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H22

Diagonally self-affine fractal cartoons. Part 1: mass, box and gap fractal dimensions, local or global

• ***Chapter foreword.*** Think of the Brownian record $B(t)$. In Wiener's original interpretation, t is time and B is a physical particle's location on a spatial axis. The two coordinates play sharply different roles, and the units of B and t (which may be cm and second) can be chosen independently. Rotation would lead to sets that are no longer records of functions, therefore, is not allowable. The expression $B(t) - \delta t$ defines a function called "Brownian motion with a drift." Here, δ is something like a velocity; $B(t) - \delta t$ mixes time and space and is conceptually a very different process from $B(t)$. The same remarks apply to the original interpretation of $B(t)$: in Bachelier 1900, t is time and B is price in francs. I have also interpreted $B(t)$ as the vertical section of a Brown landscape M 1982F {*FGN*}, Chapter 28 and also Chapter H20. There, the coordinates play different roles because gravity defines the vertical direction, makes overhangs an exception and makes it useful to represent the relief by a (single-valued) function. However, both B and t are lengths in this example, and their units can no longer be chosen independently.

Some "cartoons" previewed in Chapter 21 are studied in depth in Chapters 22, 23 and 24. Each began as a part of M 1986t. The titles have been made more descriptive and extensive forewords and/or appendices have been added.

This chapter reprints the bulk of Part I of M 1986t. The texts of Sections 1 and 2 being largely superseded by the detailed discussion of "tile self-affinity" in Chapter ⋆H2, large parts were deleted. When the remainder was copy-edited, many small gaps were filled.

After this chapter's abstract and before proceeding with the text, it may be helpful to skim the Abstracts of the two chapters that follow.

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✦ **Abstract.** This and the next two chapters consider certain diagonally self-affine fractals obtained by a recursive cascade (in most cases, a non-random one). It is shown that, in contrast to the unique fractal dimension of strictly self-similar sets, several distinct dimensions are needed.

This chapter begins with the dimensions defined via the mass in a sphere and via the covering by uniform boxes. It is shown that either case introduces two sharply distinct dimensions: a local one, valid on scales well below the set's crossover scale, and a global one, valid on scales well above the crossover. For self-affine sets that present gaps, this paper also tackles gap dimension. ✦

THE FRACTAL CONSTRUCTIONS AND PROCEDURES borrowed from mathematics involve infinite interpolation. Physical interpolation, on the contrary, cannot proceed without end, and constructions tend to proceed by extrapolation. Around 1980, when fractals were first used in physics, this contrast puzzled many physicists.

In the self-similar case, however, both the mathematicians and the physicists were pleasantly surprised to find that this contrast did not matter: the mathematicians' infinitesimal techniques lead to power laws that hold uniformly at all scales.

In the self-affine case, the situation has proven to be altogether different. Conceptually distinct scaling exponents that in the self-similar case reduce to a single number called the fractal dimension must now be replaced by multiple and numerically distinct quantities. Therefore, the "all-purpose" or "generic" notion of fractal dimension subdivides into many distinct "special purpose" or "specific notions."

1. INTRODUCTION

The gap dimension D_G (Section 4) holds uniformly for all scales from 0 to ∞ and the Hausdorff-Besicovitch dimension D_{HB} is a local concept. But many other fractal dimensions come in two flavors. The local flavor concerns small scales. The global flavor concerns large scales. Of particular significance are the global mass dimension D_{MG} and the local box dimension D_{BL}.

2. THE NOTIONS OF DIAGONAL SELF-AFFINITY

2.1. Diagonal affinities

A special role is played by affinities whose invariant set is made of straight lines that are parallel to the coordinate axes. Such an affinity, which I propose to call *diagonal*, operates in the E-dimensional affine space A^E. Each member of a collection of affinities is specified by giving a fixed point of coordinates q_m for $0 < m < E-1$, and an array of reduction ratios r_m for $0 < m < E-1$, and by considering the map

$$x_m \to q_m + r_m(x_m - q_m).$$

The ratios r_m need not be positive. And they must not be identical because otherwise the transformation would fail to be a similitude. In the simplest recursive the "bases" $1/|r_m| = b_m$, are example integers.

