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H23

Diagonally self-affine fractal cartoons. Part 2: Length and area "anomalies"

• ***Chapter foreword.*** When a mathematical or scientific notion is sensible and useful in one context, one cannot prevent free citizens (read "mathematicians or scientists") from using it in increasingly broad contexts. This process of "generalization" is fundamental and often very useful. But, even for Euclidean shapes, the notions of length and area are more delicate than generally believed. More broadly, generalization often leads to "anomalies," even in cases that seem totally innocent. This is especially so in the context of fractals. Both old and new examples are discussed in this chapter. A *Post-Publication appendix* dwells on the "Schwarz paradox" that profoundly shook mathematics in the late 1800s. •

✦ **Abstract.** For a self-similar curve, one can estimate fractal dimension by "walking a divider." For self-affine curves, it is shown that this process yields a local and a global value, both doubly anomalous. Other problems raised by length and area measurement for fractals are investigated. ✦

1. Introduction

IT IS TEMPTING TO THINK that it is always possible to measure a curve's length by "walking" an increasingly narrowly opened divider along it, or a surface's area by triangulating it increasingly finely. On standard curves, the procedure works well. On standard surfaces (e.g., of a cylinder), anomalies are known to arise, the main one being the Schwarz area paradox, which deserves to be widely known and is restated below. On self-similar curves, walking a divider yields the fractal dimension. Let us now explore the same procedure for self-affine fractals and show that the exponents it yields differ from the mass/box dimension.

2. Measuring the length of self-affine fractal curves obtained as records of functions

***2.1. The measurement of length using the Minkowski sausage yields local and global dimensions respectively identical to** D_{ML} **and** D_{MG}* . In the spirit of Minkowski and Bouligand, define a curve's approximate length $B(\eta)$ via the η-neighborhood, called "Minkowski sausage" in M 1977F and M1982{*FGN*}. This set contains all the points within a distance η of a point on the curve. For a standard rectifiable curve and $\eta \ll 1$, one has $B(\eta) = (2\eta)^{-1}$ (area of the sausage). For a self-similar curve (*FGN*, p. 36), $B(\eta) \propto \eta^{1-D}$. The results for a self-affine curve are familiar from the preceding Chapter: the area of the sausage behaves for small η as $N(\eta)\eta^{-2} \propto \eta^{H}$; hence the local sausage dimension is $2-H$. The global sausage dimension is 1.

2.2. The measurement of length by walking a divider and monitoring last exits yields local and global dimensions identical to D_{ML} and D_{MG}. One of the many methods of measuring the length of a rectifiable curve is "to walk a divider" along it. The curve may have knots, i.e., multiple points of any order, as long as its points are ordered, "in time." Pick a starting point, $P_{0.}$ The point P_1 will be the first exit of the curve from the circle of center P_0 and radius η, and so forth. If $L(\eta)$ denotes the length of the resulting broken line approximation, the curve's length is $\lim_{\eta \to 0} L(\eta)$.

One may instead select for P_1 the last, rather than the first exit along the curve. And one may move backwards.

For a self-similar curve, Chapter 6 of M1982F{*FGN*} has shown that $L(\eta) \propto \eta^{1-D}$, and again one can pick either the first or the last exit at will.

For our self-affine curves, the situation is quite different. In addition to the local dimension for $\eta \to 0$, there is a global dimension, which we shall see is equal to 1. And the local divider dimension has two totally different versions: one for last exits and one for first exits.

Before we proceed, note that arguments on self-similar records are made simpler, and the results are unchanged, if the circle around P_k is replaced by a square.

In this case, last exits become a simple matter. Cover our curve with $(b''^{k})^{2-H}$ square boxes of side $b''^{-k} \ll 1$; this yields $D \geq 2-H$. Next, add a ring of 8 identical boxes around each cell to multiply the side by three.

Clearly, b''^{2-H} "divider steps" of size $3b''^{-k}$ are sufficient to walk along the curve. Therefore, the divider dimension satisfies $D \leq 2-H$. To have both $D \geq 2-H$ and $D \leq 2-H$, we must have $D = 2-H$.

