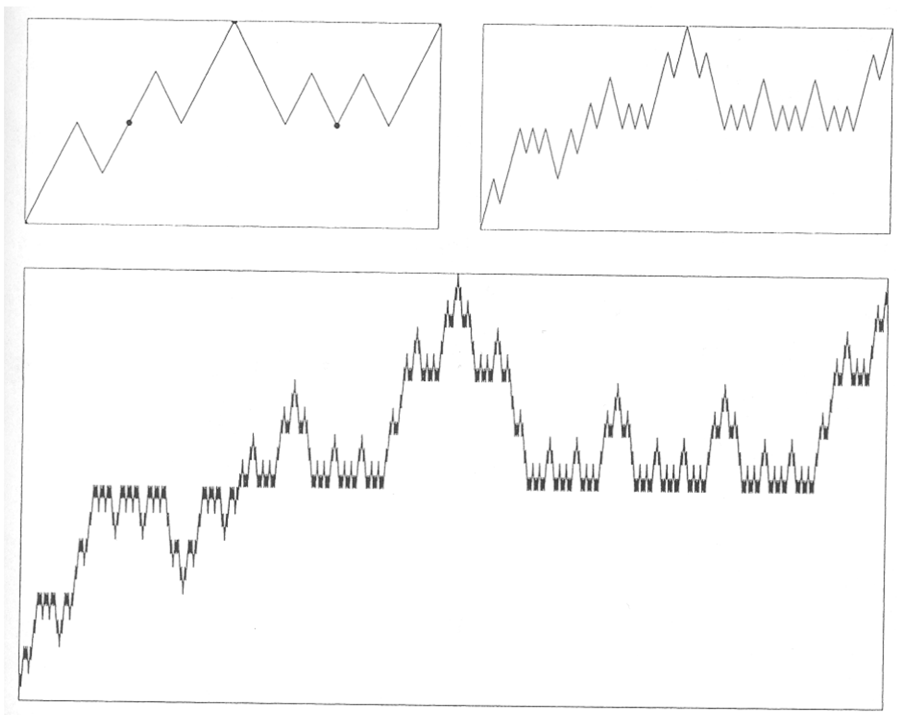

FIGURE C24-2. Starting with Figure 1, the second generator stick was inverted. For this graph of a self-affine function $f(t)$, $D_{\mathrm{HB}} \sim 1.45 < 2 - H$. The up and down oscillation is conspicuously non-Brownian exhibiting an infinity of sharp needles, all pointing down. Most values of $f(t)$ are highly localized. Is this $f(t)$ the cartoon of any intrinsically interesting process?

Perceived absence or presence of "localization" in Figures 1, 2, and 3. In Figure 1, the up and down oscillation seems unconstrained and is highly reminiscent of Wiener Brownian motion (WBM) (Chapter ⋆H3), and also of Weierstrass–Mandelbrot motion for $H = 1/2$ (Chapter ⋆H4).

To the contrary, both Figures 2 and 3 seem inhomogeneous in the sense that $f(t)$ seems, much of the time, to zigzag within tight horizontal strips. Interrupting those zigzags, quick back and forth transitions create a collection of smooth surgical needles.

In fact, each needle has a rich structure. Starting on Figure 3 from the "bump" that peaks at $x = 1/2$ and $y = 1$, smaller bumps are obtained by a series of affine reductions in the ratios of 1/2 vertically and 1/4 horizon-