Most of the examples will be sets in the affine plane A^2 (i.e., $E=2$). We shall write $b' = \max b_m$, $b'' = \min b_m$ and $H = \log b''/\log b'$. The quantity H, called the affinity exponent, will satisfy $0 < H < 1$. When $E > 2$, there are $E(E-1)/2$ affinity exponents and crossover scales.

Formally, a linear transformation is the sequence of a translation and a multiplication by a matrix. We will only tackle the cases where the matrix is diagonal and its diagonal terms are not identical. The product of two diagonal affinities is a diagonal affinity. Therefore, a collection of diagonal affinities can be used as the basis for a group.

The issues to be addressed involve the meaning of "square," "distance," and "circle" in affine geometry. These notions remain meaningful for relief cross-sections, but for records of noise or of price, the units along the t axis and along the B axis are set up independently of each other. Since there is no meaning to equal height and width, a square cannot be defined. Similarly, a circle cannot be defined, because its square radius $R^2 = \Delta t^2 + \Delta B^2$ would have to combine the units along both axes. Furthermore, one cannot "walk a divider" along a self-affine noise record, to measure its approximate length, because the distance covered by each step combines a Δt and a ΔB. On the other hand, a noise record is always represented on the same graph paper as is used for a relief section or an isotropic set. This does *not* cause the distinction between the affine A^E and the Euclidean R^E to disappear, but sometimes this distinction is elusive, and one is tempted to evaluate various "prohibited" dimensions "mechanically." One should not.

2.3. The principle of recursive constructions in a grid easily extends from self-similar to diagonally self-affine fracals

To generalize the Sierpinski carpet, take the semi-open unit square as initiator. ("Semi-open" means that the top and the right sides are open and the bottom and left sides are closed. The rectangles to be considered will also be semi-open). As generator, take the array in Figure 1.

Divide the initiator into 3.4 = 12 subrectangular parts, and erase the middle two parts, shown in black. Then erase the middle two of the 12 sub-subrectangular parts, etc... The resulting self-affine carpet is the union of $N=10$ "tenths." Each tenth is obtained from the whole by a diagonal affinity with $r'_n=1/3, r''_n=1/4$ for $n=1$ to $n=N=10$. The signs of the reduction ratios can be illustrated by placing arrows along the diagonals of the ten rectangles. In the present example, arrows must be placed as marked to insure that the "tenths" of this carpet do not overlap. The fixed points are the four vertices, the midpoints of the left and right sides, and the points 1/3 and 2/3 along the top and bottom sides. Indeed, an affinity's fixed point is the point of intersection of the four straight lines that join the vertices of the whole to these vertices' transforms, each of which is the vertex of the part.

A general fractal generator in a self-affine lattice is obtained by drawing $b' \times b''$ subrectangles, and keeping $N < b'b''$ of them. Again, $|r'_n| = 1/b'$ and $|r''_n| = 1/b''$ for all n. The orientation of the n-th affinity expressed by the signs of the r'_n and r''_n may depend on n. And I propose that it be represented by a diagonally placed vector. The two variants shown in Figures 2 and 3 play especially important roles. Surprisingly, Chapter 24 will show that D_{HB} depends on which variant is chosen!

When both the rectangles and their diagonals are kept, the resulting fractal is obtained as a limit of nested collections of rectangles, "nested" meaning that each is contained in the preceding one. When only the diagonals are kept, and form a curve, a self-affine fractal curve is obtained as a limit of broken lines. Connectedness of the stick generator imposes a constraint on the retained subrectangles.

As in *FGN*, Chapter 13, the stick generator may split into several curves, creating "islands" and/or "lakes."

Important special case: When a stick-generated fractal curve is the record of a (one-valued) continuous function, one has $N = b'$.

3. THE SIMILARITY DIMENSION HAS NO MEANING FOR SELF-AFFINE SETS

The "(self-)similarity dimension" D_S is a notion specifically addressed to self-similar fractal sets, which are made of N parts, each obtained from the whole by a similitude of ratio r_n. When all reduction ratios are identical, with $|r_n| = 1/b$, it is well-known that $D_S = \log N/\log b$. But for self-affine fractal sets, the base b is replaced by two bases b' and b'', or many bases b_m. In all cases, and the similarity dimension $\log N/\log b$ is meaningless under self-affinity.

