COARSE HYPERBOLICITY AND CLOSED ORBITS FOR QUASIGEODESIC FLOWS

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Abstract. We prove Calegari’s conjecture that every quasigeodesic flow on a closed hyperbolic 3-manifold has closed orbits.

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1. Introduction

In 1950, Seifert asked whether every nonsingular flow on the 3-sphere has a closed orbit [20]. Schweitzer gave a counterexample in 1974 and showed more generally that every homotopy class of nonsingular flows on a 3-manifold contains a $C^1$ representative with no closed orbits [19]. Schweitzer’s examples were generalized considerably and it is known that the flows can be taken to be smooth [17] or volume-preserving [16].

On the other hand, there are certain geometric constraints on flows that ensure the existence of closed orbits. Taubes’ 2007 proof of the 3-dimensional Weinstein conjecture shows that every Reeb flow on a closed 3-manifold has a closed orbits [21]. Reeb flows are geodesible, i.e. there is a Riemannian metric in which the flowlines are geodesics. In 2010, Rechtman showed that real analytic geodesible flows on closed 3-manifolds have closed orbits, unless the manifold is a torus bundle with reducible monodromy [18].

Geodesibility is a global geometric condition. In contrast, a flow is said to be quasigeodesic if the flowlines lift to quasigeodesics in the universal cover, a local condition. In this paper we will show that every quasigeodesic flow on a closed hyperbolic 3-manifold has closed orbits.

1.1. Flows, transverse structures, and closing. Our proof works by studying the transverse structure of a quasigeodesic flow. For motivation we will outline a parallel picture that works for Anosov and pseudo-Anosov flows.

A flow $\Phi$ on a 3-manifold $M$ is Anosov when the tangent bundle splits into three one-dimensional sub-bundles: the tangent bundle $T\Phi$ to the flow, a stable bundle $E^s$, and an unstable bundle $E^u$. The flow exponentially contracts the stable bundle and exponentially expands the unstable bundle. The two-dimensional sub-bundles $T\Phi \oplus E^s$ and $T\Phi \oplus E^u$ integrate to a pair of transverse two-dimensional foliations, the weak stable and weak unstable foliations. The flowlines in a weak stable leaf are all forward asymptotic, while the flowlines in a weak unstable leaf are all backwards asymptotic. Furthermore, the weak stable and unstable leaves are themselves foliated by strong stable and unstable leaves, obtained by integrating the one-dimensional sub-bundles $E^s$ and $E^u$. Two flowlines lying in a single stable leaf are forwards asymptotic, and the points where these flowlines intersect a strong stable leaf are asymptotic on the nose.

Example 1.1. The simplest examples of Anosov flows are obtained as suspension flows of Anosov diffeomorphisms. An Anosov diffeomorphism $\phi : T^2 \to T^2$ preserves a pair of 1-dimensional foliations, $\mathcal{F}^{s/u}$, which are respectively contracted and expanded by $\phi$.

Let $M_\phi$ the be the 3-manifold obtained from $T^2 \times [0, 1]$ by gluing the top face to the bottom using $\phi$. The semi-flow on $T^2 \times [0, 1]$ that moves points at unit speed in the interval direction glues up to a flow $\Phi$ on $M_\phi$ called the suspension flow of $\phi$. This is an Anosov flow whose weak stable and unstable foliations are simply the suspensions of $\mathcal{F}^s$ and $\mathcal{F}^u$.

More generally, a flow is pseudo-Anosov if it is Anosov everywhere except near some isolated closed orbits, where it is modeled on the suspension of a pseudo-Anosov diffeomorphism. Pseudo-Anosov flows have singular weak stable and unstable foliations, which look like Figure 1 near the singularities.
The Anosov Closing lemma leverages the transverse contracting-expanding behavior of a pseudo-Anosov flow to find closed orbits. An *almost-cycle* is a long flow segment whose endpoints are close together. The Anosov Closing Lemma says, roughly, that a sufficiently long almost-cycle whose endpoints are sufficiently close lies near a closed orbit.