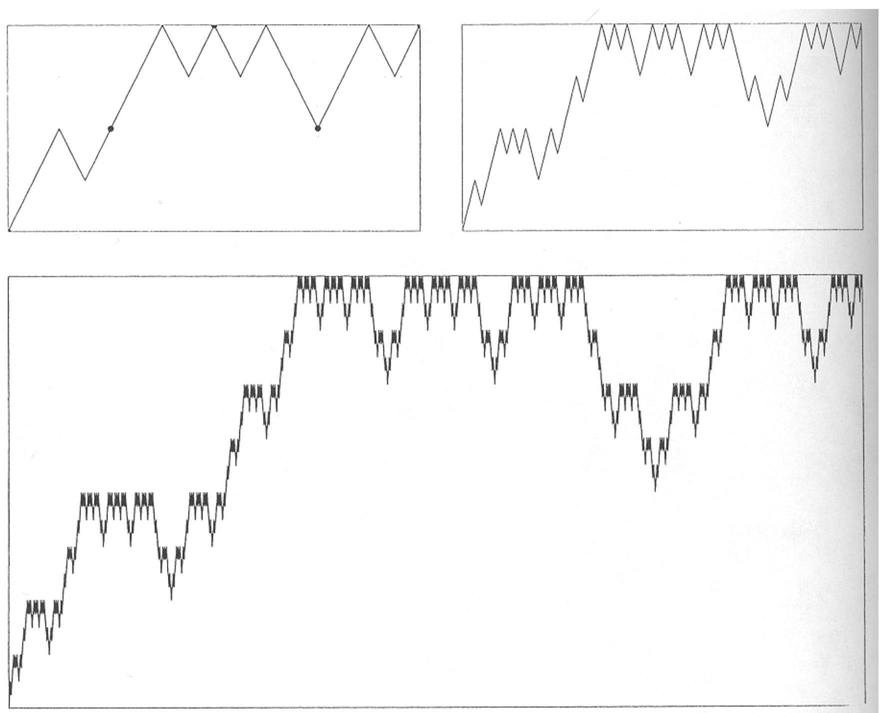

FIGURE C24-3. Starting again with Figure 1, the second and fourth generator sticks were inverted. This variant is, like Figure 2, highly non-Brownian but, unlike Figure 3, clearly not asymmetric. As a matter of fact, denote by $N^{+}(k)$ and $N^{-}(k)$ the number of intervals that point up and down after the kth stage. It is easy to show that, as $k \rightarrow \infty$, the ratio $N^{+}(k)/N^{-}(k)$ converges to 1. In this sense, Figures 1 and 3 are both asymmetric.

tally. The bumps become increasingly needle-like but, in an affine sense, their sides' extreme "unsmoothness" remains unchanged.

Moreover, the zigzags seem to be interrupted by quick transitions to other horizontal strips. Seen closely, each transition is not instantaneous but has a rich affine structure. On a higher level of complexity, the up and down oscillations in Figures 2 and 3 strongly recall the "singular non-decrease" that characterizes the Cantor and "Besicovitch" staircases; the

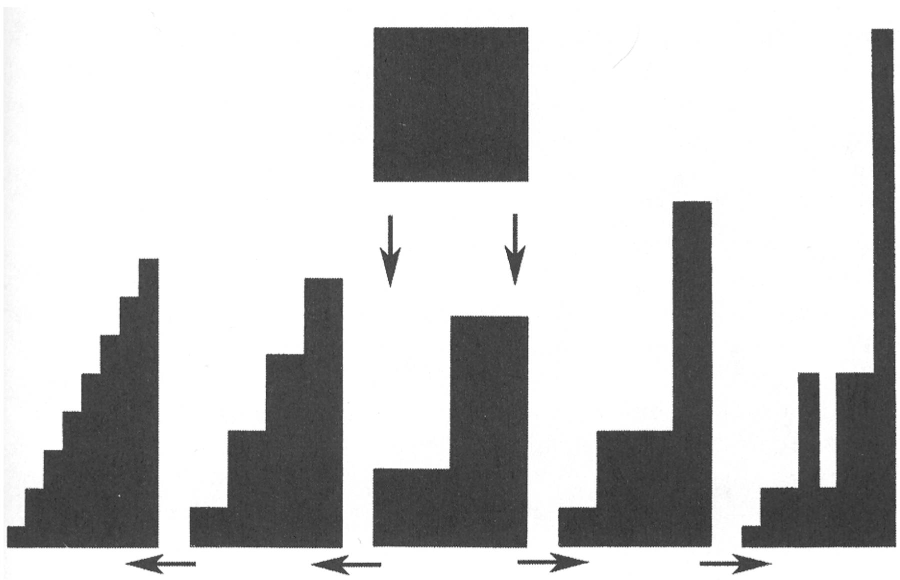

FIGURE C24-4. This six-panel figure features three approximations to the y-projected measures of the functions drawn in Figures 1 and 2.

The middle panels from top to bottom show the uniform initiator and the first approximation, which are common to both measures.

To the left are two further approximations relative to the nonlocalized variant $f(t)$. They suggest, strongly and correctly, that the limit projected measure has a smooth density.

To the right are two further approximations relative to the localized variant. Here, the $f(t)$ projected measure is binomial; this is the simplest of all multifractal measures, yet it quickly becomes extraordinarily uneven.

latter are defined as indefinite integrals of the binomial measures that we shall encounter soon. "Singular nondecrease," the technical term that Lebesgue attached to those functions, was well chosen, since the new meaning followed an everyday sense. Before the Cantor staircase became fully understood, past "intuition" had to be retrained and refined. Further retraining is now required to fully understand the behavior of Figures 2 and 3, which combine continuity with localized values.

The needles are the key to anomalous Hausdorff–Besicovitch dimensions. The needles give play to a feature that is needed in the mathematical theory of D_{HB} but seems at first to lack intuitive justification. Indeed, D_{HB} involves a notion called Hausdorff measure, and it is necessary to insure that Hausdorff measure satisfies all the conditions to be a measure. For that, the D_{HB} of a set S involves the most efficient covering of S by disks of radius $\leq \rho$, rather than $= \rho$. In the case of Cantor dusts, the availability of disks of radius $< \rho$ makes no difference. But in the case of localized graphs, it allows the neighborhoods of sharp needle points to be covered more tightly, which in turn affects D_{HB}. Other fractal dimensions, such as the local box dimension (see Chapter H22), use more restricted coverings of radius $= \rho$ which sharp needles do not affect.