It is tempting to "save it" formally by replacing b by some suitably "effective base" $\tilde{b}$ and then attempting to find a useful interpretation for $\tilde{D}_G = \log N/\log \tilde{b}$. Taking for $\tilde{b}$ the geometric mean of the b_m, namely $\tilde{b} = (b_1 b_2, \ldots b_E)^{1/E}$, Section 4 will show that D_G is indeed a dimension in the cases where one can define either gaps or islands.

There are other purely formal generalizations of D_S to the self-affine case; most prove unjustifiable. For example, it is widely known {P.S. 2000; see Chapter ⋆H3} that the Hausdorff-Besicovitch dimension D_{HB} of the Brown record is 3/2. When this D_{HB} is written as 1 + 1/2, its value happens to coincide numerically with $1 + \log_{b'} b''$, where b' is the larger and b'' is the smaller base. This has led to the suggestion that the similarity dimension could be generalized by the expression $1 + \log_{b'} b''$. In fact, as already recalled, the fractional Brownian motion $B_H(t)$ yields the completely different value $D_{HB} = 2 - \log_{b'} b''$.

Other guesses are less obviously absurd but no more justifiable.

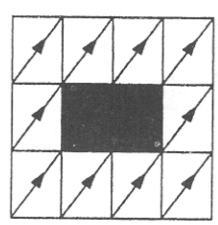

FIGURE C22-1.

4. THE GAP DIMENSION IS GIVEN BY $D_G = E \log N / \log(b_1 b_2 \ldots b_E)$

We begin with a fractal dimension that is simple but of narrow validity and interest. The formula for D_G is invariant under the interchange of b' and b''. We shall see that the other and more important D's are *not* symmetric.

The notion of a gap dimension applies to the self-similar fractals in $\mathbb{R}^E$ exemplified by the Cantor dust on the line, and by the Sierpiński gasket and carpet in the plane. These shapes have the following two properties. Their E-dimensional measure vanishes ("fat fractals" – the topic, without the name, of M 1982F{*FGN*}, Chapter 15 – are not considered here). And their complement splits into an infinity of *gaps* (maximal connected open sets) which are domains in $\mathbb{R}^E$, similar to each other and differing solely by their linear scale. In all these cases, the following relation is known to hold for all L:

$$\{\text{number of gaps of linear scale } > L\} \propto L^{-D_G} \text{ with } D_G < E.$$

The exponent D_G is called the gap dimension, and all other definitions of the fractal dimension of a self-similar fractal give the same value. Now consider self-affine fractals that have gaps. There is some good news to report, and some bad news.

The good news is that it is still true that the number of gaps scales like N^k and the volume scales like $b_1 b_2 \ldots b_E$. Define *linear size* as the $(1/E)$-th power of volume, i.e., as the geometric mean of the sides. With this definition, the number-size relation for gaps or islands continues to be a power law valid for all sizes L. The exponent independent of L can continue to be called a gap dimension; it is the D_G defined in Section 3.

The bad news is that D_G bears no direct relation to D_{HB}. For example, consider the generator in Figure 4. Here, $D_G = \log 8 / \log \sqrt{24} = 1.30$. But the Chapter after next will show that $D_{HB} = 1.34$, and we shall soon see that the basic fractal dimensions are $D_{BL} = 1.38$ and $D_{MG} = 1.20$.

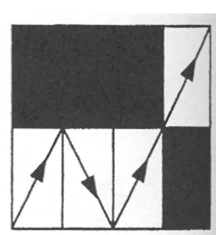

FIGURE C22-2.

Let us digress to consider the special case of thin fractal dusts (dusts of Lebesgue measure 0) on the line, such as the Cantor dust. For these dusts, the gap dimension is a stronger form (valid for all scales) of an exponent that Besicovitch and Taylor introduced for small scales and showed to be identical to D_{HB} (M 1982F{*FGN*}, p. 359). The generalization of the identity $D_{BT} = D_{HB}$ to $E > 1$ is known to be correct in some self-similar cases (Sierpiński carpet; *FGN*, p. 134). But we now see that in the self-affine cases, D_G stands alone.

5. DIAGONALLY SELF-AFFINE PLANAR FRACTAL CURVES DEFINED AS GRAPHS OR RECORDS OF FUNCTIONS

The letter H for $\log_{b'} b''$ is appropriate in the context of this section because H is the "Hölder exponent."