The idea behind the Anosov Closing Lemma is illustrated in Figure 2. The left side of the figure depicts the local structure near the ends of an almost-cycle $[x_-, x_+]$, while the right side depicts the local structure near a point $x$ in the middle. Since $x_-$ is close to $x_+$, the local stable/unstable leaf through $x_-$ intersects the local unstable/stable leaf through $x_+$. Take a point $y$ where the stable leaf through $x_-$ intersects the unstable leaf through $x_+$. Flowing forward, we arrive at a point $y_+$ which lies very close to $x$ along its stable leaf. Flowing backwards, we arrive at a point $y_-$ which lies very close to $x$ along its unstable leaf. This produces an almost-cycle $[y_-, y_+]$ whose length is comparable to $[x_-, x_+]$, but whose ends are much closer. Repeating this, we obtain a sequence of better and better almost-cycles, which limit to a closed orbit.

Anosov and pseudo-Anosov flows are defined by their transverse structure. In contrast, a quasigeodesic flow is defined by a tangent condition. When the ambient manifold is hyperbolic, however, we will see that a quasigeodesic flow has a remarkably similar sort of transverse structure.

Given a quasigeodesic flow $\Phi$ on a closed hyperbolic manifold $M$, Calegari constructed a pair of flow-invariant decompositions of $M$ into positive leaves and negative leaves. In the universal cover, all points in a positive/negative leaf are forwards/backwards asymptotic to a single point at infinity.

These decompositions are generally quite different from foliations. Leaves may have nontrivial interior, and may not be path-connected or even locally connected. Nevertheless, we can understand the separation properties of leaves by thinking of them as subsets of the flowspace, the orbit space of the lifted flow. This is a topological plane, and each leaf corresponds to a closed, connected, unbounded subset.
We will see that the positive and negative leaves are coarsely contracted and expanded by the flow. Moreover, we will construct strong positive and negative decompositions, analogous to the strong stable and unstable foliations of a pseudo-Anosov flow.

Using coarse contraction-expansion, we will prove a Homotopy Closing Lemma. This allows us to approximate each recurrent orbit homotopically by closed orbits, provided that it lies in a reasonable topological configuration. We show that every quasigeodesic flow on a closed hyperbolic 3-manifold has a recurrent orbit in the appropriate topological configuration, and hence has closed orbits; many, in fact.

1.2. **Organization.** We review the basic theory of quasigeodesic flows in Section 2, and outline our main results in Section 2.5.

Section 3 is concerned with the transverse structure of a quasigeodesic flow. We show that quasigeodesic flows behave coarsely like pseudo-Anosov flows, and build the strong decompositions.

Section 4 studies the topological properties of the decompositions. In particular, we study Hausdorff limits and complementary regions of leaves. We motivate this by sketching a special case of our closing lemma in Section 4.1; the reader who is familiar with quasigeodesic flows may wish to look at this first.

Section 5 is the meat of the paper. We prove the closing lemma, and use this to show that quasigeodesic flows have closed orbits.

In Section 6 we ask whether our results can be extended to a larger class of coarsely hyperbolic flows. We propose an alternative method for finding closed orbits using ideas from geometric group theory.

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2. **Background**

In this section we will review some of the basic topological theory around quasigeodesic flows. See [3], [8], and [9] for more details.
A flow on a manifold $M$ is a continuous map
$$\Phi(\cdot) : \mathbb{R} \times M \to M$$
with the property that
$$\Phi_s(\Phi_t(x)) = \Phi_{t+s}(x)$$
for all $x \in M$ and $t, s \in \mathbb{R}$.

