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Fractional Brownian motions, fractional noises and applications (M & Van Ness 1968)

THE TERM "FRACTIONAL BROWNIAN MOTIONS" and the abbreviation FBMs will be used to denote a family of Gaussian random functions defined as follows. Let $B(t)$ be ordinary Brownian motion, and H be a parameter satisfying $0 < H < 1$. Then the FBM of the exponent H is a moving average of $dB(t)$ in which past increments of $B(t)$ are weighted by the kernel $(t-s)^{H-1/2}$. We believe that FBMs provide useful models for a host of natural time series and that their curious properties deserve to be presented to scientists, engineers and statisticians.

1. INTRODUCTION

The basic feature of FBMs is that the "span of interdependence" between their increments can be said to be infinite. In contrast, the study of random functions has been overwhelmingly devoted to sequences of independent random variables, Markov processes and other random functions having the property that sufficiently distant samples of these functions are independent or nearly independent. On the contrary, empirical studies of random chance phenomena often suggest a strong interdependence between distant samples.

One class of examples arises in economics. It is known that economic time series "typically" exhibit cycles of all orders of magnitude; the slowest cycles have periods of duration comparable to the total sample size. The sample spectra of such series show no sharp "pure period" but a spectral density with a sharp peak near frequencies close to the inverse of the sample size.

Another class of examples arises in the study of fluctuations in solids. Many such fluctuations are called "$1/f$ noises" because their sample spectral density takes the form λ^{1-2H}, where λ is the frequency and H is a number always satisfying $1/2 < H < 1$, and often close to 1. However, since values of H far from 1 are also frequently observed, the term "$1/f$ noise" is inaccurate. It is also unwieldy. With some trepidation, due to the availability of several alternative expressions, we take this opportunity to propose that "$1/f$ noises" be relabeled *fractional noises* (see M 1967i).

A third class of phenomena with extremely long interdependence is encountered in hydrology: Hurst 1951, 1956 found that the range (to be defined below) of cumulated water flows varies proportionately to d^H with $1/2 < H < 1$. This fact will be linked in this paper to the presence of an infinite span of interdependence between successive water flows. Hurst's law is likely to acquire significant practical importance in the design of water systems.

These and related empirical findings suggest that it is desirable to identify and to study in detail many specific, simple families of random functions that are in some way "typical" of asymptotic dependence. Since our purpose is *not* to contribute to the development of analytical techniques of probability, we selected FBM because it allows us to derive results of practical interest with a minimum of mathematical difficulty. Extensive use is made of the concept of "self-affinity," a form of invariance with respect to changes of time scale. A few self-affine processes other than FBMs are considered in passing. From the purely mathematical viewpoint, our work is, in retrospect, largely expository. While writing our paper we discovered that FBMs were already considered (implicitly) in Kolmogorov 1940, Hunt 1951, Lamperti 1962, Yaglom 1958 p. 423, Yaglom 1965 p. 262. These little known references contain a wealth of material to which the applications we listed should draw general interest.

2. DEFINITION: FRACTIONAL BROWNIAN MOTION AS MOVING AVERAGE DEFINING A FRACTIONAL INTEGRO-DIFFERENTIAL TRANSFORM OF THE WIENER BROWNIAN MOTION

As usual, t designates time ($-\infty < t < \infty$) and ω designates the set of all values of a random function (where ω belongs to a sample space Ω). The ordinary Brownian motion $B(t, \omega)$ of Bachelier, Wiener and Lévy, is a real random function with independent Gaussian increments; it is such that $B(t_2, \omega) - B(t_1, \omega)$ has mean zero and variance $|t_2 - t_1|$, and is such that, if the intervals (t_1, t_2) and (t_3, t_4) do not overlap, $B(t_2, \omega) - B(t_1, \omega)$ is independent of $B(t_4, \omega) - B(t_3, \omega)$. The fact that the standard deviation of the increment $B(t + T, \omega) - B(t, \omega)$ with $T > 0$ is equal to $T^{1/2}$ is often referred to as the "$T^{1/2}$ law."

Definition 2.1. Let $0 < H < 1$, and let b_0 be an arbitrary real number. We call the following random function $B_H(t, \omega)$ the *reduced fractional Brownian motion* with parameter H and starting value b_0 at time $t = 0$. For $t > 0$, $B_H(t, \omega)$ is defined by

$$B_H(0, \omega) = b_0$$

$$B_H(t, \omega) - B_H(0, \omega) = \frac{1}{\Gamma(H + 1/2)} \left\{ \int_{-\infty}^{0} [(t - s)^{H - 1/2} - (-s)^{H - 1/2}] dB(s, \omega) + \int_0^t (t - s)^{H - 1/2} dB(s, \omega) \right\}.$$

An analogous rule holds for $t < 0$. The integration is taken in the pointwise sense (as well as in the mean square sense), using the usual methods involving integration by parts.

Note that if $b_0 = 0$, then $B_{1/2}(t, \omega) = B(t, \omega)$. For other values of H, $B_H(t, \omega)$ is called a *fractional derivative* or a *fractional integral* of $B(t, \omega)$ (Weyl 1917).

The definition of B_H becomes easier to remember and more symmetric after it is restated as a convergent difference of divergent integrals,

$$B_H(t_2, \omega) - B_H(t_1, \omega) = \frac{1}{\Gamma(H + 1/2)} \left\{ \int_{-\infty}^{t_2} (t - s)^{H - 1/2} dB(s, \omega) - \int_{-\infty}^{t_1} (t - s)^{H - 1/2} dB(s, \omega) \right\}.$$

The FBMs corresponding to $0 < H < 1/2$, $1/2 < H < 1$, and $H = 1/2$, respectively, differ in many significant ways.

Paul Lévy 1953 (p. 357) briefly commented on a similar but better known moving average of $B(t, \omega)$, namely, the Holmgren-Riemann-Liouville fractional integral

$$B_H^0(t, \omega) = \frac{1}{\Gamma(H + 1/2)} \int_0^t (t - s)^{H - 1/2} dB(s, \omega).$$

Here, H may be any positive number for many applications. This integral gives excessive weight to the location of the origin which is why Weyl's integral was introduced (see comments in Zygmund 1959, Section XII.8).

If $B(t, \omega)$ is replaced by a complex-valued Brownian motion, the integral that defines B_H yields the complex fractional Brownian motion.

3. SELF-AFFINITY PROPERTIES OF FBM

Definition 3.1. The notation $\{X(t, \omega)\} \overset{\Delta}{=} \{Y(t, \omega)\}$ means that the two random functions $X(t, \omega)$ and $Y(t, \omega)$ have the same finite joint distribution functions (a fortiori, the same space).

Definition 3.2. The increments of a random function $X(t, \omega)$ defined for $-\infty < t < \infty$ are said to be *self-affine* with the exponent $H \geq 0$ if, for any $h > 0$ and for any t_0,

$$\{X(t_0 + \tau, \omega) - X(t_0, \omega)\} \overset{\Delta}{=} \{h^{-H}[X(t_0 + h\tau, \omega) - X(t_0, \omega)]\}.$$

The following theorem motivated the introduction of FBM.

Theorem 3.3. The increments of $B_H(t, \omega)$ are stationary and *self-affine* with parameter H.