2.3. When length is measured by walking a divider and monitoring first exits, it yields "anomalous" dimensions. The divider dimension's local value for small η is 1/H. This is the fractal dimension of a fractal trail related to the record. For large η, the divider dimension is 1. This section is summarized from M 1985l{H21}. When $\eta \gg t_c$ (e.g., when the unit of B_H is very small), the record is effectively a horizontal line. The divider walked along the curve remains mostly parallel to the t-axis, and $L(\eta)$ varies little with η. If one insists on writing $L(\eta) \propto \eta^{1-D}$, the fact that $L(\eta)$ is constant yields 1 as the global dimension, irrespectively of H.

When, to the contrary, $\eta \ll t_c$ (e.g., when the unit of B_H is large), the situation is different: a divider walked along the curve remains mostly parallel to the B-axis. This yields $1/H$ as the dimension.

This extremely strange value can exceed 2 and is doubly anomalous insofar as it contradicts the value $2-H$ of the other local definitions of a fractal dimension. On the other hand, those familiar with the fractional Brownian motion will identify $1/H$ as being the fractal dimension of the trail of a motion whose E coordinates are independent realizations of $B_H(t)$. This trail is drawn in an E-dimensional Euclidean space $\mathbb{R}^E$, with E satisfying $E > 1/H$.

In this case, an attempt to take an unusual path to measure "the" fractal dimension for one set actually measures the value that all paths yield for a different set.

2.4. Affine box dimensions. This section relates length measurements to a line of thought started in Section 8 of the previous chapter. In both limiting cases, $\eta \ll 1$ and $\eta \gg 1$, the number of divider steps, $L(\eta)/\eta$, is for all practical purposes the number of rectangular cells $\eta = b''^{-k}$ high and b'^{-k} wide, used to cover our fractal. Using the customary definitions of dimension, cells are squares and numbers of cells are written as functions of cell diameter. This wording can be extended to $L(\eta)/\eta$ if the diameter of a rectangular cell is taken to be its longer side. Locally, the longer side is vertical, and one finds, as in Section 2.3, that the dimension is $1/H$. Globally, the longer side is horizontal, and one finds the dimension 1.

3. Measuring the length of other self-affine curves, including Peano motion trails

The only interesting case is the counterpart of the divider walking argument in Section 2.3.

Local value. Walking a divider of length $b''^{-k} \ll 1$ will take N^k steps; hence the approximate length exponent is $\log_{b''}(b''N^{-1}) = 1 - \log_{b''}N$, and the dimension is $\log_{b''}N$. In particular, the Peano case $N = b'b'$ gives the dimension $1 + 1/H$.

Global dimension. This value is $\log_{b'}N$; in the Peano case, it is $1 + H$.

4. The concept of area of an ordinary surface and the Schwarz paradox

Triangulating ordinary surfaces is far more difficult than expected. For example, Schwarz 1890, Vol. II. p 309-311 studied the unit cylinder of radius and height 1, and found that seemingly innocuous triangulation methods can yield any value from the true value 2π to infinity!

{P.S. 2000. This section is shortened because Schwarz's triangulation is described fully in this Chapter's *Post-publication appendix.* To summarize, measured areas above 2π correspond to triangles that a) become increasingly "thin," i.e., have at least one angle that $\to 0$, and b) lie in planes that tend to become orthogonal to the cylinder. The resulting approximation is in fact increasingly "corrugated" and increasingly removed from the actual surface.}

Ordinary curves exhibit no analog to the Schwarz paradox. Self-similar fractal curves exhibit no analog, either *FGN* points out that length measurements to various precisions ε can be performed in many different ways, but all approximations grow at the same rate ε^{1-D}. But for self-affine curves, Sections 2.1 to 2.3 prove that the situation is more complex. While the length grows like ε^{1-D}, one has $D = D_{BL}$ via the Minkowski approach, but $D = D_{CL} > D_{BL}$ via the divider-walking approach.

Can D take values other than the two described above?

5. Measuring the area of self-affine fractal surfaces obtained as records of functions

5.1. Area of a fractal relief $B_H(x,y)$ via the Minkowski comforter. Using the Minkowski comforter, one falls back on the dimensions D_{BL} and D_{BG}

5.2. Area of a fractal relief via triangulation. Begin with square tiles in the (x, y) plane satisfying $\Delta x = \Delta y = 1/b$. Each cell's 4 vertices define 4 values of B_H. There are two ways of approximating a piece of the area by two "twin triangles." Take the average of these two approximations for each cell and add the averages for the b^2 cells.