For a fixed $t \in \mathbb{R}$, the time-$t$ map $\Phi_t : M \to M$ is a homeomorphism, since $\Phi_{-t}$ acts as its inverse. We can therefore think of a flow as an action of $\mathbb{R}$ on $M$. When the flow is fixed, we will use the flow and action notation interchangeably, writing $x \cdot t = \Phi_t(x)$.

### 2.1. Quasigeodesic flows.

**Definition 2.1.** A curve $\gamma$ in a metric space $X$ is $(k, \epsilon)$-quasigeodesic if
$$d_\gamma(x, y) \leq k \cdot d(x, y) + \epsilon$$
for all $x, y \in \gamma$, where $d$ is the metric on $X$ and $d_\gamma$ is the distance along $\gamma$.

In $\mathbb{H}^n$, each quasigeodesic has well-defined and distinct endpoints in the Gromov boundary $S^\infty_{n-1}$. Furthermore, each $(k, \epsilon)$-quasigeodesic is contained in the $C$-neighborhood of the geodesic between its endpoints, where $C$ is a constant depending only on $k$, $\epsilon$, and $n$. See [12], [14], or [2].

**Definition 2.2.** A flow $\Phi$ on a manifold $M$ is quasigeodesic if each flowline lifts to a quasigeodesic in the universal cover $\tilde{M}$.

Calegari showed that every quasigeodesic flow on a closed hyperbolic 3-manifold is uniformly quasigeodesic, i.e. each lifted flowline is a $(k, \epsilon)$-quasigeodesic for uniform $k$ and $\epsilon$.

On a closed hyperbolic 3-manifold, quasigeodesic flows are exactly those that can be studied “from infinity” in the following sense.

**Proposition 2.3 ( [7] Theorem B and [3] Lemma 4.3).** Let $\Phi$ be a flow on a closed hyperbolic 3-manifold $M$. Then $\Phi$ is quasigeodesic if and only if

1. each lifted flowline has well-defined and distinct endpoints in $S^2_{\infty} = \partial_\infty \mathbb{H}^3$, and
2. the maps $e^\pm : \mathbb{H}^3 \to S^2_{\infty}$ that send each point to the positive/negative endpoint of its flowline are continuous.

Quasigeodesic flows are quite common. Cannon and Thurston showed that the suspension flow of a pseudo-Anosov diffeomorphism is quasigeodesic [5]. Zeghib generalized this, showing that any flow transverse to a fibration of a 3-manifold is quasigeodesic [23]. Even more generally, Fenley and Mosher showed that any taut, finite-depth foliation on a closed hyperbolic 3-manifold admits a transverse or “almost-transverse” quasigeodesic flow [7].

Gabai showed that a closed hyperbolic 3-manifold with nontrivial second betti number has a taut, finite-depth foliation [11], so there are many such examples.
2.2. Flowspaces and decompositions. Fix a quasigeodesic flow $\Phi$ on a closed hyperbolic 3-manifold $M$, and lift to a flow $\tilde{\Phi}$ on the universal cover $\tilde{M} \simeq \mathbb{H}^3$.

**Definition 2.4.** The *flowspace* $P$ is the space of lifted flowlines, with the topology induced by the quotient map

$$\pi_P : \mathbb{H}^3 \to P = \mathbb{H}^3 / x \sim x \cdot t$$

that collapses each flowline to a point.

In other words, $P$ is the orbit space of the lifted flow $\tilde{\Phi}$. Using uniform quasi-geodesity, Calegari showed that $P$ is Hausdorff, and therefore homeomorphic to a plane. The action of $\pi_1(M)$ on $\mathbb{H}^3$ by deck transformations preserves the oriented foliation by flowlines, so it descends to an orientation-preserving action of $\pi_1(M)$ on $P$.

Each point $p \in P$ corresponds to a flowline, which we denote by $p \times \mathbb{R}$. Each subset $A \subset P$ corresponds to a union of flowlines, denoted $A \times \mathbb{R}$.