5.1. The local mass dimension for small R is $2 - H$; the global mass dimension for large R is 1

When a set S is a self-similar fractal, the mass $M(R)$ contained in the intersection of S with a disc (or ball) of radius R behaves like $M(R) \propto R^{D_M}$. One can also replace the disc or ball by a square or cube whose sides are parallel to the axes, and of length $2R$. For some physicists, the relation $M(R) \propto R^{D_M}$ has become almost a definition of the notion of fractal. But we shall now show that *this scaling rule fails to generalize to self-affine fractals.* For them, the single D_M splits into a global D_{MG} and a local D_{ML}.

Large radii, satisfying $R \gg t_c$. The physicists do not usually think of fractals as objects that one can interpolate without limit. They think primarily of objects that one can extrapolate to long radii, of log-log plots of mass versus radius that remain linear as $R \to \infty$ and of a mass dimension defined through $D_M = \lim_{R \to \infty} \log M(R) / \log R$. This is sufficient motivation to evaluate this limit for a record of *any* diagonally self-affine function, such as a fractional Brownian motion $B_H(t)$ or a recursively defined function. When $R \gg t_c$, any one of these records is effectively a horizontal

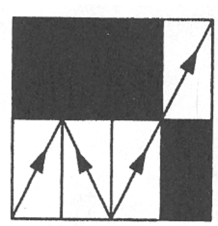

FIGURE C22-3.

interval. Within a square of side $2R$, it occupies a very thin horizontal slice. Therefore, if we follow Section 2.5 and weigh our record proportionately to the time elapsed, we find that $M(R) \propto R$. That is, we obtain the striking result that $D_{MG} = 1$.

Small radii, satisfying $R \ll t_c$. The mathematicians, contrary to the physicists, are primarily concerned with the case $R \ll t_c$. In that range, a self-affine function is effectively a collection of vertical intervals, one for each zero. Some algebra yields for the local mass dimension the value $D_{ML} = 2 - H$, which is the familiar value of D_{HB} for the records of $B_H(t)$.

Conclusion. We discover that two limits that are identical for self-similar fractals can differ for self-affine fractals. Since $1 < 2 - H$ when $0 < H < 1$, one has $D_{ML} > D_{MG}$. The rest of the paper extends this discovery.

5.2. The box dimension's local value is $2 - H$; its global value is 1

Cover the unit square by a lattice consisting of boxes of side $r = 1/b$, and let $N(b)$ denote the number of lattice boxes that intersect the set. Self-similar sets satisfy $N(b) \propto b^{D_B}$ and D_B is the "box dimension," an abbreviation for "box-counting dimension".

Again, this scaling rule fails to generalize to self-affine fractals. For them, the single D_B splits into a global D_{BG} and a local D_{BL}.

Small boxes. The mathematician is only interested in local behavior and interprets the exponent as $\lim_{b \to \infty} \log N(b) / \log b = D_{BL}$. In the case of recursively defined function records, cover a piece $b'^{-k} = b^{-1}$ wide and b''^{-1} high using boxes of side b^{-k}. Clearly, these boxes must be piled into vertical stacks. The number of boxes in each stack is $(b''/b')^{-k} = b^{1-H}$, and the number of stacks is b. Hence, the exponent $D_{BL} = 2 - H$. Similarly, in the case of $B_H(t)$, the heuristic box argument given in *FGN* (bottom left of page 237) yields $D_{BL} = 2 - H$.

Large boxes. The physicist, however, may also want to consider the global limit when $r \to \infty$, leading to an unbounded record. The portion of a self-affine record from 0 to $t \gg 1$ is covered by a single box. Hence,

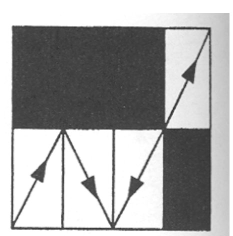

FIGURE C22-4.

$$\lim_{b \to 0} \log N(b) / \log b = D_{BG} = 1.$$

5.3. The dimensions of cuts of the graph of $B_H(t)$; the skew cuts are bounded fractals that are only locally self-similar

As a rule (*FGN*, p. 135), when a self-similar fractal in the plane is cut by a line, the fractal dimension decreases by one. This rule is fundamental, but has numerous exceptions. To apply it to the record of $B_H(t)$, which is not self-similar, care is necessary and well rewarded. We assume that $B_H(t)$ has the intrinsic scale $t_c = 1$.