When needles exist and point toward the exterior of the loop, they "attract lightning," and more prosaically concentrate electric charges on a subset of dimension 1. Figures 1 and 3 appear the same as seen from top or bottom, while Figure 2 is very asymmetric. There are two ways to draw a half-circle centered at $t = f(t) = 1/2$ and including the points (0, 0) and (1, 1). Either half-circle, combined with the graph on Figures 1, 2, or 3, creates a loop ("Jordan curve"). Set this loop at an electric potential 0 and a far-away point at potential 1, and solve the Laplace equation. For every loop, one can say that, roughly speaking, the electric charge will concentrate on a "Makarov set" with $D_{HB} = 1$. It will be fun to evaluate this potential and inspect this set on Figures 1, 2, and 3.

The distribution of the y-projected measure: it is multifractal under localization and very smooth without localization. So far, the distinction between the absence or presence of localization was visual. Could another visual aid help confirm the distinction confirmed in a more quantitative fashion?

A natural measure on the graph of any function of t attaches to the arc between t' and t'' a mass equal to $|t' - t''|$. The idea is to start with a uniform measure on the time axis, "lift" it on the graph of $f(t)$, and then project this natural measure on the ordinate axis. The first stages of construction for Figures 1 and 2 are shown in Figure 4.

To the nonlocalized Figure 1, corresponds the left of Figure 4. Here, the asymptotic measure is smooth and graphed by a straight line. This is an opportunity to recall the Moore "crinkly curves" of Chapter ⋆H2. The y-projected measure in that case is a projection of the Cesaro triangle sweep, therefore in the limit its density varies linearly.

To the localized Figure 2 corresponds the right side of Figure 4. Here, to the contrary, the asymptotic measure is anything but smooth. It has no density and concentrates on a set of y whose length ("linear Lebesgue measure") is zero. The reader familiar with multifractals recognizes the first stages of the construction of a binomial measure corresponding to the masses $m' = 2.5$ and $m'' = 0.75$. This and other multifractal measures are the topic of M 1999F and the projected M 2001T. Marked "localization" is already conspicuously present in those first stages. Asymptotically, a theorem guessed by Besicovitch and proved by Eggleston asserts that the bulk of the variation of $f(t)$ can be viewed as concentrated in a subset of Hausdorff–Besicovitch dimension $D_{HB} = -m' \log_2 m' - m'' \log_2 m'' \sim 0.81$.

To characterize the graph, one still needs the overall Hölder exponent, which is $H = 1/2$, and was knowingly inputted together with the generator. But additional numerical characteristics are provided by some indirect consequences of the generator. They include the Hölder exponents of the y-projected intrinsic measure, namely, the above D_{HB} and they range from $\alpha_{max} = -\log_2(1/4) = 2$, in the regions of high localization, to $\alpha_{min} = -\log_2(3/4) \sim 0.41$ in the low-measure regions in-between.

If the sticks carry randomly selected arrows, an anomalous D_{HB} and localization are almost surely absent. That is, both effects manifest very peculiar resonances. But they deserve broad attention. •

✦ **Abstract.** For certain self-affine fractals constructed recursively, McMullen has shown that the Hausdorff–Besicovitch dimension D_{HB} takes a doubly "anomalous" value: it is a fraction and is strictly smaller than the local box dimension D_{BL}. This raises interesting questions: does this discrepancy point toward deep new developments? Does it cast doubt upon the special position of D_{HB} in fractal geometry? This paper addresses these questions and also comments on the dimensions of sections for certain self-affine fractals. ✦

FOR RECURSIVELY CONSTRUCTED SELF-AFFINE FRACTALS, the Hausdorff–Besicovitch dimension D_{HB} may be strictly smaller that the local box dimension D_{BL}.

1. Introduction

For a very simple self-affine construction, a theorem of McMullen 1984 yields for D_{HB} a value that came as a surprise and has interesting implications. Recall from the chapter before last that a cartoon involves a box generator and a stick generator. When b', b'' and the box generator are given, the stick generator, that is, the combination of the signs of r'_n and r''_n can be chosen to yield $D_{\text{HB}} < D_{\text{BL}}$. This inequality is "anomalous," that is, came as a surprise. Additional results on these and related structures are given in Bedford 1984. However, we shall prove or argue that other combinations of signs – including some random combinations – yield $D_{\text{HB}} = D_{\text{BL}}$. The mathematical implications of this variability will be seen – unexpectedly – to involve multifractal measures. Conceptual issues will also be discussed; they may mean that D_{HB} is not a physical notion, contrary to my previous conviction.