There are continuous endpoint maps

$$e^\pm : P \to S^2_\infty$$

that take each point to the positive/negative endpoint of the corresponding flowline.

Given subsets $A, B \subset S^2_\infty$, let $\{ \to A \}$ and $\{ \to B \}$ be the sets of flowlines that start in $A$ and end in $B$, respectively. Similarly, $\{ A \to \} = \{ A \to \} \cap \{ \to B \}$.

As subsets of $P$,

$$\{ A \to \} = (e^-)^{-1}(A),$$
$$\{ \to B \} = (e^+)^{-1}(B),$$

and

$$\{ A \to B \} = (e^-)^{-1}(A) \cap (e^+)^{-1}(B).$$

Consequently, if $A$ and $B$ are closed then each of these are closed. Moreover,

**Lemma 2.5.** If $A, B \subset S^2_\infty$ are closed and disjoint then $\{ A \to B \}$ is a compact subset of $P$.

**Proof.** Each flowline in $\mathbb{H}^3$ is contained in a uniformly bounded neighborhood of its corresponding geodesic. Consequently, there is a compact set $C \subset \mathbb{H}^3$ that intersects every flowline with one end in $A$ and the other in $B$. Then $\pi_P(C)$ is a compact subset of $P$ that contains $\{ A \to B \}$.

For each $z \in S^2_\infty$, each component of $\{ \to z \}$ is called a positive leaf, and each component of $\{ z \to \}$ is called a negative leaf. In contrast to the preceding lemma,

**Lemma 2.6** ([3], Lemma 4.8). Each positive or negative leaf is unbounded.

If $X$ is a space, a decomposition of $X$ is a collection $D$ of closed subsets that fill $X$. The collections

$$D^+ := \{ \text{components of } \{ \to z \} \subset P \mid z \in S^2_\infty \}$$

and

$$D^- := \{ \text{components of } \{ z \to \} \subset P \mid z \in S^2_\infty \}$$

of positive and negative leaves are called the positive and negative decompositions of $P$. If $p \in P$, we write $D^+(p)$ and $D^-(p)$ for the positive and negative leaves through $p$.

By the preceding lemmas, our decompositions have two important properties.
(1) Each leaf is closed, connected, and unbounded.
(2) The intersection of a positive leaf with a negative leaf is compact.

Property (1) will allow us to treat the decompositions as if they were foliations. Property (2) is a weak form of transversality.

2.3. The compactified flowspace. In [8] we showed that \( P \) has a universal compactification to a closed disc \( \hat{P} = P \cup S^1_u \), whose boundary is Calegari’s universal circle.

Notation. If \( A \) is a subset of \( P \) or \( \hat{P} \), we will write \( \overline{A} \) for the closure of \( A \) in \( \hat{P} \). If \( B \) is a subset of \( \hat{P} \), we define \( \partial_u B := B \cap S^1_u \).

The closure of each leaf \( K \in D^\pm \) intersects the universal circle in a totally disconnected set \( \partial_u K \) which we’ll call the ends\(^1 \) of \( K \). Each of the sets

\[
\bigcup_{K \in D^\pm} \partial_u K,
\]

consisting of all ends of positive/negative leaves are dense in \( S^1_u \). The compactification \( \hat{P} \) is universal in the sense that any other disc compactification with these properties is a quotient of \( \hat{P} \).

The action of \( \pi_1(M) \) on \( P \) extends naturally an orientation-preserving action on \( \hat{P} \). This restricts to Calegari’s universal circle action on the boundary.

Remark 2.7. Calegari-Dunfield showed that the fundamental group of the Weeks manifold admits no faithful orientation-preserving actions on the circle [4]. Consequently, the Weeks manifold has no quasigeodesic flows. This is the only way we know to show the non-existence of quasigeodesic flows.

Remark 2.8. Anosov and pseudo-Anosov flows also have universal circles, first constructed by Calegari-Dunfield [4]. The flowspace of a pseudo-Anosov flow is topologically a plane, and Fenley showed that one can use the universal circle compactify the flowspace [6]. In fact, this may be done in the same manner as for quasigeodesic flows.