Coarse triangulation. If one neglects details below the critical values $x_c = y_c$, my Brown model of the Earth's relief nearly has an area that is well-defined and not much larger than the area of its projection over an idealized plane (or sphere). This result stands in sharp contrast to all I have argued about island coastlines.

Consider in this light the two non-Gaussian landscapes in M 1982F{*FGN*}, Plate C13. They are obtained from the same Gaussian landscape by nonlinear transformations that are meant to insure that t_c is very small in the "valley" of Plate C13 top and on the "mesa" of Plate C13 bottom but is very high in the "sierra" of Plate C13 top and the "canyon" of Plate C13 bottom. Also, I used to point out in lectures that good airports runways are as rough as the Himalayas, but their vertical scale is smaller. We see now that these differences in measured vertical scale have qualitative effects. First, as suggested by the eye and "common sense," an airport does have a well-defined area for most sensible yardsticks. As to the Himalayas, the usual photographs tend to fall in the crossover region. Therefore, areas measured with different yardsticks are expected to yield a doubly logarithmic graph that is definitely *not* straight. {P.S. 1999. This paragraph was obscure in the original and was shortened.}

Fine triangulation. The area will be certainly arbitrarily large, but how rapidly will it grow with triangle "size"? Each twin triangle within a cell has sides of length $\sim b^{-Hk}$ but a height of size $\sim b^{-k}$; it is very thin, and its area is $\sim b^{-(H+1)k}$. The total number of triangles is $b^{2k} = \alpha^{-2/(H+1)}$, and the approximate area $\sim \alpha^{1-2/(H+1)}$. This is the counterpart to the relation $L(\eta) \sim \eta^{1-1/H}$ for curves, but the anomalous dimension is now $2/(H+1)$ instead of $1/H$.

The next grid to consider is itself self-affine and involves $(b'b'')^k$ rectangles b'^{-k} wide and b''^{-k} high, with $b' > b''$. Each triangle's area now becomes $\sim (b''^{-1}b'^{1-H})^k$. The anomalous dimension $\log(b'b'')/\log(b''b'^H)$ can range between $2/(H+1)$ and $1/H$, and its indeterminateness is a fractal form of the Schwarz area paradox.

Acknowledgement. This paper arose from puzzlement at the conflicting published estimates of various rough surfaces' fractal dimensions. This led me to write a section that became part of M 1985l {H21}. Penetrating comments by Michael V. Berry led me to expand the three papers in which this is the middle one.

&&&&& POST-PUBLICATION APPENDIX &&&&&

THE AREA PARADOX OF HERMANN AMANDUS SCHWARZ: ILLUSTRATION AND DETAILED ALGEBRA

The Schwarz paradox impressed me greatly when I was a student. To my disappointment, it is known to few persons today and is not found in recent mathematical textbooks in English.

A.1. The problem: what is the area of a cylinder whose basis is a circle of radius 1 and whose height is 1? "Intuition" based on past experience tells us that this area is 2π. A proof geared to this example argues that the area is unchanged if the cylinder is cut along a vertical and flattened onto a rectangle. Hence it equals the area of this rectangle, which is 2π.

Mathematicians prefer, however, not to rely on special features. The outcome was ironical: the general method yielded an absurd answer – which doomed it. (I cannot resist an aside on a mantra of mine. Many statistical techniques are billed as "objective and general," but in fact assume "mild" variability. Their use on "wildly" variable phenomena is not justified and indeed often yields absurd answers; recognition of this absurdity dooms their reputation for generality. The most treacherous cases are the wrong but not absurd ones.)

A general definition of area had been put forward in a 19th century treatise on analysis by Serret. "Intuition" tells us to triangulate any surface with triangles of well defined areas. Then one should take the limit of the total area of this polyhedral surface as the triangles become vanishingly small.

Approximating the cylinder's base by an m-sided polygon approximates the area by that of m very thin rectangles, or $2m$ very thin triangles. This triangulation yields the correct 2π for the area. But Serret observed that it remained to show that the same limit holds for all sequences of triangulating polyhedra.

Unfortunately – a shattering discovery reported by Hermann Amandus Schwarz – different triangulations can yield different results. The resulting genuine paradox deeply contributed to events that occurred around 1900 and were described as ushering the "crisis of intuition" that M 1982F{*FGN*} discusses at length. Overall, the mathematicians' response was to rely upon the driest rigor and distrust intuition. My counterresponse was to rely on improved visual understanding, because intuition can be retrained.