As background, this paper includes some results that seem novel, concerning cuts of certain self-similar fractals.

2. McMullen's theorem yielding D_{HB} and a corollary

Theorem A (McMullen 1984, as extended by Curtis T. McMullen in private communication.) Consider a recursive self-affine fractal generator in a lattice, with b' and b'' the horizontal and vertical bases and with $H = \log b''/\log b'$. Assume $r'_n = \pm 1/b'$ and $r''_n = 1/b''$ for all n (this means that all arrows point up). Denote by b'_j the number of cells contained in the j-th horizontal row of the generator. Then the value of D_{HB} for the limit fractal is the solution of

$$b''^D = \sum b'^H_j.$$

Corollary B (McMullen 1984, Bedford 1984) For the self-affine fractals covered in Theorem A, one may have $D_{BH} < D_{ML} = D_{\text{BL}}$, where D_{ML} and D_{BL} are the local mass and box dimensions discussed in the chapter before last.

Numerical example. $D_{HB} < D_{BL}$ when the generator is Figure 2 in Chapter H22. Here, $b' = 4$, $b'' = 2$, $H = 1/2$, $b'_1 = 1$ and $b'_2 = 3$. Theorem A yields $2^D = 1 + \sqrt{3}$; hence $D_{HB} = 1.44998$, but $D_{BL} = 2 - H = 1.5$.

Observation. In this example, the Y-projected measure is singular. Indeed, it is the well-known binomial multifractal measure (a Besicovitch measure in the terminology of *FGN*, p. 377) with $p_1 = 1/4$ and $p_2 = 3/4$.

3. Expression for the vertical dimension anomaly, defined as $A'' = D_{BL} - D_{HB}$

The "vertical dimension anomaly" will be defined by $A'' = D_{BL} - D_{HB}$. This is an intrinsic measure of the dispersion of the nonvanishing values of b'_j. For example, the stick generator of the record of a continuous function may either make a few large swings or many small ones; the discrepancy is larger in the second case. (The reader is encouraged to construct specific illustrations.)

First, we express the anomaly A'' in terms of $p_j = b'_j/N$, where the notation obviously intentionally mimics the probabilities corresponding to the Y-projected measure. The vertical dimension anomaly A'' becomes

$$A'' = (1 - H)(\log_{b''} N'' - I''_H), \text{ where } I''_H = \frac{(\log_{b''} \sum p_j^H)}{(H - 1)}.$$

The first term in A'', namely $\log_{b''} N''$, is the dimension of the set that supports the Y-projected measure; this support is an interval when $N'' = b''$, but it is a Cantor dust when $N'' < b''$. The second term in A'' – namely, I''_H – is familiar to the reader acquainted with the binomial measures: it is the critical ("generalized") dimension of the exponent $q = H$ of the Y-projected measure.

Next, we note that the binomial measure is a special case of the random multifractal measure introduced in Mandelbrot 1974f{N15}. This is made clearer by adopting the notation $W_j = b'_j b''/N = p_j b''$ and $\langle W^h \rangle = \sum b''^{-1} W_j^h$. This notation attributes equal probabilities to the b'' possible values of a (random) "weight" W that satisfies $< W > = 1$. Now,

$$A'' = -\log_{b''} \langle W^H \rangle = -\log_{b'} \langle W^H \rangle^{1/H}) > 0.$$

Corollary C. (McMullen, private communication.) When $b' \neq b''$, a necessary and sufficient condition for the anomaly to vanish is that all b'_j that do not vanish must be identical.

Observation. The Y-projected measure is uniform on its support when $A'' = 0$ it is singular on its support when $A'' > 0$.

4. Fractal dimension of horizontal cuts

4.1. Background: horizontal cuts of certain recursive self-similar fractals and expressions for their anomaly. To appreciate the results in the next subsection, it is necessary to understand fully the corresponding results relative to self-similar fractals, which are self-affine fractals that correspond to $b'' = b'$. Those results seem new but are also interesting in themselves. Specifically, it is known that, for the ordinary Sierpiński carpet, the dimension obtained from the overall dimension $\log_3 8$ by subtracting 1 is not observed along any horizontal cut. Horizontal cuts are therefore "anomalous." To study their dimensions, let us place a uniform mass on the carpet, each construction stage spreading it into 8 pieces of density 9/8. Clearly, the X- and Y-projected measures are both multinomial measures, with $b = 3$, $p_1 = 3/8$, $p_2 = 2/8$ and $p_3 = 3/8$. Let me show that the dimensions of the cuts are related to these measures.