Corollary 3.4. A T^H law for the standard deviation of B_H is stated as follows:

$$\mathscr{E}[B_H(t + T, \omega) - B_H(t, \omega)]^2 = T^{2H} V_H,$$

where

$$V_H = [\Gamma(H+1/2)]^{-2}\left\{\int_{-\infty}^{0}[(1-s)^{H-1/2}-(-s)^{H-1/2}]^2 ds + \frac{1}{2H}\right\}.$$

Definition 3.5. Let $X(t, \omega)$ be a real-valued random function. Its *cumulative range* is defined to be

$$M(t, T, \omega) = \sup_{t \le s \le t+T}[X(s, \omega) - X(t, \omega)] - \inf_{t \le s \le t+T}[X(s, \omega) - X(t, \omega)].$$

Define $M(T, \omega)$ as $M(0, T, \omega)$. If $X(t, \omega)$ has continuous sample paths (as B_H does by Proposition 4.1) and t and T are finite, one can replace sup by max and inf by min.

Corollary 3.6. A T^H law for the sequential range of a process of *self-affine* increments is stated as follows: if $X(t, \omega)$ has *self-affine* increments with parameter H, then

$$M(T, \omega) \triangleq T^H M(1, \omega).$$

For example, if $X(t, \omega) = B(t, \omega)$, then $T^{-1/2}M(t, T, \omega)$ has a distribution independent of both t and T, which is described in Feller 1951.

Remark. Definition 3.1 means that, when $t \ge t_0$, $X(t, \omega) - X(t_0, \omega)$ is a "semistable stochastic process" in the sense of Lamperti 1962. Semistability is a weaker property than self-affinity of the increments. For example, Lévy's Riemann-Liouville fractional integral of $B(t, \omega)$ is semistable for all $H > 0$. {P.S. 1999. But it is not self-affine.}

If $X(t, \omega)$ is semistable with parameter H and has stationary increments, then $X(t, \omega)$ is the restriction to $t \ge 0$ of a process with self-affine increments with parameter H.

One might think that definition 3.2 could be generalized by replacing the h^{-H} in Definition 3.1. by a more general factor $A(h)$. However, $A(h)$ must satisfy $A(h'h'') = A(h')A(h'')$. If $A(h)$ is measurable, or under suitable other conditions, one must have $A(h) = h^{-H}$, as we have postulated.

3.1. Some partial converses

Proposition 3.7. If $X(t, \omega)$ has self-affine and stationary increments and is mean square continuous, then $0 \le H < 1$.

Proof. By Minkowski's inequality, for any τ_1 and $\tau_2 > 0$,

$$\left\{\mathscr{E}\left[X(t+\tau_1+\tau_2)-X(t)\right]^2\right\}^{1/2} \leq \left\{\mathscr{E}\left[X(t+\tau_1+\tau_2)-X(t+\tau_1)\right]^2\right\}^{1/2} + \{\mathscr{E}\left[X(t+\tau_1)-X(t)\right]^2\}^{1/2}.$$

By hypothesis, there is a constant V such that

$$\mathscr{E}\left[X(t+\tau,\omega)-X(t,\omega)\right]^2 = V\tau^{2H}.$$

Therefore,

$$V^{1/2}[\tau_1+\tau_2]^H \leq V^{1/2}[\tau_1^H+\tau_2^H],$$

which implies that $H<1$. Mean square continuity requires that $H \geq 0$.

Proposition 3.8. If $X(t,\omega)$ is a Gaussian random function satisfying the conditions of Proposition 3.7. and is not constant, then it must be FBM.

Proof. A Gaussian process is determined by its covariance and mean properties.

3.2. Digression concerning some non-Gaussian self-affine processes

$X(t,\omega)$ may satisfy the conditions of Proposition 3.7 without being Gaussian. This is indicated by an example given in Rosenblatt 1960 (pp. 434-435).

If the requirement of continuity is abandoned, many other interesting self-affine processes suggest themselves. For example, one may replace $B(t)$ by a non-Gaussian process whose increments are stable random variables, as defined by Paul Lévy. This leads to "fractional Lévy-stable random functions," which also have an infinite span of interdependence.

3.3. First digression concerning data analysis: Hurst's empirical results concerning $M(T,\omega)$

M 1965h{H9} singled out FBM to account for some empirical results reported in Hurst 1965, concerning the range M in the records of water flows through the Nile and other rivers, the price of wheat and other physical series, such as rainfall, temperatures, pressures, thickness of tree rings, thickness of valves (stratified mudbeds) and sunspot numbers.

In the first approximation, Hurst's empirical conclusion is that the range is proportional to T^H, where $1/2 < H < 1$. This result was a source of great surprise for statisticians. Indeed, processes of the form

$$X(t, \omega) = \int_0^t Y(s, \omega) ds,$$

where $Y(s, \omega)$ is stationary with summable covariance function, have a sequential range asymptotically proportional to $\sqrt{t}$. Thus, the statisticians who discussed Hurst's findings were led to conclude that the river flows cannot be represented by stationary stochastic processes. As shown in the next section, the existence of FBM with $1/2 < H < 1$ indicates that this conclusion is not necessarily correct. However, we shall have to return to Hurst's evidence, because his empirical evaluation actually deals with the sequential range after removal of the sample mean (see Section 5.10).

4. CONTINUITY AND NON-DIFFERENTIABILITY OF FBM

The variance given by Corollary 3.4 tends to zero with τ; hence $B_H(t, \omega)$ is mean square continuous. However, this property taken alone does not tell us anything about the sample paths.

Proposition 4.1 Almost all sample paths of $B_H(t, \omega)$ are continuous (t in any compact set).

Proof. If $H > 1/2$, the statement follows immediately from the expression in Corollary 3.4 combined with a theorem of Kolmogorov (see Loève 1960, p. 519). In any case, we can choose k such that $0 < k < H$ and note that (dropping the ω in the notation)

$$\Gamma(H + 1/2)^{1/k} \mathcal{E} \left| B_H(t + \tau) - B_H(t) \right|^{1/k} = \Gamma(H + 1/2)^{1/k} \mathcal{E} \left| B_H(\tau) - B_H(0) \right|^{1/k}$$
$$= \mathcal{E} \left| \int_{-\infty}^{\tau} [(\tau - s)^{H - 1/2} - N(s)(-s)^{H - 1/2}] dB(s) \right|^{1/k},$$

where $N(s) = 1$ if $s \le 0$ and $N(s) = 0$ if $s > 0$. Making a change of variables, the expression above takes the form

$$|t|^{H/k}\mathscr{E}\left|\int_{-\infty}^{1}\left[(1-s)^{H-1/2}-N(s)(-s)^{H-1/2}\right]dB(s)\right|^{1/k}=|t|^{H/k}V(H,k),$$

and we again apply Kolmogorov's theorem.

The process $B_H(t, \omega)$ is not mean square differentiable (this follows by an obvious modification of the next proposition), and it almost surely does not have differentiable sample paths.

Proposition 4.2 Almost all sample paths of $B_H(t, \omega)$ are not differentiable for any t; in fact,

$$\lim_{t \to t_0} \sup \left| \frac{B_H(t, \omega) - B_H(t_0, \omega)}{t - t_0} \right| = \infty \text{ with probability one.}$$

Proof. Assuming $B_H(0) = 0$, the identity in Definition 3.2 yields

$$\frac{B_H(t, \omega) - B_H(t_0, \omega)}{t - t_0} \triangleq (t - t_0)^{H-1}\{B_H(t_0 + 1, \omega) - B(t_0, \omega)\}$$
$$\triangleq (t - t_0)^{H-1} B_H(1, \omega).$$

Define the events

$$A(t, \omega) = \left\{ \sup_{0 \le s \le t} \left| \frac{B_H(s, \omega)}{s} \right| > d. \right\}$$

For any sequence such that $t_n \downarrow 0$, we have

$$A(t_n, \omega) \supset A(t_{n+1}, \omega);$$

thus,

$$P\{\lim_{n \to \infty} A(t_n)\} = \lim_{n \to \infty} P\{A(t_n)\}$$

and

$$P\{A(t_n)\} \ge P\left\{ \left| \frac{B_H(t_n)}{t_n} \right| > d \right\} = P\{|B_H(1)| > t_n^{1-H} d\},$$

which tends to 1 as $n \to \infty$. Note that this proof assumes nothing beyond self-affinity.