More precisely, the emergency police action recommended by Charles Hermite was to outlaw thin triangles. (Hermite was a great man, but always the fearful conservative. In 1893 he was to write to Stieltjes of "turning away in fear and horror from this lamentable plague of functions with no derivatives;" see *FGN*, pp 35-36 for reference and elaboration.)

More reasonably, the "Schwarz area paradox" stimulated Hermann Minkowski to devise his safe definition of length and area via the volumes of increasingly thin Minkowski "sausages" of curves and "comforters" of surfaces. These are the ε-neighborhoods made of all points within ε of a point on the curve or surface. Minkowski defines the area of an ordinary surface as $\lim_{\varepsilon \to \infty} (1/2\varepsilon)$ (volume of the ε-comforter) and for the unit cylinder he obtains as unambiguous 2π.

(By the way, this procedure continues to be very useful: the notion of "negative dimension" introduced in M 1984e is best presented by extending the idea behind Minkowski sausages; see M 1991k, 1995h.)

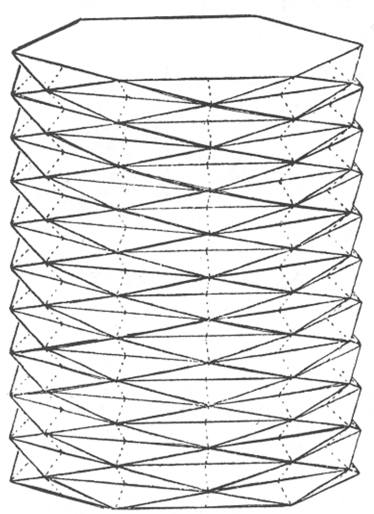

FIGURE C23-1. Original illustration of the special triangulation of the unit cylinder that led to the Schwarz paradox. Do the approximating triangles lie flat on the surface, in the limit? Not necessarily!

A.2. Schwarz's triangulation of the cylinder and calculation of the area. Figure 1 reproduces Schwarz's illustration. His argument never appeared in his own words, but as reported by Charles Hermite in his winter 1882 lectures.

To begin, we divide the cylinder height into n layers by the planes $z = p/n$ (p being an integer with $0 < p \leq n$), divide the circumferences of odd-numbered levels by the points $\theta = (2q+1)\pi/m$ (q an integer), and divide the circumferences of even-numbered levels by the points $\theta = 2q\pi/m$. Join each point (z, θ) to the points $(z \pm 1/n, \theta \pm \pi/m)$. In this fashion, the unit cylinder is triangulated by $2mn$ isosceles triangle. Their common base b and height h are given by

$$b = 2 \sin \frac{\pi}{m} \text{ and } h^2 = \frac{1}{n^2} + (1 - \cos \frac{\pi}{m})^2.$$

The second term of h^2 is all too easily missed, or even deliberately omitted as negligible. However, the exact algebra that follows shows that it may be essential. Omitting it would amount to dividing both odd- and even-numbered levels by the points $\theta = q\pi/m$, thus considering a different triangulation based on right triangles. The whole point is that distinct triangulations, like these two, may disagree, even when the curvature is as plain as that of the unit cylinder!

Each of Schwarz's triangles has the area

$$t = \frac{1}{2} bh = \sin \frac{\pi}{m} \left\{ \frac{1}{n^2} + 4 \sin^4 \frac{\pi}{2m} \right\}^{1/2}.$$

Hence the area of the polyhedron corresponding to this triangulation is

$$S(m, n) = 2mnt = 2mn \sin \frac{\pi}{m} \left\{ \frac{1}{n^2} + 4 \sin^4 \frac{\pi}{2m} \right\}^{1/2}.$$

To "true area" should be obtained when $m \to \infty$ and $n \to \infty$. Since the integers m, n are independent of each other, various constraints may be placed on them as they increase. Simple constraints involve a constant $\lambda > 0$.

Case where $n = \lambda m$, *implying* $h/b \to 2\pi/\lambda$ *as* $m \to \infty$.

$$S(m,n) = 2m^2\lambda \sin\frac{\pi}{m}\left\{\frac{1}{\lambda^2 m^2} + 4\sin^4\frac{\pi}{2m}\right\}^{1/2}$$

$$= 2m^2\lambda\ \frac{\pi}{m}\ \frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\left\{\frac{1}{\lambda^2 m^2} + 4\,\frac{\pi^4}{2^4 m^4}\left[\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right]^4\right\}^{1/2}.$$

The second term in the brace becomes indeed an asymptotically negligible correction, as compared to the first. Hence, this first estimate converges to 2π as $m \to \infty$, *independently of* λ.