To carry out the argument, take general integer values of b, N and the corresponding $p_j = b_j/N$, not excluding the possibility of some $p_j = 0$. Write the intersecting horizontal line's ordinate in base b as $y = 0.y_1 y_2 \ldots$ and let k_j be the number of repetitions of j in the first k digits. Define $\beta(y_h)$ as equal to b'_j if $y_h = j$. In the k-th prefactal approximation, the horizontal line of ordinate $\boldsymbol{y}$ intersects a number of cells of side b^{-k} that is equal to the product of the $\beta(y_h)$ from $h = 1$ to k. Thus, the kth approximation to the horizontal cut has the dimensional exponent

$$\log N_k(y) / \log(b^k) = \sum (k_j/k) \log_b b_j.$$

When y is such that $k_j/k \to q_j$ for every j, this expression has a limit; the limit is a box dimension, and that dimension depends on y. Now, choose a point using the intrinsic measure on our fractal: choosing t with uniform measure attributes to y the Y-projected multinomial singular measure. In this case, $(k_j/k) \to p_j$. Subtracting and adding $\Sigma(k_j/k) \log_b N - 1$ to both sides yields the result that the cut's dimension is almost surely

$$D = (\log_b N - 1) + (1 - I), \quad \text{where} \quad I = -\sum p_j \log_b p_j.$$

Since $\log_b N$ is the dimension of our original planar fractal, $\log_b N - 1$ is the value given by the rule that taking a cut decreases the dimension by one. We know that the rule fails. We see that the anomaly depends on the 1-information dimension of the Y-projected measure, and it can either be equal to or greater than zero.

The anomaly vanishes if (and only if) $I = 1$, which requires $p_j \equiv N/b$.

If $p_j = 1/N''$ for N'' values of j and $p_j = 0$ for the other values, the anomaly is $1 - \log_b N''$. In that case, the Y-projected measure is uniform over a Cantor dust. Consider the horizontal cut with respect to the uniform measure on [0,1] it is a.s. empty but with respect to the uniform measure on the Cantor dust it is a.s. of dimension

$$\log_b - \log_b N''.$$

This is a very interesting way to generalize the standard rule to cuts that are conditioned to be nonempty.

4.2. Horizontal cuts of certain recursive self-affine fractals. The results to be described here are parallel to those of Section 4.1.

When the anomaly A'' vanishes, the horizontal cuts of the fractal in Theorem A are either empty or Cantor dusts of dimension $\log_{b'}(N/N'')$. When $N'' = b''$, the cut is never empty and is of dimension

$$\log_{b'}(N/b'') = D_{\mathrm{BL}} - 1,$$

which fulfills the standard "subtract one" rule about cuts' dimensions. When $N'' < b''$ and the cut is conditioned to be nonempty, it is of dimension

$$\log_{b'}(N/N'') = (D_{\mathrm{BL}} - 1) + \log_{b''} N'',$$

which expresses the generalized standard rule of Section 4.1 in self-affine terms. But one can also write

$$\log_{b'}(N/N'') = D^*_{\mathrm{BL}} - 1,$$

which fits the standard rule in a manner appropriate for self-affinity, by replacing D_{BL} by D^*_{BL}, which is the dimension the fractal takes after it has been squeezed vertically to eliminate its gaps.

When $A'' > 0$, the cut's dimension is almost surely

$$D = \log_{b'} N - \sum p_j \log_{b'} p_j.$$

It is convenient to restate this in terms of dimensions. Since

$$\log_{b'} N = D_{\mathrm{ML}} - (1/H - 1) \log_{b'} N'' = D_{\mathrm{ML}} - (H - 1) \log_{b''} N'',$$

we have

$$D = D_{\mathrm{BL}} - (1 - H) \log_{b''} N'' - HI''_1,$$

where $I''_1 = -\sum p_j \log_{b''} p_j = 1 - \left\langle W \log W \right\rangle$. This is the information dimension of the Y-projected measure. It follows that the cut's dimensional anomaly takes the form

$$D - (D_{\mathrm{BL}} - \log_{b''} N'') = H[\log_{b''} N'' - I''_1].$$

This anomaly is > 0, and is $> H[\log_{b''} N'' - I''_H]$, because $I''_1 < I''_H$.

5. Definitions: $D*$ as a global counterpart for D_{HB}, and the horizontal anomaly $A' = D* - D_{\mathrm{HB}}$

It was observed in the Chapter before last that the formulas for $D_{\mathrm{BL}} = D_{\mathrm{ML}}$ and $D_{\mathrm{BG}} = D_{\mathrm{MG}}$ are obtained from each other by exchanging the roles of b' and b'' and the roles of N' and N''. Love of symmetry immediately led me to seek a global counterpart to D_{BH} in the solution $D*$ of the equation

$$b'^D = \sum b''^{(1/H)}_j.$$

The corresponding quantity $A' = D_{\mathrm{BG}} - D* = -\log_{b''} \left\langle W^{1/H} \right\rangle < 0$ can be called a "horizontal dimension anomaly."