A pseudo-Anosov flow \( \Psi \) on a closed 3-manifold \( M \) comes with 2-dimensional stable and unstable singular foliations. The lifts of these to the universal cover \( \hat{M} \) project to 1-dimensional singular foliations of the flowspace \( P_\Psi \). The leaves of these foliations are properly embedded lines and \( n \)-prongs that intersect transversely. In particular, they are closed, connected, unbounded sets that intersect compactly, and we can use them to produce a universal compactification \( \hat{P}_\Psi \). The deck action induces an action on \( P_\Psi \), which extends to \( \hat{P}_\Psi \).

If \( \Psi \) is both pseudo-Anosov and quasigeodesic then the stable and unstable foliations are exactly the positive and negative decompositions.

In [9] we showed that the endpoint maps \( e^\pm \) extend continuously to \( \pi_1 \)-equivariant maps

\[
\hat{e}^\pm : \hat{P} \to S^2_\infty
\]

---

\(^1\)Our usage of the word “end” differs slightly from that of [8] and [9], where it refers to a Freudenthal end. In fact, our set of ends \( \partial_u K \) is the closure of the image of \( K \)’s Freudenthal ends (see [8], Lemma 7.8).
on the compactified flowspace. Furthermore, $\hat{e}^+$ agrees with $\hat{e}^-$ on the boundary circle, where it restricts to a $\pi_1$-equivariant sphere-filling curve

$$\hat{e} : S^1_u \rightarrow S^2_\infty.$$ 

This generalizes the Cannon-Thurston Theorem, which produces such curves for suspension flows [5].

Notation. Given $A,B \subset S^2_u$ we define

$$\langle A \rightarrow \rangle := (\hat{e}^-)^{-1}(A),$$

$$\langle \rightarrow B \rangle := (\hat{e}^+)^{-1}(B),$$

and

$$\langle A \rightarrow B \rangle := \langle A \rightarrow \rangle \cap \langle \rightarrow B \rangle.$$ 

Given $A \subset S^2_\infty$, we define

$$\langle A \rangle := \hat{e}^{-1}(A).$$ 

The endpoints of each flowline are distinct, so $e^+(p) \neq e^-(p)$ for each $p \in P$. Equivalently, $\langle z \rightarrow z \rangle = \langle z \rangle \subset S^1_u$ for each $z \in S^2_\infty$. Furthermore, if $A,B \subset S^2_\infty$ are disjoint, then $\langle A \rightarrow B \rangle = \{A \rightarrow B\} \subset P$.

2.4. Extended leaves. Distinct positive leaves are disjoint, but their closures may meet in $S^1_u$. The extended endpoint maps provide a convenient way to organize such leaves.

The positive and negative extended decompositions are

$$\hat{D}^+ := \{\text{components of } \langle \rightarrow z \rangle \subset \hat{P} \mid z \in S^2_\infty\}$$

and

$$\hat{D}^- := \{\text{components of } \langle z \rightarrow \rangle \subset \hat{P} \mid z \in S^2_\infty\}.$$ 

The elements of $\hat{D}^\pm$ is called a positive/negative extended leaves. If $p \in \hat{P}$, we write $\hat{D}^+(p)$ and $\hat{D}^-(p)$ for the positive and negative extended leaves through $p$.

It would be convenient if the set of ends $\partial_u \hat{K}$ of an extended leaf $\hat{K}$ were totally disconnected. We can take this to be true using the following construction.

Construction 2.9. Let

$$C = \{\text{components of } \langle z \rangle \mid z \in S^2_\infty\}.$$ 

The quotient of $S^1_u$ obtained by collapsing the elements of $C$ is still a circle. Similarly, the quotient of $\hat{P}$ obtained by collapsing the elements of $\hat{C}$ is still a closed disc.

