Categorified Quantum $\mathcal{U}(\mathfrak{su}_2)$ at Prime Roots of Unity

- Why do we want to categorify $\mathcal{U}(\mathfrak{su}_2)$?
- Reshetikhin-Turaev-Witten:
  $\mathcal{U}(\mathfrak{su}_3)$ is the quantized gauge group of 3d Chern-Simons theory.
- Crane-Frenkel:
  Categorify 3d Chern-Simons to a 4d-TQFT.
  $\mathcal{U}_2(\mathfrak{su}_2)$: quantized 2-gauge group?

- Quantum $\mathcal{U}(\mathfrak{su}_2)$ at roots of unity.

We are interested in the idempotented version of $\mathcal{U}_2(\mathfrak{su}_2)$. It is generated over $\mathbb{Z}[q,q^{-1}]$ by pictures of the form

$$\begin{array}{ll}
\lambda+2 & \lambda \\
E & \\
\lambda-2 & \lambda \\
F & \\
\end{array}$$

($\lambda \in \mathbb{Z}$)

with the algebra structure

$$\begin{array}{ll}
\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \lambda & \cdot \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \downarrow \uparrow \downarrow \mu^2 = \delta_{\lambda \mu} \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \downarrow \uparrow \downarrow \uparrow \downarrow \mu^2 \\
\end{array}$$

(etc)

Modulo relations (at a 2k-th root of unity, $k$ odd)

$$\begin{array}{ll}
\begin{array}{ll}
\uparrow \downarrow \downarrow \lambda & = & \downarrow \uparrow \downarrow \lambda + [\lambda] \downarrow \uparrow \uparrow \uparrow \lambda \\
E & F & E & \end{array} & (\lambda \geq 0) \\
\begin{array}{ll}
\downarrow \uparrow \uparrow \lambda & = & \uparrow \downarrow \uparrow \lambda + [\lambda] \downarrow \uparrow \uparrow \uparrow \lambda \\
F & E & F & \end{array} & (\lambda \leq 0) \\
\begin{array}{ll}
\uparrow \cdots \uparrow \downarrow \lambda \\
k\text{-mary} & = & 0 = \downarrow \cdots \downarrow \lambda \\
k\text{-mary} & \end{array} & (\text{Nilpotency relation})
\end{array}$$
• **Categorification of** $U_q(u_2)$

Below we present Lauda's diagrammatic calculus for $U_q(u_2)$ at a generic $q$-value. The rough idea is that:

• Pictures = Isomorphism class/symbol of some modules
• Sum of pictures = symbol of direct sum of modules
• Equalities of pictures = isomorphisms of modules.

In general, isomorphisms are rare between modules. Instead, study homomorphisms between them. Intuitively, homomorphisms = evolution of pictures, which is not necessarily reversible.

• Maps just among $E$'s (or $F$'s) (Khovanov-Lauda-Rouquier)

(Nil-Hecke algebra)

• To categorically Drinfeld-double $E$'s, Lauda introduces cups and caps

Together with the nilHecke algebra generators, cups and caps satisfy certain relations

(i) Biaadjointness  E.g.  

(ii) Bubble positivity (degrees of $k = \frac{1}{2}(m-1-\lambda) \geq 0$  

\[ l = \frac{1}{2}(m+\lambda) \geq 0 \]
(iii). NilHecke relations

\[ \begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\uparrow \quad \uparrow \\
\uparrow \quad \uparrow
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\uparrow
\end{array}
\end{array} \quad \begin{array}{c}
\downarrow
\end{array} = 0 \quad \begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow
\end{array}
\end{array} \]

(iv). Reduction to bubbles

\[ \begin{array}{c}
\begin{array}{c}
\uparrow \\
\uparrow
\end{array}
\end{array} = - \sum_{a \in b = - \lambda} \begin{array}{c}
\begin{array}{c}
\downarrow \\
a
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \\
b
\end{array}
\end{array} = \sum_{a \in b = \lambda} \begin{array}{c}
\begin{array}{c}
\downarrow \\
a
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\downarrow \\
b
\end{array}
\end{array} \]

(v). Identity decomposition

\[ \begin{array}{c}
\begin{array}{c}
\uparrow \\
\uparrow
\end{array}
\end{array} = - \begin{array}{c}
\begin{array}{c}
\downarrow \\
\downarrow
\end{array}
\end{array} + \sum_{a \in b c = \lambda - 1} \begin{array}{c}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\end{array} + \sum_{a \in b c = \lambda - 1} \begin{array}{c}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\end{array} \]

Thm. (Lauda) This graphical calculus is non-degenerate and categorifies $U_q(\mathfrak{sl}_2)$ at a generic $q$-value.

Rmk: Lauda's calculus is a 2-dim'1 idempotent algebra, i.e. it has two compatible multiplication structures (vertical and horizontal). Such idempotent algebras are also known as a 2-category.