In the case of function records, one has $D* = 1$. In Chapter 22, this value has become familiar for all other global dimensions. The anomaly $A' = 0$, and the vertical cuts' dimension is 0, as it should be. The anomaly A' plays for vertical cuts the role that A'' plays for horizontal cuts, but does $D*$ have anything else to recommend itself as a dimension?

6. Self-affine continuous records not covered by Theorem A

6.1. An example where D_{HB}=D_{BL} and a question. The two generators shown on Figures 2 and 3 in Chapter H22, differ only by the direction of the arrow placed on the second stick, but this seemingly minor feature has drastic consequences. Figure 3 is the coordinate of the Peano motion and its Y-projected measure is differentiable with the density $2(1-x)$. For it, C. McMullen has shown (private communication) that $D_{\text{HB}} = 1.5$.

More generally, suppose that the Y-projected measure has right and left derivatives $f'(y+)$ and $f'(y-)$. A heuristic application of the formula in Theorem A to the generator after k stages yields the following formula, with the sum from $p = 1$ to b''^k:

$$b''^{kd} = \sum_{y = pb''^{-k}} (\Delta f(y))^{H_{b''}k} \sim \sum b''^{-k} (f'(y+))^{H_{b''}(2-H)k} \sim b''^{(2-H)k} \int (f'(y+))^H \, dy.$$

When $k \gg 1$,

$$D = 2 - H + (1/k) \log_{b''} \int (f'(y+))^H \, dy.$$

Asymptotically for $k \to \infty$, $D = 2 - H$.

The same result extends, obviously, to the case where the Y-projected measure is absolutely continuous on a Cantor dust of ys.

Reminder. Consider the fractional Brownian process $B_H(t)$. In that case, $D_{\text{BH}} = 2 - H$; hence $A'' = 0$. It is also known that the Y-projected measure is differentiable. Observe that, for random processes, the abscissas of the horizontal level cuts through the ordinate y form the set of recurrences to the point y (also called the "local time").

Question. Is there a one-to-one relation between A being 0 and the H-information dimension of the Y-projected measure being 1?

6.2. Random generators. Randomly selected generators are manageable when they involve geometrically imbedded birth processes. (The study of these processes deserves to be extended beyond the properties sketched here.)

The simplest function $M_H(t)$, used in M 1985l {H21}, is obtained by taking $b'=4, b''=2$ and the same stick generator as in Figure 1 but positioning all four arrows at random, with equal probabilities for "up" and "down." A partly heuristic argument to be given momentarily suggests that the Y-projected measure of $M_H(t)$ is differentiable. Assuming that the question at the end of Section 6.1 should be answered in the affirmative, this measure's differentiability suggests that $D_{\mathrm{HB}}=D_{\mathrm{ML}}=2-H$. This answer is asserted without evidence in the *Scripta* paper, before I knew that the issue is a difficult one.

The first half of the argument is rigorous. The mass in a y interval of length b''^{-k} of the y-coordinate is contributed by rectangles of width b'^{-k}, of height b''^{-k} and of weight $N^{-k}=b'^{-k}$. The number of these rectangles is given by a simple birth random process with an average of $N/b''=b'/b''>1$ offspring per generation. Consider for each y the sequence of nested b-adic intervals of length b''^{-k} that defines y (if y is not b''-adic) or the sequences that define $y+$ and $y-$ (if y is b''-adic). The average measures over the intervals in one of these sequences are of the form $(b'/b'')^{-k}$ $\times$ (number of offspring in the k-th generation). A standard theorem on birth processes (Harris 1963, p. 12) tells us that these average measures almost surely converge to a limit, which is the value of a strictly positive random variable W satisfying $\left\langle W\right\rangle=1$.

Therefore, $\mu(y)$, defined as the cumulative measure between 0 and y, has the property that over the nested intervals of length b''^{-k} that define y $y+$, or $y-$, the average slope of $\mu(y)$ converges to a limit.

The next step becomes much easier to state for $b'=4$, $b''=2$, and $N=4$. Select a finite $k\gg 1$ and consider the k-th approximate measure $\mu_k(y)$ over a dyadic interval (y', y'') of length 2^{-k}. In the left half of this interval, the number of offspring at stage $k=1$ is the sum of 2^kW independent random variables of expected value 2 and finite variance σ^2. This number can be written as $2^{k+1}W+(2.2^kW)^{1/2}\sigma G$, where G is a normalized Gaussian variable. In the right half of the same interval, the same expression holds except that + is replaced by –. Therefore, over the two halves of our binary interval the slopes of $\mu_{k+1}(y)$ are $W\pm\sqrt{WG}\sqrt{2^{-(k-1)/2}}$. The approximate measure $\mu_{k+1}(y)$ is obtained from $\mu_k(y)$ by midpoint displacement (see M 1982F{*FGN*}, Chapter 26). That is, the midpoint value $\mu_k((y'+y'')/2)$ is displaced by the amount $2^{-k-1}\,\sigma\sqrt{WG}\,2^{-3(k-1)/2}$. When k