Horizontal cuts. These cuts are self-similar dusts, meaning that their local dimension also applies globally. The dimension is $1 - H = (2 - H) - 1$. Thus, it is obtained from D_{BL} by subtracting 1, and the fact that $D_{BG} = 1$ does not matter.

Vertical cuts. These cuts reduce to a single point, whose dimension is 0. This case until now was considered to be an isolated exception to the rule concerning the dimension of intersections.

Skew cuts by $Y = \sigma t$, with $0 < \sigma < \infty$. When a cell of size $b'^{-k} \times b''^{-k}$ is upsized to a unit square, $Y = \sigma t$ is replaced by $Y = \sigma (b''/b')^k t$. Thus, $Y = \sigma t$ is locally indistinguishable from $Y = 0$. The skew cut by $Y = \sigma t$ is locally self-similar; indeed locally identical to the horizontal cut.

But this does not close the problem. The study of self-affinity has sensitized us to also check the global properties and the intrinsic scale.

Globally, a skew cut is *not* self-similar. In fact, it is bounded; therefore $D_{MG} = D_{BG} = 0$. This is confirmed by observing that, globally, $Y = \sigma t$ scales up to a vertical line. Indeed, $D_{MG} = D_{BG} = 0$ are the values obtained from the basic rule if it is applied to the original curve's dimensions $D_{MG} = D_{BG} = 1$.

I confess having been, until now, insensitive to the special status of bounded, locally self-similar fractals. They are extremely common in nature, the best examples being individual island coastlines and diffusion limited aggregates.

For a bounded dust, an intrinsic scale is the length of the smallest interval that contains it. As $\sigma \to \infty$, this scale approaches zero, which is why a vertical cut reduces to one point and why the only factor that matters in the limit is the global dimension, which is 0. In other words, vertical cuts have now ceased to be exceptions. In addition, as $\sigma \to 0$, this intrinsic scale approaches infinity, which is why the horizontal cut is unbounded self-similar. This explains why horizontal cuts are not at all affected by global quantities.

6. SELF-AFFINE RECURSIVE PLANAR FRACTALS FOR WHICH BOTH AXIAL PROJECTIONS ARE LINE-FILLING

6.1. The projections of a self-affine recursive fractal on the axes

When a projection of the generator fills the corresponding side of the initiator, the projection of the limit fractal also fills that interval. When the projection of the generator fails to fill the side of the initiator, the projection of the limit fractal is a dust that leaves uncovered gaps.

6.2. The global mass dimension is $D_{MG} = 1 - 1/H + \log_{b''} N = \log_{b''}(Nb'/b'')$ and the local mass dimension is $1 - H + \log_{b'} N = \log_{b'}(Nb''/b')$

Once again, we begin with the notion of greatest interest to physics: the extrapolative global mass dimension. We use a unit square as the initiator and attach a unit mass to it. The extrapolation is uniquely specified in a sequence of increasing boxes of mass N^k and area $(b'b'')^k$. This seems to suggest that the gap dimension $D_G = \log N / \log \tilde{b}$ is a mass dimension; but this situation is more complex, because the mass-radius relation requires the mass to be evaluated within square boxes and not within specially adapted rectangular boxes. A square box of side b''^k, if chosen at random within a rectangular box of sides b'^k and b''^k, contains on the average the mass $N^k(b''/b')^k$. Hence, the surprising new result

$$D_{MG} = \log(b''N/b')^k / \log b''^k = 1 + \log_{b''}(N/b') = 1 - 1/H + \log_{b''} N$$

In the case of function records (Section 5) $N = b'$, and hence this D_{MG} duly yields the already known value $D_{MG} = 1$.

A similar argument applied to local behavior yields

$$D_{ML} = 1 - H + \log_{b'} N.$$

Again, in the case of function records, $N = b'$, hence D_{MG} yields the familiar value $2 - H$.

The above formulas could not have been guessed. A first striking feature is that both are asymmetric functions of b' and b''. A second feature is that they are symmetric of each other: interchanging b' and b'' replaces H by $1/H$ and interchanges D_{MG} and D_{ML}.