4.1. Fractional Gaussian noises and approximations thereto

The fact that FBM has no derivative is inconvenient. As is well-known, ordinary Brownian motion is also non-differentiable. Several methods, not always rigorous, have evolved that attempt to define the concept of the "derivative of Brownian motion." These constructs are called "white Gaussian noises." Analogous approaches can be followed with the fractional Brownian motions and lead to what may be called "fractional Gaussian noises."

The most elementary method of circumventing the FBMs lack of derivative is to smooth B_H by selecting $\delta > 0$ and forming the random function

$$B_H(t, \omega; \delta) = \delta^{-1} \int_t^{t+\delta} B_H(s, \omega) ds = \int_{-\infty}^{\infty} B_H(s, \omega) \varphi_1(t-s) ds,$$

where

$$\varphi_1(t) = \begin{cases} \delta^{-1} \text{ if } 0 \le t \le \delta, \\ 0 \quad \text{otherwise} . \end{cases}$$

The function $B_H(t, \omega; \delta)$ does have a stationary derivative, namely,

$$B'_H(t, \omega; \delta) = \delta^{-1} [B_H(t+\delta, \omega) - B_H(t, \omega)] = -\int_{-\infty}^{\infty} B_H(s, \omega) d\varphi_1(t-s).$$

The derivative is almost surely continuous, but surely non-differentiable.

When δ is sufficiently small, $B_H(t, \omega)$ and $B_H(t, \omega; \delta)$ are indistinguishable for all "practical purposes" that disregard the high frequency effects to which the non-differentiability of $B_H(t, \omega)$ is due (see Section 7).

To insure higher-orders of differentiability, one can proceed step by step, replacing φ_1 by increasingly smoother kernels, or use an infinitely differentiable kernel φ, which vanishes outside some finite interval and integrates to one. Then, the kth derivative of

$$\int_{-\infty}^{\infty} B_H(s, \omega)\varphi(t-s)ds$$

is

$$(-1)^k \int_{-\infty}^{\infty} B_H(s, \omega)\varphi^{(k)}(t-s)ds,$$

which is continuous and stationary for all positive integers k. This approach interprets B'_H as not being a random function but a "generalized random function," also called a *Schwartz distribution* (see Gelfand & Vilenkin 1964). For practical purposes, Schwartz distributions are better avoided. We shall be concerned with determining whether finite differences of B_H are reasonable approximations of B'_H.

4.2. Digression concerning some non-Gaussian fractional noises

The non-Gaussian fractional functions of Section 3.2 are, in most cases, also non-differentiable. However, several ways of defining a generalized differential, or a differential after smoothing do exist. Such constructs, when possible, may be called "fractional non-Gaussian noises." There is no doubt that such noises are required to model some of the phenomena listed in the Introduction.

5. SOME CORRELATIONS AND THEIR APPLICATIONS TO THE EXTRAPOLATION AND INTERPOLATION OF $B_H(t, \omega)$

In this section, we pause to examine certain interesting properties which fractional Brownian motion has with regard to extrapolation and interpolation. This excursion will familiarize the reader with these processes, and will identify problems for which FBM is a good model.

5.1. The correlation between two non-consecutive increments of $B_H(t, \omega)$

Let T, T_1 and T_2 be fixed and nonnegative. Then (dropping the ω in the notation) compute the correlation between the increments of $B_H(t)$ over the following time intervals: $T/2$ to T_1 and $-T/2$ to T_2. One has

$$\{[B_H(T/2+T_1)-B_H(T/2)][B_H(-T/2)-B_H(-T/2-T_2)]$$
$$2\mathscr{E} = \mathscr{E}[B_H(T/2+T_1)-B_H(-T/2-T_2)]^2 + \mathscr{E}[B_H(T/2)-B_H(-T/2)]^2$$
$$- \mathscr{E}[B_H(T/2+T_1)-B_H(-T/2)]^2 - \mathscr{E}[B_H(T/2)-B_H(-T/2-T_2)]^2.$$

Thus, the desired correlation is

$$C(T, T_1, T_2) = \frac{1}{2}\,\frac{(T+T_1+T_2)^{2H}+T^{2H}-(T+T_1)^{2H}-(T+T_2)^{2H}}{T_1^H T_2^H}$$

If $T>0$, we can write $s_1 = T_1/T$ and $s_2 = T_2/T$, and can see that the correlation is only a function of the reduced variables s_1 and s_2 (as expected from self-affinity):

$$C(s_1, s_2) = \frac{1}{2}\,\frac{(1+s_1+s_2)^{2H}+1-(1+s_1)^{2H}-(1+s_2)^{2H}}{(s_1 s_2)^H}.$$

For all s_1 and s_2, this correlation is positive if $1/2 < H < 1$ and negative if $0 < H < 1/2$. This is the first example in a series where the sign of $H - 1/2$ is an important distinguishing factor.

5.2. Failure of FBM to be strongly mixing

Now, consider the least upper bound of the absolute value of the correlation $C(T, T_1, T_2)$ over various sets of values of T, T_1 and T_2. Fixing T_1 and T_2, we see that this absolute value attains a maximum for $T=0$. Then, varying T_1/T_2, we see that for $T_1 = T_2$ it attains a maximum equal to $|2^{2H-1}-1| > 0$. If T is fixed and >0, $|2^{2H-1}-1|$ is not an attainable maximum, but remains a least upper bound (corresponding to $T_1 = T_2 = \infty$).

This leads us to the condition of strong mixing in Rosenblatt 1960, which is a form of asymptotic independence. In the Gaussian case, Kolmogorov and Rozanov showed that strong mixing requires a certain maximal correlation coefficient to tend to zero as the distance between the two time points tends to infinity. In the case of FBM, by self-affinity and from the form of $C(T, T_1, T_2)$ in Section 5.1, this coefficient is bounded below by $|2^{2H-1}-1| > 0$. Therefore, strong mixing does not hold for the increments of FBM, except in the classical Brownian case $H = 1/2$.

Strong mixing was originally introduced as one of several conditions that insure a random process satisfies the central limit theorem. This question of limit does not arise here, since the increments of FBM constitute a Gaussian process and satisfy the central limit theorem trivially. Therefore, the practical importance of strong mixing is to be found elsewhere. Stating that the increments of an FBM are not strongly mixing happens to be a convenient way of expressing the idea that the span of interdependence between such increments is infinite (see end of Section 6.3).