Cases when $n = \lambda m^B$, with $1 < B < 2$, implying $h/b \to 0$ as $m \to \infty$. Asymptotically, again, the second term in the brace becomes a relatively negligible correction and the second estimate again converges to 2π.

Case where $n = \lambda m^2$, *implying* $h/b \to 0$ *as* $m \to \infty$.

$$S(m,n) = 2m^3\lambda\ \frac{\pi}{m}\ \frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}}\left\{\frac{1}{\lambda^2 m^4} + 4\,\frac{\pi^4}{2^4 m^4}\left[\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right]^4\right\}^{1/2}.$$

Now the so-called "correction" becomes essential because the two terms in the brace remain of comparable value in the limit. More precisely, this third estimate of the area converges to $2\pi\sqrt{1+\pi^4\lambda^2/4}$ as $m \to \infty$ and $\lambda > 0$. One can obtain *any* finite limit $> 2\pi$. (Yes!)

Case where $n = \lambda m^B$, *with* $2 < B$, *implying again* $h/b \to 0$ *as* $m \to \infty$. Now, the "small correction" becomes predominant and the approximate area may increase as a power of m, n or the triangle area $\sim 1/mn$. A cylinder can be found to mimick a fractal!

Indeed, the first term in the brace becomes relatively negligible for $m \to \infty$. Taking $B = 3$ as an example, we see that

$$S(m,n) = 2\pi\,\frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}}\left\{1 + \frac{\pi^4}{2^2}\,m^2\lambda^2\left[\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right]^4\right\}^{1/2}.$$

This last estimate converges to ∞ as $m \to \infty$.

A.3. Comments. The paradoxical conclusion is that only for $B < 2$ does the approximation $S(m, n)$ converge to the true value 2π.

Reason for the paradoxical conclusion made intuitively obvious. When $n = \lambda m$, the triangles' height-to-base ratio satisfies $h/b \to 2\pi/\lambda$. As $m \to \infty$, the triangulation involves triangles whose shape (i.e. the set of angles) tends to a limit. When $n/m \to \infty$, to the contrary, $h/b \to 0$, so that the triangulation involves increasingly elongated triangles. When the convergence of h/b to 0 is slow, its rate does not matter. To the contrary, when h/b converges rapidly to 0, the cylinder's curvature prevents the very small triangles from "lying flat." Instead, they "stand" at an increasing angle from the cylinder's surface, as revealed by the breaks in the dotted lines in Figure 1. This feature has lead to Figure 1 being called a *pine-cone (pomme de pin* in French). When $n = \lambda m^2$, the angle between the triangles and the cylinder tends to a limit that is > 0 and $< \pi/4$, and when $n = \lambda m^3$, this angle increases to $\pi/4$.

An enlightening "brain teaser" concerning the rectification of curves and the length of the unit interval. This brain-teaser dates to the 18th century and was familiar to generations of school children, at least in France. The interval $[0, 1]$ can be approximated by a saw curve made of $n = 1/b$ isosceles triangles of respective sides b (the base), $\lambda_b b$ and $\lambda_b b$, where $2\lambda_b > 1$. The natural distance between the interval and the saw approximants is the height of the teeth, namely $b(\lambda_b^2 - 1/4)^{1/2}$.

Now let $b \to 0$. If the height of the teeth also $\to 0$, the approximants converge to the interval. The total length of the saw curve is $2\lambda_b$. If $2\lambda_b$ is independent of b, the approximant length is independent of b and may take any real value > 1, in addition to the true value 1. If $\lambda_b \to \infty$ but $\lambda_b b \to 0$, converges to an interval but its length tends to ∞.

Conclusion: When curve A converges to curve B, the length of A need not converge to the length of B. However, the resemblance to the Schwarz paradox is misleading. For a curve, triangulation is replaced by rectification (meaning straightening from "rectus" which means straight and "facere" which means to do.) Rectification first approximates a curve by a broken line made of intervals of length $\leq \varepsilon$ then lets $\varepsilon \to 0$. But the interval's endpoints must remain on the curve; this is a condition our brain-teaser fails to satisfy.