From now on, we will replace $S^1_u$ and $\hat{P}$ by these quotients. For each $z \in S^2_\infty$, $\langle z \rangle$ is now totally disconnected.

An extended leaf is called trivial if it is contained entirely in $S^1_u$. Each trivial extended leaf consists of a single point, since it is a connected component of a totally disconnected set $\langle z \rangle$.

On the other hand, each nontrivial extended leaf $\hat{K}$ is the closure of a union of leaves, the subleaves of $\hat{K}$.

The following observation is ubiquitous in the sequel; we use it without further mention.

Lemma 2.10. If $\hat{K}$ and $\hat{L}$ are the extended leaves through a point $p \in P$, then $\partial_u \hat{K}$ and $\partial_u \hat{L}$ are disjoint.
Proof. If $\partial_u\hat{K} \cap \partial_u\hat{L} \neq \emptyset$, then $\hat{e}^+(\hat{K}) = \hat{e}^-(\hat{L})$, because the endpoint maps agree on $S^1_k$. Then $e^+(p) = e^-(p)$, a contradiction. □

2.5. Results. We will now summarize our main results.

Fix a quasigeodesic flow $\Phi$ on a closed hyperbolic 3-manifold $M$. If a point $x \in M$ is forward recurrent, then we can build a sequence of elements $g_i \in \pi_1(M)$ that approximates the homotopy class of its forward orbit. Simply take $g_i$ to be the homotopy class of a long forward flow segment closed up with a short arc, chosen so that as $i$ increases the flow segments get longer and the arcs get shorter. This is called an $\omega$-sequence for $x$ (see Section 3.4).

**Homotopy Closing Lemma.** Let $(g_i)_{i=1}^\infty$ be an $\omega$-sequence for a forward recurrent point $x \in M$. If the extended leaves through $x$ are linked, then the $g_i$ represent closed orbits when $i$ is sufficiently large.

By the extended leaves through $x$ we really mean those through a lift of $x$. These are called linked when their ends are linked in the universal circle.

**Recurrent Links Lemma.** Every quasigeodesic flow on a closed hyperbolic 3-manifold has some recurrent point whose extended leaves are linked.

Our main theorem follows immediately from these two lemmas.

**Closed Orbits Theorem.** Every quasigeodesic flow on a closed hyperbolic 3-manifold has closed orbits.

3. Coarse geometry

So far, we have only seen the topological picture of a quasigeodesic flow. In this section we will study its geometry, showing that a quasigeodesic flow has a coarsely hyperbolic transverse structure.

3.1. The comparison map. Fix a quasigeodesic flow $\Phi$ on a closed hyperbolic 3-manifold $M$. We will build a correspondence between the lifted flow on $H^3$ and the geodesic flow on the unit tangent bundle $T^1H^3$.

Each lifted flowline $x\cdot\mathbb{R}$ is an oriented quasigeodesic that shares its endpoints with an oriented geodesic $(x\cdot\mathbb{R})_G$. The nearest-point projection $\rho_{(x\cdot\mathbb{R})_G} : (x\cdot\mathbb{R}) \to (x\cdot\mathbb{R})_G$ moves each point a bounded distance independent of $(x\cdot\mathbb{R})$. To see this, recall that $(x\cdot\mathbb{R})$ is contained in the $C$-neighborhood of $(x\cdot\mathbb{R})_G$ for some uniform constant $C$. This can be pictured as a "banana" foliated by the radius-$C$ hyperbolic discs perpendicular to $(x\cdot\mathbb{R})_G$, and $\rho_{(x\cdot\mathbb{R})}$ simply slides points along these discs to $(x\cdot\mathbb{R})_G$.

Sending each point $x \in H^3$ to the nearest point $\rho_{x\cdot\mathbb{R}}(x)$ in $(x\cdot\mathbb{R})_G$ we obtain a continuous, $\pi_1(M)$-equivariant map

$$G' : H^3 \to H^3.$$  

This takes each flowline to its associated geodesic, and moves points a uniformly bounded distance.