5.3. Extrapolation and interpolation of $B_H(t, \omega)$ from its values $B_H(0, \omega) = 0$ and $B_H(T, \omega)$ with $T > 0$ to its values for $-\infty < t < \infty$

Recall that if G_1 and G_2 are two dependent Gaussian random variables with zero mean, then

$$\frac{\mathscr{E}[G_1 \mid G_2]}{G_2} = \frac{\mathscr{E}[G_1 G_2]}{\mathscr{E}[G_2^2]}.$$

Thus, by setting $B_H(0) = 0$, we write

$$\begin{aligned}\frac{\mathscr{E}[B_H(t) \mid B_H(T)]}{B_H(T)} &= \frac{\mathscr{E}[B_H(t) B_H(T)]}{\mathscr{E}[B_H^2(T)]} \\ &= \frac{\mathscr{E}B_H^2(t) + \mathscr{E}B_H^2(T) - \mathscr{E}[B_H(t) - B_H(T)]^2}{2\mathscr{E}[B_H^2(T)]}.\end{aligned}$$

This yields the interpolatory/extrapolatory formula

$$\frac{\mathscr{E}[B_H(t) \mid B_H(T)]}{B_H(T)} = \frac{t^{2H} + T^{2H} - |t - T|^{2H}}{2T^{2H}}.$$

Introducing the "reduced" variable $s = t/T$, this expression takes the following form, which is illustrated by Figure 1 and defines the function $Q_H(s)$

$$\begin{aligned}\frac{\mathscr{E}[B_H(sT) \mid B_H(T)]}{B_H(T)} &= \frac{1}{2}\left[s^{2H} + 1 - |s - 1|^{2H}\right] \\ &\equiv Q_H(s).\end{aligned}$$

In the case of Brownian motion with $H = 1/2$, one has

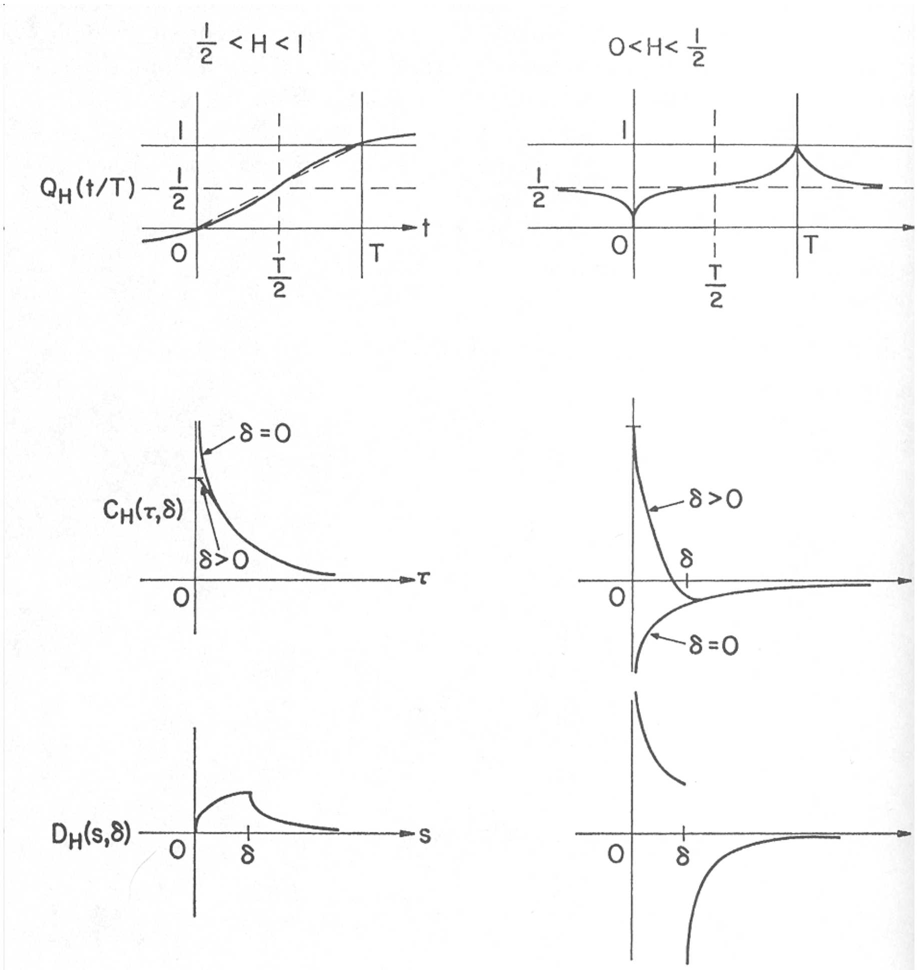

FIGURE C11-1. Freehand graphs of the shape of several important functions introduced in the text. The function Q_H occurs in the interpolation and extrapolation of B_H (Section 5.3). The function $C_H(\tau, \delta)$ is the covariance of the process of finite differences $B_H(t+\delta, \omega) - B_H(t, \omega)$, where t is a continuous time (Section 6). The function $D_H(s, \delta)$ occurs in Section 6.1. The differences between the two cases $0 < H < 1/2$ and $1/2 < H < 1$ are striking.

{P.S. 2000: accurate graphs of these functions differ significantly from this figure; they are provided, together with important additional comments, in the Post-publication appendix that follows this chapter.}

$$Q_H(s)\begin{cases} = 0 & \text{for } s < 0 \\ = s & \text{for } 0 < s < 1 \\ = 1 & \text{for } s > 0. \end{cases}$$

This $Q_H(s)$ is represented by a kinked curve consisting of straight intervals and half-lines.

When $1/2 < H < 1$, $Q_H(s)$ has a continuous derivative $Q'_H(s)$ which satisfies

$$0 < Q'_H(0), \quad Q'_H(1) < 1 \text{ and } Q'_H(1/2) > 1.$$

Finally, when $0 < H < 1/2$, $Q_H(s)$ is differentiable everywhere except at the abscissas $s = 0$ and $s = 1$, where it has a cusp.

5.4. The extrapolating function $Q_H(s) = \frac{1}{2}[s^{2H} + 1 - |s-1|^{2H}]$ for $s \to \infty$

For the Brownian case $H = 1/2$, we have

$$\mathscr{E}[B(t, \omega) | B(T, \omega)] = B(T, \omega) \text{ for all } t > T.$$

Thus, as is well-known, the best forecast is that $B(t, \omega)$ will not change.

In the second case, $1/2 < H < 1$,

$$Q_H(s) \sim H|s|^{2H-1} \text{ for large } s,$$

and the extrapolation involves a *nonlinear* "pseudo-trend" that diverges to infinity.

In the remaining case, $0 < H < 1/2$,

$$Q_H(s) \sim 1/2 \text{ for large } s,$$

and the extrapolation has a *nonlinear* "pseudo-trend" that converges to

$$\frac{1}{2}[B_H(0, \omega) + B_H(T, \omega)].$$

5.5. Extrapolation for large t when $\mathscr{E}[B_H(t, \omega)] \neq 0$ and the problem of "variable trends"

In analyzing a time series $X(t, \omega)$ without "seasonal effects," it is customary to search for a decomposition into a "linear trend component" and an "oscillatory component." The former is usually an estimate of

$$\mathscr{E}[X(t+\tau, \omega) - X(t, \omega)],$$

and it is interpreted as being due to major "causal" changes in the mechanism generating $X(t, \omega)$. To the contrary, the latter is interpreted as an "uncontrollable" stationary process, hopefully free of low-frequency components.

It is obvious that, in the case of FBM with $H \neq 1/2$, difficult statistical problems are raised by the task of distinguishing the linear trend Δt from the nonlinear "trends" just described. In reality, FBM falls outside the usual dichotomy between causal trends and random perturbations. {P.S. 1999: Further comments on this topic are found in the post-publication appendix that follows.}

5.6. Digression concerning data analysis

Those who analyze time series know that to decompose into trend and oscillation is difficult. For example, in ex-post factum analyses of long samples of data, the interpolated trend often appears to vary between successive subsamples. To avoid the difficulty one can assume that there are nonlinear trends or that the series is otherwise nonstationary. Examples are found in the literature of economics and in the discussions of Hurst's work.