is very large, one can neglect variations of W between construction stages, and one finds for $\mu(y)$ the series

$$\mu(y) \sim \mu*(y) = \mu_k(y) + a\sqrt{W}\sum_{h>k} a^h \sigma(2^h y).$$

Here $a = (2\sqrt{2})^{-1}$ and $\sigma(y)$ is a random sawtooth function, namely a function that vanishes for integer y's, takes independent normalized Gaussian values at half integer y's and is linear over dyadic intervals of length 2^{-k-1}. The resulting function $\mu*(y) - \mu_k(y)$ is familiar. For $a > 1/2$, it is continuous but non-differentiable (and serves in several rough fractal algorithms meant to model mountains). For $a < 1/2$, and hence for our present value $a = (2\sqrt{2})^{-1}$, the function is continuous and right and left differentiable. (In fact, the derivative $\mu*'(y+)$ is near the Brownian function; it is a variant of the Rademacher series, and is close to the Fourier series of the Brown-Wiener process.)

I expect but did not check that the heuristics in the last paragraph can be either made rigorous or replaced by a rigorous short argument.

Conclusion. Since the present application involves $a < 1/2$, $\mu(y)$ is differentiable: the above W is its derivative for non- b''-adic y's, and is its right or left derivatives for b''-adic y's. (At the b''-adic points, the right and left derivatives are negatively correlated.)

6.3. More general $M_H(t)$-like random functions. The discussion is written in terms of b', b'' and N, instead of 4, 2 and 4, because the same argument holds more generally whenever the stick generators that yield continuous records are assigned certain special probabilities. (When $b'' > 1$ and $b' \gg 1$, randomly generated stick generators will suffice).

For other probability assignments, however, the situation is more complex. An interval of length b''^{-k} may be nested in either of b'' "locations" within an interval of length b''^{-k+1}, and the expected number of offspring usually depends on the "location." The corresponding Y-projected measure is *not* expected to be differentiable.

Conjecture. Scattered examples suggest to me that both $D_{\text{HB}} < D_{\text{BL}}$ and $D_{\text{HB}} = D_{\text{BL}}$ can a.s. be achieved by recursive self-affine continuous random records and that $A'' = D_{\text{BL}} - D_{\text{HB}}$ is a continuous function of the probabilities allocated to the acceptable stick generators.

7. Random self-affine sets not covered by theorem A

7.1. Generators obtained by conservative curdling with $N/b'' > 1$ or canonical curdling with $pb' > 1$. The idea of selecting the generator completely at random can take one of several forms. One can attribute equal probabilities to every way of drawing N among the $b'b''$ cells. When all the choices are statistically independent, the ultimate fractal is obtained by the self-affine counterpart of the "microcanonical" {P.S.2000, also called "conservative"} version of the process of multifractal curdling introduced in M 1974f{N15} and M 1974c {N16} (see also M 1982F{*FGN*}, Chapter 10).

Alternatively, one can form a generator at random, each cell having a given probability p of belonging to the generator. Each generator then includes $pb''b'$ cells on the average. When all the choices are statistically independent, the ultimate fractal is obtained by the self-affine counterpart of the "canonical" version of the process of multifractal curdling that I also introduced in 1974 (*FGN*, Chapter 10).

The expected number of offspring per generation is N/b'' in the first model, and pb'' in the second. The birth process becomes a birth and death process, but the argument of Section 3 remains generally valid if $N/b'' > 1$ in the first model, and $pb'' > 1$ in the second. The novelty is that the derivative can now be zero with a probability that is between 0 and 1 . In a given sample, it may vanish over some intervals of y.

7.2. Generators obtained by conservative curdling with $N/b'' < 1$ or canonical curdling with $pb' < 1$. In the combined generator after k stages, $b'_j = 0$ for most values of j. In a first examination, let us disregard these values and consider only the j such that $b_j > 0$. A standard theorem in the theory of birth processes (Harris 1963, p. 12) is that the conditional distribution of b'_j, knowing that $b'_j > 0$, tends for $k \to \infty$ to the distribution of a limit random variable. If we denote this limit by W, as in Section 3, then the measure carried by a nonempty interval of length b''^{-k} is again the product of W by the measure N^{-k} of a cell of area $b''^{-k} \times b'^{-k}$. The average number of nonvanishing b'_j is, therefore, $N^k / \langle W \rangle$. In the limit as $k \to \infty$, the y-projected measure is carried by a fractal dust of dimension $\log_{b''} N$, (respectively, $\log_{b''}(pb'b'')$). This result could have been guessed. Less easy to guess is the fact that, on this dust, the distribution of the measure is uniform except for the factor W. This shows a situation that is parallel to that in Section 7.1, with the exception that $\langle W \rangle > 1$ here.