6.3. The global (resp., local) box and mass dimensions

The formula for D_{BL} was obtained (implicitly) long ago, in Kline 1945, in a case whose degree of generality lies between those of Sections 5 and 6.

6.4. One has $D_{MG} = D_{BG} < D_G < D_{BL} = D_{ML}$

Proof: $D_{MG} < D_{BL}$ follows from $b' > b''$. Hence,

$$\log(Nb'/b'')/\log b' < \log(Nb''/b'')/\log b''.$$

Averaging the numerators and the denominators separately yields the quantity $\log N/\log\sqrt{b'\,b''} = D_G$, which is strictly contained between D_{MG} and D_{BL}.

6.5. Generalizations to $E > 2$

In the affine space A^E, with $E > 2$, the crossover scales may be widely scattered. In addition to global and local exponents, several other exponents may be needed.

7. SELF-AFFINE PLANAR RECURSIVE FRACTALS FOR WHICH AT LEAST ONE OF THE PROJECTIONS IS A CANTOR DUST

7.1. Definitions and example

This section's title restricts attention to the case where (a) $r'_n > 0$ and $r''_n > 0$ for all the intervals of the generator and (b) the X- and/or Y-projections of the limit fractal are Cantor dusts made, respectively, of N' parts with $r' = 1/b'$ and of N'' parts with $r'' = 1/b''$.

For example, use the generator in Figure 5. Globally, the resulting fractal is the dust obtained by "mid-thirds removal." The global dimensions are, therefore, $\log_3 2$. Locally, the resulting fractal is the Devil's staircase minus its flat steps. The local dimensions are known to be 1.

7.2. The global mass dimension is $D_{MG} = (1 - 1H) \log_{b'} N' + \log_{b''} N$. The local mass dimension is $D_{ML} = (1 - H) \log_{b''} N'' + \log_{b'} N$.

The argument concerning D_{MG} follows exactly the argument in Section 6.2, until the point where the average mass in a box of side b''^k is evaluated. This mass is now allowed to vanish. But massless squares cannot be part of our hierarchy of boxes. Therefore, it is necessary to exclude the massless boxes and to take a conditional average mass, which is *larger* than the average mass. Observe that $b'^{kH} = b''^k$. The strip of width b'^k and height b''^k decomposes into $(b'/b'')^k$ boxes of side b''^k, of which $N'^{k(1-H)}$ boxes are not empty. Hence, the conditional average mass is $N^k N'^{1-k(1-H)}$, and the global mass dimension is given by

$$D_{MG} = -(1 - H) \log_{b''} N' + \log_{b''} N = (1 - 1/H) \log_{b'} N' + \log_{b''} N.$$

A similar argument applied to local behavior yields

$$D_{ML} = (1 - H) \log_{b''} N'' + \log_{b'} N = (1/H - 1) \log_{b'} N'' + \log_{b'} N.$$

The first (respectively, second) expression for D_{ML} is symmetric of the second (respectively, first) expression for D_{MG}.

7.3. One has $D_{ML} > D_{MG}$, with equality, if and only if, $N < N'N''$

Writing $1/\log b'' - 1/\log b' = F$, we obtain

$$\begin{aligned} D_{MG} - D_{ML} &= \log_{b''} N - (\log_{b'} N' - \log_{b''} N') - \log_{b'} N - (\log_{b'} N'' - \log_{b''} N'') \\ &= F(\log N - \log N' - \log N'') \\ &= F \log(N/N'N''). \end{aligned}$$

Since $F > 0$ and $N < N'N''$, we have $D_{MG} \leq D_{ML}$.

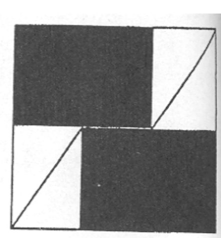

FIGURE C22-5.

Equality prevails only if $N = N'N''$, in which case our planar Cantor dust is the Cartesian product of two linear Cantor dusts. A linear Cantor dust is self-similar; thus the simplicity of the self-similar situation also exists in the self-affine case, where it is obtained as a Cartesian product.

7.4. The global and local box dimensions take the same values as the mass dimensions

This is obvious by inspection.

8. SELF-AFFINE SURFACES

This section comments briefly on two functions: $Z(x,y)$, where the (x,y) plane is isotropic and the scaling parameter is H, and $T(x,y,z)$, where the (x,y,z) space is itself affine and the scaling parameters are G and H.