However, the same phenomena can be explained also by assuming that $X(t, \omega)$ has the overall characteristics of FBM. A confirmation of this conjecture is found in the empirical observation that, for these series, the estimated spectral density is very "red." That is, no matter how large the sample duration T, the spectrum has a large amount of energy in frequencies that are not much greater than $1/T$ (see Adelman 1965 and Granger 1966). Although these two difficulties were observed independently, they are closely related, and FBM provides an excellent context in which to study their interplay.

5.7. Interpolation by $Q_H(s) = \frac{1}{2}[s^{2H} + 1 - |s-1|^{2H}]$ for $0 < s < 1$

In the Brownian case, the interpolate of $Q_H(s)$ is linear, which is well-known.

When $1/2 < H < 1$, the interpolate has the form illustrated in Figure 1. The slope $Q'_H(s)$ is largest at $s = 1/2$, where $Q'_H(1/2) = H2^{2-2H}$. This maximum slope is largest for $H = 1/2 \log_2 e$, when it is equal to 1.06. Thus, $Q_H(s)$ for $0 < s < 1$ is nearly linear if $1/2 < H < 1$.

When $0 < H < 1/2$, the interpolate has an S-shape which is inverted with respect to that of the previous case (see Figure 1).

5.8. The variance of $B_H(0, \omega)$ and $B_H(T, \omega)$

The usual formulas for the Gaussian case tell us that, given $B_H(0, \omega)$ and $B_H(T, \omega)$, the variance of $B_H(sT, \omega)$ is smaller for the interpolate and extrapolate, and equals

$$V_H(Ts)^{2H}\left\{1 - \frac{\left[1 + s^{2H} - |1-s|^{2H}\right]^2}{4s^{2H}}\right\}.$$

For large values of s, this tends to $V_H(Ts)^{2H}$. Thus, σ, defined as the standard deviation of $B_H(sT, \omega)$, is asymptotically proportional to s^H. Moreover, as $s \to \infty$,

$$\frac{\sigma}{\mathscr{E}[B(sT, \omega) \mid B(T, \omega)]} \sim \begin{cases} s^{1-H} & \text{if } 1/2 < H < 1, \\ s^H & \text{if } 0 < H < 1/2. \end{cases}$$

Note that, as s increases, this ratio always increases without bound.

5.9. Property of conditional self-affinity

While on the subject of conditional random variables, it is appropriate to discuss a property that we call *conditional self-affinity*. This concept plays an important role in the theory developed in M 1967b{N10}. Consider the random function

$$U_H[(h, \omega); T, B_H(T, \omega)] = T^{-H}\{[B_H(T, \omega) \mid B_H(T, \omega)] - Q_H(h)B_H(T, \omega)\},$$

where the vertical slash denotes conditioning, as usual. For example, if $B_H(T, \omega) = b$, then $[B_H(Th, \omega) \mid B_H(T, \omega)]$ is the restriction of $B_H(Th, \omega)$ to $\{\omega \mid B_H(T, \omega) = b\}$ with the corresponding conditional probability measure. Since U_H is Gaussian, it is determined by its mean and covariance matrix. The former vanishes and the latter is independent of T and $B_H(T, \omega)$. This interesting property of self-affinity differs from that discussed in Section 3 by the presence of the variable conditioning event $B_H(T, \omega)$.

Consider the random functions of the form

$$T^{-H}\{[B_H(hT, \omega) \mid B_H(T, \omega)] - Q(h)B_H(T, \omega)\}.$$

The choice $Q(h) = Q_H(h)$ minimizes the variance of this function, and is the only way to insure that its value is independent of $B_H(T, \omega)$.

5.10. Second data analysis digression concerning Hurst's problem

In Hurst's study of the range (as in the study of the trends in Section 5.5), the mean of $X(t, \omega)$ is not known. To model this situation, continue to assume $B_H(0, \omega) = 0$ and consider the expression

$$\tilde{B}_H(t, \omega; \Delta) \equiv B_H(t, \omega) + t\Delta.$$

When the constant Δ is unknown, it must be estimated from data. By symmetry, a reasonable estimate is

$$\tilde{\Delta} = \frac{1}{T}\tilde{B}_H(T, \omega; \Delta).$$

When substituted into the interpolatory/extrapolatory formula, $\tilde{\Delta}$ yields

$$\hat{\mathscr{E}}[\tilde{B}_H(hT, \omega; \Delta) \mid \tilde{B}_H(T, \omega; \Delta)] = h\tilde{B}_H(T, \omega; \Delta).$$

However, the value of Δ does not affect the range, because the quantity

$$M^*_H(T, s, \omega) = \max_{0 \le h \le s} [\tilde{B}_H(hT, \omega; \Delta) - h\tilde{B}_H(T, \omega; \Delta)] - \min_{0 \le h \le s} [\tilde{B}_H(hT, \omega; \Delta) - h\tilde{B}_H(T, \omega; \Delta)]$$

is independent of T and satisfies the s^H law. This further explains the empirical finding of Hurst.

The results of Section 5 generalize easily to cases where one knows the value of the process at more than two points. The formulas become much more complicated, but it is worth noting again that they are functions of certain "reduced" variables.

6. THE DERIVATIVE OF THE SMOOTHED PROCESS $B_H(t, \omega; \delta)$

The derivative process, $B'_H(t, \omega; \delta)$, is itself interesting as a stochastic model. Being stationary, it has a covariance of the form

$$C_H(\tau; \delta) = \mathscr{E}[B'_H(t, \omega; \delta) \mid B'_H(t + \tau, \omega; \delta)].$$

Without loss of generality, assume $B_H(0, \omega) = 0$. Then

$$C_H(\tau; \delta) = \frac{1}{2} V_H \delta^{2H-2} \left\{ \left(\frac{|\tau|}{\delta} + 1 \right)^{2H} - 2 \left| \frac{\tau}{\delta} \right|^{2H} + \left| \frac{|\tau|}{\delta} - 1 \right|^{2H} \right\}.$$

If $\tau \gg \delta$,

$$C_H(\tau, \delta) \sim V_H H(2H-1) |\tau|^{2H-2}.$$

This has the same sign as $H - 1/2$. It tends to zero as $|\tau| \to \infty$, which (by a theorem of Maruyama 1949) means that $B'_H(t, \omega; \delta)$ is weakly mixing and ergodic. However, from our remarks in Section 5, $B'_H(t, \omega; \delta)$ is not strong mixing (this also follows from the representation given in Section 4.2).

For $\tau = 0$,

$$C_H(0, \delta) = V_H \delta^{2H-2};$$

for small values of $|\tau|/\delta$,

$$C_H(0; \delta) - C_H(\tau; \delta) \sim V_H \delta^{-2} \tau^{2H}.$$

If $1/2 < H < 1$, $C_H(\tau; \delta)$ is positive and finite for all τ, and

$$\int_0^\infty C_H(s, \delta) ds = \infty.$$

If $0 < H < 1/2$, $C_H(\tau; \delta)$ changes sign once from positive to negative, at a value of τ proportional to δ, and

$$\int_0^\infty C_H(s, \delta) ds = 0.$$

6.1. Extrapolation of $B'_H(t, \omega; \delta)$ and $B_H(t, \omega)$

Given $\{B(s, \omega), -\infty < s < t\}$, and defining $N(s)$ as in the proof of Proposition 4.1, the least squares estimate of $B_H(t+\tau, \omega)$ is

$$\hat{B}_H(t+\tau, \omega) = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{t} \left[(t+\tau-s)^{H-1/2} - (-1)^{H-1/2} N(s) dB(s, \omega) \right].$$