The map $G'$ is not necessarily injective or even monotone along flowlines. To fix this, we work with a monotonized version

$$G : H^3 \to H^3$$

defined by

$$G(x) = \sup_{t \leq 0} G'(x \cdot t).$$
This still moves points a bounded distance, because flowlines have bounded backtracks. That is, if \( x \) and \( x \cdot t \) intersect the same disc in the foliated banana around \((x \cdot \mathbb{R})_G\), then \(|t|\) is less than some uniform bound depending only on the quasigeodesic constants.

Think of \( T^1\mathbb{H}^3 \) as the space of pairs \((\gamma, x)\) where \( \gamma \) is an oriented geodesic and \( x \) is a point in \( \gamma \). We can lift \( G \) to a map

\[
F : \mathbb{H}^3 \to T^1\mathbb{H}^3
\]

by defining

\[
F(x) = ((x \cdot \mathbb{R})_G, G(x)).
\]

3.2. Strong leaves. In a pseudo-Anosov flow, each weak stable/unstable leaf is foliated by strong stable/unstable leaves. We can use the map \( F \) to build an analogous structure for quasigeodesic flows.

Let \( K \in D^+ \) be a positive leaf, which corresponds to a 2-dimensional positive leaf \( K \times \mathbb{R} \). Given a point \( x \in K \times \mathbb{R} \), let \( S^+(x) \) be the horosphere in \( T^1\mathbb{H}^3 \) determined by the vector \( F(x) \). The strong positive leaf through \( x \) is the preimage of this horosphere, intersected with \( K \times \mathbb{R} \). That is,

\[
k_x = \{ z \in K \times \mathbb{R} | S^+(z) = S^+(x) \}.
\]

Each flowline in \( K \times \mathbb{R} \) intersects \( k_x \) nontrivially, so \( \pi_P(k_x) = K \). Since \( F \) is monotone along flowlines, each flowline in \( K \times \mathbb{R} \) intersects \( k_x \) in either a point or a closed interval. Each 2-dimensional positive leaf \( K \times \mathbb{R} \) is decomposed into an \( \mathbb{R} \)'s worth of strong positive leaves, corresponding to the \( \mathbb{R} \)'s worth of horospheres centered at \( e^+(K) \).

Similarly, each 2-dimensional negative leaf \( L \times \mathbb{R} \) is decomposed into strong negative leaves \((l_y)\). For \( y \in L \times \mathbb{R} \), let \( S^-(y) \) be the horosphere determined by \(-F(y)\). The strong negative leaf through \( y \) is

\[
l_y = \{ z \in L \times \mathbb{R} | S^-(z) = S^-(y) \}.
\]

The collections of all strong positive and strong negative leaves form decompositions of \( \mathbb{H}^3 \) called the strong decompositions. These are preserved by deck transformations, so they project to decompositions of \( M \).

3.3. Coarse transverse hyperbolicity. The strong decompositions are not necessarily invariant under the flow. This will not matter since we are interested in the flowlines themselves, not their parametrizations.

Let \( k \) and \( k' \) be two strong positive leaves in \( K \times \mathbb{R} \), where \( k' \) lies in front of \( k \). Given flowlines \( \gamma_1 \) and \( \gamma_2 \) in \( K \times \mathbb{R} \), let \( x_i \in \gamma_i \cap k \) and \( x'_i \in \gamma_i \cap k' \) for \( i = 1, 2 \). The corresponding points \( F(x'_i) \) are obtained by flowing the points \( F(x_i) \) forward under the geodesic flow, so

\[
d(F(x'_1), F(x'_2)) < \lambda \cdot d(F(x_1), F(x_2))
\]

where \( \lambda < 1 \) is arbitrarily small when \( k \) and \( k' \) are sufficiently far apart. The comparison map \( F \) moves points by at most some fixed constant \( D \), so

\[
d(x'_1, x'_2) < \lambda \cdot d(x_1, x_2) + 2D.
\]