8.1. Graphs of fractal functions of a point in an isotropic plane and models of relief

The simplest model of the Earth's relief (*FGN*, Chapter 28) is a fractional Brownian surface $B_H(x,y)$, where the point with coordinates x and y is in an isotropic plane. All characteristics of this surface depend on the single parameter H. It is easy to see that $D_{BG} = D_{MG} = 2$, while $D_{BL} = D_{ML} = 3 - H$. Also, $D_{HB} = 3 - H$.

Dimensions' behavior under vertical or horizontal plane sections. Recall the rule that, when a fractal is intersected by a plane, the dimension decreases by 1. This rule is fundamental but seems to have many exceptions; let us show how some of these exceptions may be eliminated.

The *vertical* sections of $B_H(x,y)$ have both local and global properties, and the rule applies to both D_{BL} and D_{BG}, without any problem.

The *horizontal* sections are the coastlines of all the islands taken together. They are self-similar and have only one dimension, which coincides with the local dimension of vertical sections. Horizontal sections have an infinite intrinsic scale.

Thus, a dimension that yields much information about the horizontal sections tells only half of the story about the vertical sections.

Dimensions' behavior under skew plane sections. As previously discussed for the skew lines in Section 5.3, a skew plane $Z = \sigma x$ downsizes locally to a horizontal plane, and upsizes globally to a vertical plane. Both the local and the global dimensions are decreased by 1. {PS 1999: A characteristic feature of these "coastlines" is that, for $\sigma > 0$, the left-hand half-plane is mostly ocean and the right-hand half-plane is mostly land; small details (including bays, capes, islands and deadvalleys) are nearly isotropic.}

8.2. Fractal functions of a variable in an isotropic plane: clouds and rain

My fractal model of coastlines was extended in Lovejoy 1982 to cloud boundaries of vertical projection on the Earth's surface. This in turn led Lovejoy & M 1985 to a two-dimensional model of rain areas or clouds. It is based on "fractal sums of pulses" (FSP), a self-explanatory new term for a family of self-affine surfaces that I had previously introduced for other purposes. In the FSP model, some quantity (like temperature, opaqueness or rain intensity) is ruled by a self-affine function $Z_H(x,y)$, where the (x,y) plane is isotropic. One can apply $B_H(x,y)$ to a relief because a mountain's altitude is mostly a continuous function, but rainfall intensities are sharply discontinuous in time and space. In the simplest case, only the parameter H is necessary to fully characterize the function.

8.3. Fractal functions of a variable in an affine plane: clouds

Large clouds of many kinds are like pancakes parallel to the Earth's surface. The conventional argument given by meteorologists for this common appearance is that the atmosphere is three-dimensional on small scales and two-dimensional on large scales, with a crossover scale in-between. In a counter-argument, Schertzer & Lovejoy 1986 argue from available empirical evidence that the atmosphere itself is self-similar in x and y but is self-affine in x (or y) and z.

I think this is an excellent proposal, and I like the way D. Schertzer and S. Lovejoy adapt various models of mine to make them self-affine, or more completely self-affine. The dimensional properties of the corresponding fractals are therefore worthy of exploration. Unfortunately, Schertzer & Lovejoy 1986 gives numbers with little or no motivation, and shows no awareness of the interesting complications that the topic presents.

With little additional complication, one can immediately consider self-affine functions $T(x,y,z)$, where the horizontal variables (x, y, z) are isotropic. The basic self-affinity property is invariance under a map whose diagonal terms can be written as r, r^G, and r^{GH} with $G < 1$. In addition, using the awkward but self-explanatory notation of op. cit., one has

$\delta T(\delta x) \propto (\delta H)^{GH}$, $\delta T(\delta y) \propto (\delta y)^{GH}$, and $\delta T(\delta z) \propto (\delta z)^{H}$; hence, $H<1$. It is easy to find that for the record of T, $D_{MG} = D_{BG} = 3$ irrespectively of H and G. However, other dimensions of the record, and the dimensions of other objects related to T, usually depend on the object itself and on H and/or G.