If $\tau > 0$, $\hat{B}_H$ is infinitely differentiable (mean square or a.e.) in τ. Thus,

$$\frac{d\hat{B}_H(t+\tau, \omega)}{d\tau} = \frac{H-1/2}{\Gamma(H+1/2)} \int_{-\infty}^{t} (t+\tau-s)^{H-3/2} dB(s, \omega).$$

Define the decay kernel $D_H(t, \delta)$ as follows:

$$D_H(t; \delta) = \begin{cases} [\delta\Gamma(H+1/2)]^{-1} t^{H-1/2} & \text{for } t \le \delta, \\ [\delta\Gamma(H+1/2)]^{-1} [t^{H-1/2} - (t-\delta)^{H-1/2}] & \text{for } t > \delta. \end{cases}$$

Then,

$$B'_H(t, \omega; \delta) = \int_{-\infty}^{t} D_H(t-s; \delta) dB(s+\delta, \omega),$$

which is a one-sided moving average. It follows that the least squares predictor $B'_H(t+\tau, \omega; \delta)$, conditioned by a value of $\{B(s, \omega), -\infty < s \leq t\}$, is

$$\hat{B}'_H(t+\tau, \omega; \delta) = \int_{-\infty}^{t-\delta} D_H(t+\tau-s; \delta) dB(s+\delta, \omega),$$

which tends to $\hat{B}'_H(t+\tau, \omega)$ as $\delta \to 0$.

A fundamental relation between the "dynamic" law of relaxation of perturbations D and the "static" law of the distribution of the spontaneous fluctuations as expressed by the covariance $C_H(t; \delta)$ is the well-known formula

$$C_H(t; \delta) = \int_0^{\infty} D_H(s, \delta) D_H(s+t, \delta) ds.$$

6.2. Second digression concerning data analysis

A primary reason for the practical importance of fractional Brownian motion as a model arises from the fact that power function decay laws have often been observed by experimentalists. In fact, it seems likely that they will be useful even in cases which are currently modeled by the exponential law, $D(s) \sim e^{-s/a}$. The exponential decay law characterizes the classical case when $X(t, \omega)$ is a stationary Markov-Gauss process. It tends to be adopted because of its tractability and because the span of observable events is too short to conclude reliably otherwise. In the exponential case, the percentage attenuation of a perturbation between the times 0 and t can be obtained as the product of two successive independent attenuations: between the times 0 and t_0 (where $0 < t_0 < t$) and between the times t_0 and t.

Things are very different in the case of fractional Brownian motion, when the age is critically important in assessing future behavior. In economics, for example, the age-dependent law $s^{H-3/2}$ of the "derivative" of $B_H(t, \omega)$ seems preferable to the exponential law, both as a law of depreciation and as an expression of the attenuation of the effects of long past "causes."

6.3. Some conditional expectation least squares predictors

Given $B_H(0, \omega) = 0$ and $B'_H(0, \omega; \delta)$, it is illuminating to reexamine, in terms of $C_H(t, \delta)$, certain extrapolation problems discussed in Section 5. It is clear that

$$\frac{\mathscr{E}[B'_H(s, \omega; \delta) \mid B'_H(0, \omega; \delta)]}{B'_H(0, \omega; \delta)} = \frac{C_H(s; \delta)}{C_H(0; \delta)} .$$

Integrating from 0 to t, we obtain

$$\frac{\mathscr{E}[B_H(t, \omega; \delta) - B_H(0, \omega; \delta) \mid B'_H(0, \omega; \delta)]}{B'_H(0, \omega; \delta)} = \frac{\int_0^t C_H(s, \delta) ds}{C_H(0; \delta)} .$$

Now consider the limit as $t \to \infty$ of the expectation written on the left-hand side. This limit is infinite when $1/2 < H < 1$, and it vanishes when $0 < H < 1/2$. It is interesting in this light to recall that Taylor 1921 proposed that the integral of the covariance can be used as a quantitative measure of memory. If $1/2 < H < 1$, this measure correctly asserts that the memory of the process is infinite. However, when $0 < H < 1/2$, the Taylor measure asserts that the memory vanishes; in fact (as we saw in discussing strong mixing), it is infinite.

7. SPECTRA

A very interesting frequency representation of the increments of fractional Brownian motion was obtained by Hunt 1951 (p. 67):

$$B_H(t_2, \omega) - B_H(t_1, \omega) = V^*_H \int_0^\infty (e^{2\pi i\lambda t_2} - e^{2\pi i\lambda t_1}) \lambda^{-H-1/2} dB(\lambda, \omega),$$

where V^*_H is a constant. This suggests that $B_H(t, \omega)$ has a "spectral density" proportional to λ^{-2H-1}. However, spectral densities of nonstationary random functions are difficult to interpret. It is tempting to differentiate B_H and claim that B'_H has a spectral density proportional to λ^{1-2H}. If $1/2 < H < 1$, this formal density becomes infinite for $\lambda = 0$. Spectral densities proportional to λ^{1-2H} near $\lambda = 0$, where $1/2 < H < 1$, are very important in electronics (M 1967i{N }). The proportionality of the spectral density to λ^{1-2H} also suggests that there is infinite energy at high frequencies. Both the derivative B' and its spectrum can be interpreted via Schwartz distributions. However, these are not needed to examine the spectrum of $B'_H(t, \omega; \delta)$.

The spectral density of $B'_H(t, \omega; \delta)$ is

$$G'_H(\lambda; \delta) = 4\int_0^\infty C_H(s; \delta) \cos (2\pi\lambda s)ds$$
$$= 2V_H\delta^{-2}\int_0^\infty \left[(s+\delta)^{2H} - 2s^{2H} + |s-\delta|^{2H}\right] \cos(2\pi\lambda s)ds.$$

A sort of self-affinity property of B'_H is expressed by the fact that one can define a function G^* by writing

$$G'_H(\lambda; \delta) = 2V_H\delta^{2H-1}G^*_H(\delta\lambda).$$

For small values of $\lambda\delta$, one has

$$G^*_H(\delta\lambda) \sim K_H(2\pi\delta\lambda)^{1-2H},$$

with

$$K_H = \frac{\pi H(2H-1)}{\Gamma(2-2H)}\left[\cos \pi(H-1)\right]^{-1} > 0.$$

Thus, $G'_H(\lambda; \delta)$ behaves like $2K_HV_H(2\pi\lambda)^{1-2H}$. For fixed $\lambda > 0$, $\lim_{\delta\to 0} G'_H(\lambda, \delta)$ is positive, finite and equal to the formal density of B'_H. In other words, changing the value of δ involves modifications whose energy is primarily concentrated in high frequencies.

Acknowledgement. The work of J. W. Van Ness was supported by the Army Research Office under Grant DA-ARO (D)-31-124-G363 at Stanford University and by the National Science Foundation under Grant GP7519 at the University of Washington. Figure 2. Figure 3.Figure 4. Figure 5. Figure 6. •