To see the plausibility of this categorification, we consider how $EF^{1,1}$ can "evolve" into $FE^{1,1} \oplus 1^{\text{odd}}$.
These elements \( \{ u_{\lambda} \}, \{ u_i \} \) satisfy

\[
\begin{align*}
  u_i u_j u_i &= u_i, \\
  u_i u_j u_i &= u_i, \\
  u_i u_j &= 0 \quad (i \neq j),
\end{align*}
\]

which follows from the identity decomposition relation. Consequently \( \{ u_i \mid i = 0, \ldots, \lambda \} \) form an orthogonal set of idempotents in \( \text{End}_A(\mathcal{E}\Phi\Lambda) \)

(\text{Factorization of idempotents})

- **Enhancing \( \mathcal{U} \) with a \( p \)-differential**

As we have heard from Mikhail's talk, if \( A \) is a \( p \)-DG algebra, then the derived category of \( p \)-DG modules over \( A \) is a module-category over the homotopy category of \( p \)-complexes.

\[
\begin{array}{ccc}
\text{\( \mathbb{k}[\mathcal{E}\Phi\Lambda] \)-gmod \times \mathcal{D}(A, \mathcal{E}) \)} & \xrightarrow{\otimes} & \mathcal{D}(A, \mathcal{E}) \\
\downarrow & & \downarrow \\
\mathcal{D}(A, \mathcal{E}) & \xrightarrow{\otimes} & \mathcal{D}(A, \mathcal{E}) \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{O}_p & \times & \mathcal{K}_0(A, \mathcal{E}) \\
\downarrow & & \downarrow \\
\mathcal{O}_p(A, \mathcal{E}) & \xrightarrow{\otimes} & \mathcal{K}_0(A, \mathcal{E})
\end{array}
\]
Def. Let \((\mathcal{U}, \partial)\) be Lauda's 2-dimensional algebra equipped with the differential \(\partial\)-action on generators given by

\[
\begin{align*}
\partial(\uparrow \downarrow) &= \uparrow \downarrow \\
\partial(\downarrow \uparrow) &= \downarrow \uparrow - 2 \\
\partial(\downarrow \downarrow) &= \downarrow \downarrow - 2 \\
\partial(\downarrow \lambda) &= \downarrow \lambda + \lambda \uparrow \\
\partial(\lambda \uparrow) &= (\lambda + 1) \lambda
\end{align*}
\]

Lemma. The above \(\partial\) preserves all relations of \(\mathcal{U}\), and it is \(p\)-nilpotent over a field of characteristic \(p > 0\).

Thm. (Elia - Q.) The derived module category \(\mathcal{D}(\mathcal{U}, \partial)\) is Karoubian, and it categorifies \(\hat{\mathcal{U}}_q(\mathfrak{g})\) at a \(p\)-th primitive root of unity.

\[K_0(\mathcal{U}, \partial) \cong \hat{\mathcal{U}}_q(\mathfrak{g})\]

Decomposition vs. filtration.

In Lauda's abelian categorification, the relations in \(\hat{\mathcal{U}}_q(\mathfrak{g})\) are usually realized as different ways of decomposing projective \(\mathcal{U}\)-modules.

In the realm of triangulated categories, direct sum decompositions are very rare. Instead, a short exact sequence of \(p\)-DG \(\mathcal{U}\)-modules gives rise to a distinguished triangle in \(\mathcal{D}(\mathcal{U}, \partial)\).

\[
\begin{align*}
0 \rightarrow & \ A \rightarrow B \rightarrow C \rightarrow 0 \quad \text{in} \quad (\mathcal{U}, \partial)\text{-mod} \\
\downarrow & \\
A \rightarrow & \ B \rightarrow C \rightarrow A[1] \quad \text{in} \quad \mathcal{D}(\mathcal{U}, \partial) \rightarrow [B] = [A] + [C] \in K_0(\mathcal{U}, \partial)
\end{align*}
\]
More generally, a filtered p-DG module \((M, F^-)\) presents \(M\) as a convolution (Postnikov tower) of \(gr F^-\).

**Example** In the nilHecke algebra \(NH_2\):
\[
NH_2 \cong \text{Sym}_2 \cdot \left( \begin{array}{c}
\times \\
\downarrow \\
\times \\
\downarrow \\
\times
\end{array} \right)
\]
\[
\Rightarrow \quad 0 \rightarrow P_2 \{1\} \rightarrow NH_2 \rightarrow P_2 \{1\} \rightarrow 0 \text{ is a s.e.s. of } (U, \partial)\text{-modules.}
\]
\[
\Rightarrow \quad \text{In } K_0(U, \partial), \quad E^2 = \langle [(NH_2, \partial)] = q[P_2] + q'[P_2] = (q+q')E^{(2)}
\]

**Prop.** Let \(\{(u_i, u_i)\mid i \in I\}\) be factorization of idempotents in a p-DG algebra \(R\).
If there is a total ordering on \(I\) such that
\[
\begin{cases}
u_i \partial(u_i) = 0 \\
u_i \partial(u_i) = 0 \text{ (modulo lower order terms)}
\end{cases}
\]
Then if \(E = \sum_{i \in I} u_i u_i\), then the p-DG module \(RE\) admits a filtration \(F^*\) whose subquotients are isomorphic to \(Ru_i\)'s.

**Cor. (Fantastic!)** In the situation of the Prop \([RE] = \sum_{i \in I} [Ru_i u_i]\).

**Cor.** Under the differential defined earlier on \(U\), there is a filtration on \(E^F I^\lambda\).
• **Uniqueness**: a small surprise!

Lauda’s factorization of idempotents, in general, is not unique.

However, in the presence of a diagrammatically local differential (not necessarily the differential we defined here, but any $\Theta$ compatible with the local relations of $U$), we have, up to conjugation by diagrammatic automorphisms

• The differential we defined here is the unique differential such that the modules $\mathcal{E}\mathcal{E}\mathcal{L}\lambda (\lambda \geq 0)$ admit filtrations whose subquotients are isomorphic to $\mathcal{E}\mathcal{E}\mathcal{L}\lambda, \lambda \{i-\lambda, \ldots, \lambda\{\lambda-1\}\}.$

• Lauda’s factorization of idempotents is the unique choice that is compatible with the differential. (Fantastic Filtration)