Alternatively, one can study the measure over a fractal dust obtained as follows. At each stage, pick all cells of length b''^{-k} in which $b'_j > 0$ plus any number of empty cells needed to add up to N. This amounts to "dilute" W to allow $W=0$ and to make $\langle W \rangle = 1$. The resulting situation is parallel to that encountered earlier in this section. Either way, the heuristic use of Theorem A suggests that D_{HB} is arbitrarily close to D_{BL}. That is, $D_{\text{HB}} = D_{\text{BL}}$.

Conjecture. I expect $D_{\text{HB}} = D_{\text{BL}}$ to hold widely for randomly generated self-affine sets that are not constrained to be records of functions.

Comment. Canonical curdling generates a special planar multifractal. One ought to investigate the projections of more general planar multifractals, both in the self-similar and in the self-affine cases.

8. Discussion

First observation. The value of D_{HB} yielded by Theorem A is usually considered "anomalous" because it is a fraction, but this view has been exorcised by fractal geometry. The second anomaly is that the value of D_{HB} is the wrong fraction. But Theorem A was *not* contrived for this purpose. Earlier known "second anomalies" had, to the contrary, been specifically contrived. They were, for example, highly non-uniform, like the Bouligand anomaly in Section 1 above. For this reason, in every previous case of interest to physics, the fractal dimension could first be obtained by some rough and ready method, usually based on D_{BL} or D_{ML}, and later "confirmed" by more elaborate and technical calculations of D_{HB}.

Second observation: When dealing with records of functions such as $B(t)$, it is natural to attach equal measures to records that correspond to time intervals of equal duration. This property is satisfied in the case of $B(t)$ by the Hausdorff measure relative to a suitable gauge function. Does such a gauge function exist for the recursive self-affine functions covered by Theorem A? If it does, the resulting Hausdorff time is *not* real time.

My gut reaction was to view the assignment of arrows (signs to the r'_n and r''_n) in the self-affine fractal construction as being a "non-physical" fine detail, and a quantity that depends on this assignment *could not* be physical. But this fine detail turned out to affect the (unique) a.s. dimension of the horizontal cuts, which is physically meaningful. At that point, we seemed to have another repeat of what had happened repeatedly as fractal geometry transformed extreme "pathologies" into natural behaviors. But, in a third stage, the argument of Section 4.1 came to mind; now the anomaly in D_{HB} again appears non-physical. Even if it should eventually reveal some useful new physical intuition, I fear that the Hausdorff-Besicovitch definition has now lost its earlier "special standing."

Recall that earlier works of mine gave much play to D_{HB} in the study of fractals, even using it in *FGN* as basis for a "tentative" definition of the term fractal. The main reason to use it then, however, was that it was well seasoned by age and more papers were devoted to it than to its variants, although these papers were mostly irrelevant to the geometric needs that I was pursuing. These rationales for D_{HB} in fractal geometry must now undergo active reexamination.

9. Record of the Weierstrass function $W_0(t)$ and its D_{HB}

The oldest example of a continuous non-differentiable function is the Weierstrass function $W_0(t)$ discussed in M 1982F {*FGN*}, page 388. A long-standing irritant is that D_{HB} is *not known* in the case of $W_0(t)$. The same is clearly true of the self-affine variant that I introduced {*FGN*,} p. 389. The self-affinity exponent H is the Hölder-Lipschitz exponent; hence a classical result of Besicovitch & Ursell shows that D_{HB} is at most $2-H$. Everyone expects $D_{\text{HB}} = 2-H$, which *FGN* denotes by D (showing that I did not anticipate the complications now being revealed). Mauldin & Williams 1986 gave a lower bound that is strictly below D and depends on b.

{P.S. 2000. This section was part of the original draft of M 1986t but was cut from the printed text. The value of D_{HB} for the graph of $W_0(t)$ continues to be unknown. I wonder whether or not the "new" W functions of Chapter ⋆H4 would be any easier to study.}

Acknowledgment. Curt McMullen has been extremely helpful.

&&&&&&&&&&&& ANNOTATIONS &&&&&&&&&&&&

How McMullen 1984 came to be written. Heisuke Hironaka was Chairman of the Harvard Mathematics Department in 1979-80; we became good friends. Concurrently, he was Japan's "Mr. Mathematics," and as such was often seen on television. In a program devoted to fractals, he asserted that the fractal dimension of the "cartoon" in Figure 2 is 1.5. A listener asked for a reference and Hironaka asked me to help. I knew of no reference and a quick look showed that, while $D_{\text{HB}} \leq 1.5$ was easy to prove, $D_{\text{HB}} \geq 1.5$ was not easy. At that point, the problem fell into the lap of C. McMullen, then a student (and now a Professor) at Harvard.