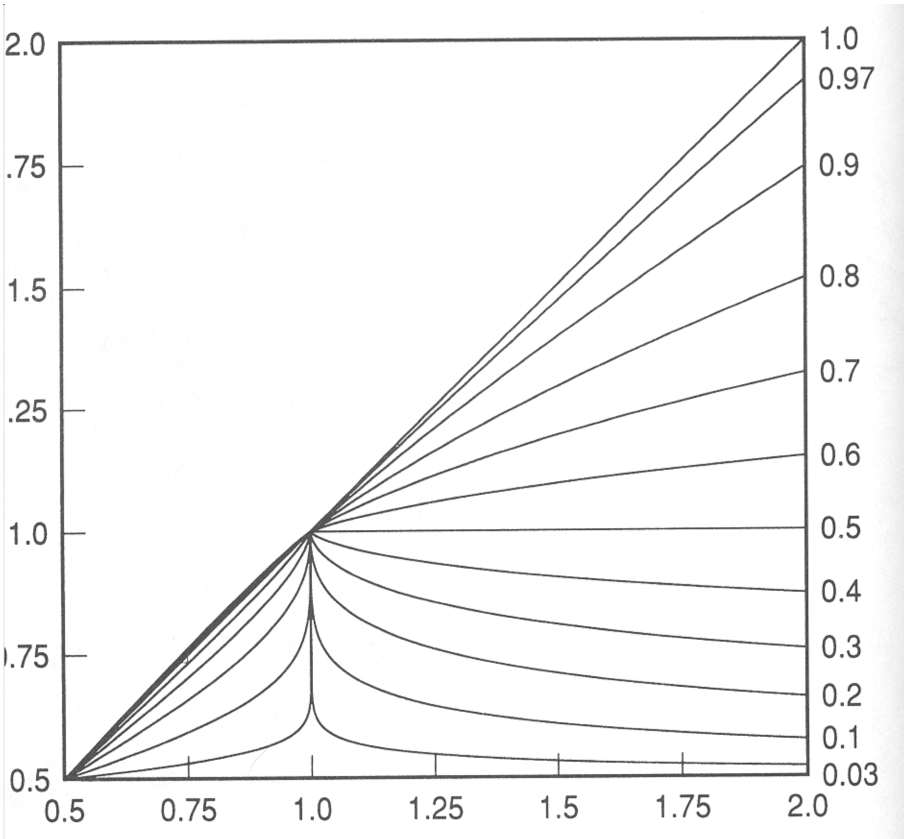

FIGURE C11-2. The top right quarters of the two graphs at the bottom of Figure 1, representing the functions $Q_H(s)$ relative to interpolation and extrapolation of FBM $B_H(t)$. The values of H are marked along the right side of this Figure. The other quarters of those graphs are either obtained by symmetry or empty.

Main observation: on the scale of this figure, all the curves relative to $H > 1/2$ nearly collapse for $s < 1$. The region of largest discrepancies is enlarged in the box inserted at the upper left on this figure. Other observation: the curves for $H < 1/2$ have an extremely sharp cusp near $s = 1$. Therefore, the effects of the imposed values of $X(0)$ and $X(1)$ are limited to $t = 0$ and $t = 1$.

&&&&& POST-PUBLICATION APPENDIX &&&&&

TOPICS IN FRACTIONAL BROWNIAN MOTION: TREND REMOVAL AND ROBUSTNESS OR FRAGILITY

This Appendix will denote M & Van Ness 1968 (namely, the body of this chapter) as *MVN*. When preparing the original for reprinting, Figure H1 was replotted on the computer. Increased precision paid, as usual. The carefully redrawn figures, prepared by Daryl Hepting, inspired no nostalgia for the pre-computer era and brought out several significant new facts, which are explored in this Appendix: Repeats of some points already made in *MVN* will increase legibility.

Section A.1 concerns the interpolation of $B_H(t)$ for $0 < t < T$. Given the values $B_H(o)$ and $B_H(T)$, the interpolate based on the conditional expectation of $B_H(t)$ is shown in *MVN* to be non-linear, except for $H = 0.5$ and $H = 1.0$. However, Figure 2 shows that in fact the interpolation is unexpectedly close to being linear if $H > 1/2$ and even (to a lesser degree) when $0.3 < H < 0.5$.

Section A.2 comments on the practice of "trend removal," which is very widespread in practical statistics and amounts to replacing complete data by their "bridges". In the case of FBM, the non-linearity of the interpolate makes this practice offensive to the theoretician. But the non-linearity is small for $H > 1/2$ hence not necessarily harmful in practice.

Section A.3 ponders the detailed form of the correlation $C_H(s)$ between distinct increments of $B_H(t)$.

Let us define Γ as follows:

$$\Gamma = \Sigma_{s=-\infty}^{-\infty} C(s), \text{ when time is discrete}$$

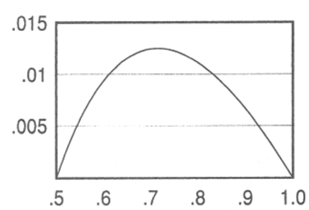

FIGURE C11-3. The "bias of linear interpolation," $Q_H(s) - s$, plotted for $s = 3/4$ as a function of $H > 1/2$.

$$\Gamma = \int_{-\infty}^{\infty} C(s)ds, \text{ when time is continuous}$$

It is shown in *MVN* that Γ has the following property

$$\begin{array}{lll} \Gamma = \infty & \text{if } H > 1/2 & \text{(persistent case)} \\ 0 < \Gamma < \infty & \text{if } H = 1/2 & \text{(Brownian case)} \\ \Gamma = 0 & \text{if } H < 1/2 & \text{(antipersistent case)} . \end{array}$$

When a theoretical $C(s)$ is slightly modified, one expects the properties $\Gamma = \infty$ or $0 < \Gamma < \infty$ to remain unchanged, but the property $\Gamma = 0$ to be easily destroyed. This is one aspect (among many) of the "fragility" of antipersistence. Nevertheless, antipersistence is robust enough to be encountered in nature, since the Kolmogorov theory of isotropic turbulence gives $H = 1/3$ for velocity. Similarly, when $C()$ and Γ are drawn for data, the precise identity $\Gamma = 0$ is very difficult to establish empirically.

A.1. When $H > 1/2$, the least mean square interpolate of FBM is not quite linear, but nearly so.

Figure 2, a more accurate combined rendering of the two bottom graphs of Figure 1 in *MVN*, illustrates the function

$$Q_H(s) = 1/2[\,|s|^{2H} + 1 - |s-1|^{2H}],$$

which gives the expected value of $B_H(s)$ when it is known that $B_H(0) = 0$ and $B_H(1) = 1$.

Only the half for $s > 1/2$ is drawn, so as to magnify the important details in the figure. *MVN* had been content to obtain the overall shape of the graph of $Q_H(s)$ from elementary calculus. Both the extrapolate for all H and the interpolate for $H < 1/2$ are as we expected, but the interpolate brings a surprise.

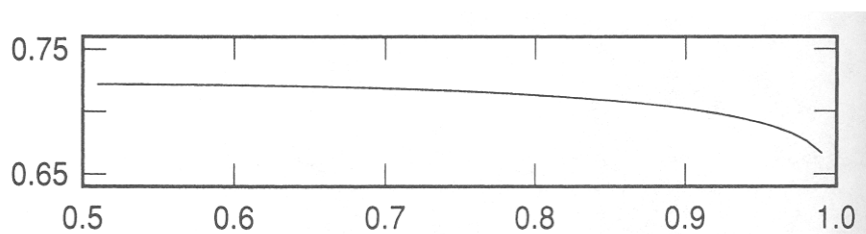

FIGURE C11-4. Value of H that maximizes the bias $Q_H(s) - s$, plotted as a function of H.

For $H > 1/2$, the interpolate is non-linear, to be sure, but the accurate Figure 2 shows that its departure from linearity is *surprisingly slight*. Of course the left and right derivatives of $Q_H(s)$ at $s = 1$ are equal for $H > ½$ (the second derivatives differ). Figure 3 plots, as function of H, the bias that would be introduced by using a linear interpolate near $s = 3/4$. At most, this bias replaces $Q = 3/4$ by $Q = 0.762$. It is easy to show that the largest bias occurs for

$$H = \frac{1}{2} \frac{\log \log(1-s) - \log \log s}{\log s - \log(1-s)} .$$

This last expression is plotted on Figure 4 and turns out to be only slightly above 0.70 over much of the range of s from 0.5 to 1. It is good to note here that in nature H is often in this range.