For example, consider D_{BL}. When $\delta x = \delta y = \delta z \ll 1$, $\delta T(\delta x)$ is dominated by $\delta T(\delta z) \propto (\delta z)^{GH}$. Therefore, covering the record of T by boxes of side $\delta x = \delta y = \delta z$ requires $(\delta x)^{-3}$ stacks with $\propto (\delta z)^{GH-1}$ boxes in each stack. Conclusion: $D_{BL} = 4 - GH$.

8.4. Coverings by rectangles and the notion of "elliptical dimension"

In the case $E=3$, $b_1 = b_2$ and $b_3 = b_1^H$, Schertzer & Lovejoy 1986 gives prominence to the quantity $D_{el} = 2 + H$, which it calls the "elliptical dimension of space." The motivation is that in the isotropic 3-dimensional case, $D_{el} = 3$, and in the isotropic 2-dimensional case, $D_{el} = 2$. Schertzer & Lovejoy 1986 then reports that "it is therefore natural to regard $2+H$ as the fractal dimension of this" self-affine space.

This attractive motivation, however, does not suffice. Moreover, it is weakened by the second supporting argument, which notes that the case $E=2$ and $b_2 = b_1^H$ with $H = 1/2$ yields "$D_{el} = 1.5$, which is the same as the fractal dimension suggested" for the record of $B(t)$. In fact, we know that for $B_H(t)$ one has $D_{HB} = 2 - H$ rather than $1 + H$; these two formulas take the same value for $H = 1/2$ because of a numerical coincidence.

The other supporting argument is that "the number of eddies of horizontal scale λ may be written as λ^{-D} with $D = D_{el}$. However, the vertical scale would have given $D = 1 + 2/H$; why choose the horizontal scale?

A search for a clearcut interpretation of $2+H$ as a dimension has involved private conversations with J. P. Kahane and J. Peyrière, who suggest investigating "intrinsic" coverings that use not cubes, but affine rectangles b''^{-k} high and b'^{-k} wide, the "radius" of a rectangle being its longer side. Local Hausdorff-like dimensions of this kind are discussed by Peyrière 1986b.

The concrete physical meaning of covering by rectangles is still unproven. Its adoption would of course involve additional local and global dimensions, many of which are found to take on very questionable values. For example, the mass dimensions in the case studied in Section 6 become $\log_{b'} N$ globally and $\log_{b''} N$ locally. Both values yield a very biased and incomplete view of these fractals' structure. Specifically, take a self-affine Sierpiński carpet with $b' = 9$ and $b'' = 3$ and one big gap, leaving a generator with $N=20$. Its global mean dimension based on intrinsic rectangles is $\log_{b'} N=1.36$. The same values continue to apply if b'' is replaced by any integer from 3 to 9 (inclusive). The carpets obtained in this fashion differ greatly from one another, except from this peculiar viewpoint.

To put this dimension in perspective, note that $D_{ML}=D_{BL}=\log_{b'} N+1-H=1.86$ and $D_{MG}=D_{BG}=\log_{b''} N+1/H=1.72$, but $D_G=1.81$. Nevertheless, Schertzer and Lovejoy 1986 give $\log_{b'} N$ (without explanation) as the only fractal dimension of this carpet.

In the limit case $N=b'b''$, the main dimensions based on rectangles simplify to $1+2/H$ locally and $2+H$ globally. In retrospect, this last value might provide an element of motivation for the "elliptic dimension." But this motivation is not an improvement of the motivation for the "twin" value $1+2/H$, which incidentally has a most undesirable feature: it is unbounded.

Observe also that, to verify that this global $2+H$ and this local $1+2/H$ are useful dimensions, one must know in advance which boxes should be used in the covering, which requires advance knowledge of H. Every dimension based on the common square boxes can be measured by direct algorithms. The expressions $1+2/H$ and $2+H$ involve an indirect measurement of H as the ratio of the measurable quantities GH and G. A less contrived motivation would be welcome.

For coverings by rectangles, see also Section 2.5 of Chapter E23.

Acknowledgement: I undertook this work while teaching fractal geometry at Harvard in the spring of 1985. The usual continuous differentiable functions are self-affine, as already defined in M 1977F, but I found that little was known about this notion. Its investigation eventually grew to explain the strange results or theoretical difficulties encountered in the study of self-affine models of surfaces (next chapter), and of clouds (Sections 8.3 and 8.4). Sections 5.1 and 5.2 (carried over from M 1985s{N21}) have greatly benefited from discussions with Richard F. Voss.