A.2. Consequence of the near-linearity of the interpolate for trend analysis

Old books of economics or practical statistics suggest that a time series should be decomposed into three components: a trend, which is a linear function; one or more seasonals, each a sine function; and a random noise component. This noise component is ordinarily fitted by a stationary random process. Because the preceding three-terms decomposition is not questioned, it has often been used for processes like Brownian or fractional

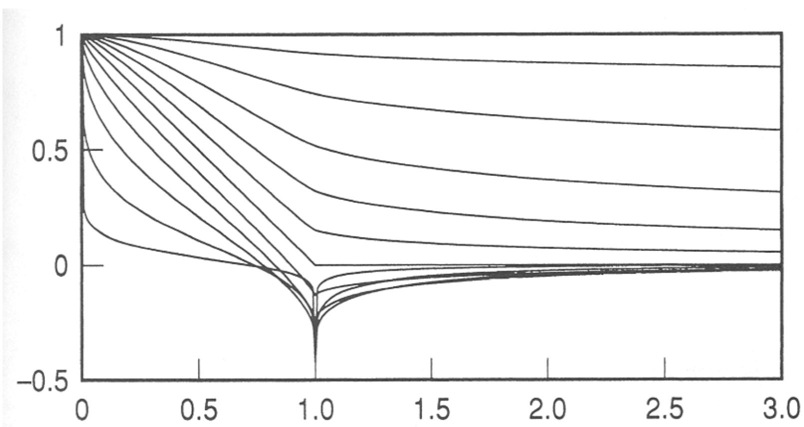

FIGURE C11-5. The correlation function $C_H(s)$ plotted as a function of s for $s > 0$. For $s = 0.5$, moving from top to bottom, the value of H is 1.0 and 0.97 near the top and 0.03 near the bottom. For the intermediate curves, H is a multiple of 0.1.

Brownian motion, which are not stationary merely of stationary increments. Let us examine what happens in those cases.

First of all, a definition is needed. Given a time series $X(t)$ from $t = 0$ to $t = T$, the linear trend can be defined in several distinct ways. It may be fitted by least squares, or it may simply be the linear function

$$X(0) + (t/T)[X(T) - X(0)]$$

.

In mathematical studies of Brownian motion the difference between $X(t)$ and this function is often called "bridge", and this book seeks to promote a wide use of this terminology. In practical statistical studies, a full list of hypotheses is not expected (and in many cases could not be provided). But the seasonals are written as periodic functions with a zero long range average and the random noise is usually taken to be of zero expectation. This implies that, when $X(0)$ and $X(T)$ are known, the linear trend is expected to represent the expected value of $X(t)$ for the given $X(0)$ and $X(T)$. In the case of fBm, for H other than $H = 1$ or 0.5, the interpolate is *not* linear, and the above-written trend again splits into two components: a non-random underlying drift in the expectation $\langle X(t)\rangle$, to be written as μt, and a random change in the fluctuation $X(t) - \langle X(t)\rangle$.

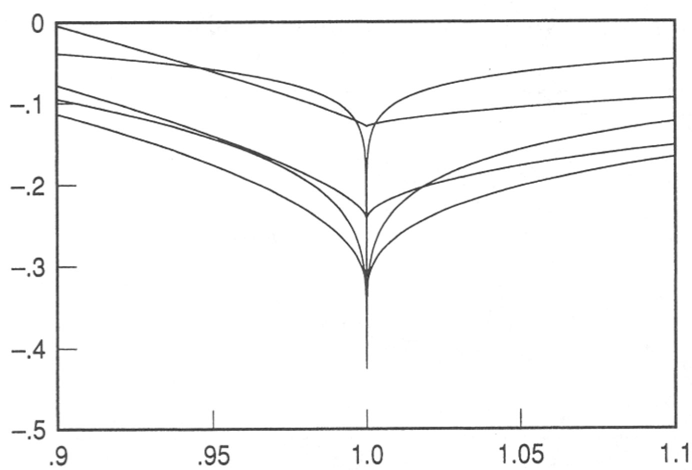

FIGURE C11-6. Blow-up of Figure 5 for $H < 1/2$ and s near 1.0. Close to $s = 1.0$, $C_H(s)$ begins by decreasing and is graphed here by three curves with a positive angle cusp at $s = 1.0$; then $C_H(s)$ increases and is graphed here by two curves with an increasingly sharp cusp. In the limit $H = 0$, the cusp is infinitely sharp, since $C_0(s) = 0$ except that $C(1) = -1/2$.

Detrending is meant to take out μt but unavoidably also takes out more. In the Brownian case $H = 1/2$, both components of the trend are linear, therefore taking them out together is mathematically legitimate. For all $H \neq 1/2$, to the contrary, the interpolate of $X(t) - \langle X(t) \rangle$ *is not* linear.

This might have been a serious practical difficulty, but in fact is not. When $H > 1/2$ and even for H slightly below $1/2$, the interpolate is near-linear. And when H is small, the value of μ can be estimated reliably from the data.

A.3. On the correlation $C_H(s) = 1/2[\,|s+1|^{2H} - 2|s|^{2H} + |s-1|^{2H}]$ between the increments $B_H(t+1) - B_H(t)$ and $B_H(t+s+1) - B_H(t+s)$

Figure 5 is a more precise replotting of the middle line of Figure 1 of *MVN*. Reading from top to bottom in the range $0 < s < 1$, the values of H are

0.97; 0.9; 0.8; 0.7; 0.6; 0.5; 0.4; 0.3; 0.2; 0.1; 0.03.

In the Brownian case $H = 1/2$, one anticipates $C_{1/2}(s) = 0$ except for $s = 0$. But this anticipation holds only for integer values of s. When s is continuous, $s > 1$ yields $C_{1/2}(s) = 0$, but $0 < s < 1$ yields $C_{1/2}(s) = 1 - s$.

It is known that $B_H(t)$ is only defined for $0 < H < 1$. But the correlation $C_H(s)$ remains meaningful for $0 \leq H \leq 1$, and it is interesting to ponder the limits values $H = 1$, or $H = 0$.

In the limit case $H \to 1$, one has $C_1(s) = 1$ for all s.

In the limit case $H \to 0$, one has $C_0(0) = 0$, $C_0(1) = C_0(-1) \to -1/2$ and $C_0(s) \to 0$ for other values of s.

For $H < 1/2$, Figure 6 enlarges Figure 5 near $s = 1$. The curves for different H cross each other repeatedly, even for large s. Indeed, for $s >> 1$ one has $C_H(s) = -2H(1-2H)s^{2H-2}$. For fixed s, this quantity is smallest when $\log s = -(2H)^{-1} + (1-2H)^{-1}$. The value s_H where $C_H(s)$ is lowest increases as $H \to 1/2$. As $H \to 0$, $C_H(s)$ increases to 0 for all s.

Once again, the identity $\Gamma = 0$ holds when $H < 1/2$ thanks to a very precise balance between positive and negative values, as made intuitive by Figures 5 and 6. These extraordinarily tight relations are easily destroyed by even a small and local change in $C(s)$.

&&&&&&&&&&&& ANNOTATIONS &&&&&&&&&&&&

Terminology and format. Throughout the body of this chapter the proper term, *self-affine*, was substituted for the original *self-similar*. Extensive and awkward footnotes were incorporated into the text.

How this paper came to be written. This story is told in Section 4.2 of Chapter H8.

Preprints. Like Chapter H9, this chapter expands upon the contents of two early IBM Internal Reports, M 1964h and M1964i.