Coarse expansion works similarly. Let \( l \) and \( l' \) be strong negative leaves in \( L \times \mathbb{R} \), where \( l' \) lies behind \( l \). Given flowlines \( \gamma_1 \) and \( \gamma_2 \) in \( L \times \mathbb{R} \), let \( y_i \in \gamma_i \cap l \) and \( y'_i \in \gamma_i \cap l' \) for \( i = 1, 2 \). We have to be a little careful: if \( \gamma_1 \) and \( \gamma_2 \) have the same positive
endpoint then $F(y_1) = F(y_2)$ and $F(y'_1) = F(y'_2)$. However, if $e^+(\gamma_1) \neq e^+(\gamma_2)$ then
\[ d(y'_1, y'_2) > \lambda \cdot d(y_1, y_2) - 2D \]
where $\lambda$ is arbitrarily large when as $l$ and $l'$ are sufficiently far apart.

3.4. Dynamics in the flowspace. In the introduction we sketched a proof of the Anosov Closing Lemma that worked directly in the manifold $M$. This idea is difficult to generalize to a quasigeodesic flow, since our contraction-expansion is only coarse. Instead, we will work in the universal cover, translating some dynamical properties of the flow to the flowspace.

Let $x$ be a point in the manifold $M$. If $x \cdot t_i \to y$ for a sequence of times $t_i \to \infty$, then $y$ is called an $\omega$-limit point of $x$. The $\omega$-limit set $\omega(x)$ is the set of all $\omega$-limit points of $x$. Since our manifold is compact, every point has a nontrivial $\omega$-limit set.

For each $x \in M$, the $\omega$-limit set $\omega(x)$ is clearly flow-invariant. Furthermore, it is an invariant of the orbit $x \cdot \mathbb{R}$. Therefore, it makes sense to write $y \cdot \mathbb{R} \in \omega(x) \cdot \mathbb{R}$ whenever $y \in \omega(x)$.

An orbit $x \cdot \mathbb{R}$ is $\omega$-recurrent if it is in its own $\omega$-limit set. In particular, closed orbits are $\omega$-recurrent.

We will work with $\omega$-limit sets on the universal cover. Given points $p, q \in P$, we’ll say that $q \in \omega(p)$ if this holds for the corresponding orbits in $M$. Notice that $q \in \omega(p)$ if and only if there is a sequence of points $x_i \in p \times \mathbb{R}$ that escape to $e^+(p)$ and a sequence of deck transformations $g_i$ such that $\lim_{i \to \infty} g_i(x_i) = x_\infty \in q \times \mathbb{R}$. Such a sequence $(g_i)$ is called an $\omega$-sequence for $q \in \omega(p)$.

A point $p \in P$ corresponds to a closed orbit in $M$ if and only if it is fixed by some nontrivial element of $\pi_1(M)$. If $g(p) = p$, then $g$ represents a power of the free homotopy class of the corresponding orbit, and we will say that $g$ represents this closed orbit. In this case, either $(g^i)_{i=1}^{\infty}$ or $(g^{-i})_{i=1}^{\infty}$ is an $\omega$-sequence for $p \in \omega(p)$.

3.5. Dynamics of $\omega$-sequences.

**Lemma 3.1.** Let $(g_i)$ be an $\omega$-sequence for a recurrent point $p \in P$. For each $i$ let $\alpha_i$ and $\rho_i$ be the attracting and repelling fixed points for $g_i$ in $S^2_\infty$. Then $\lim_{i \to \infty} \alpha_i = e^-(p)$ and $\lim_{i \to \infty} \rho_i = e^+(p)$.

**Proof.** Since $(g_i)$ is an $\omega$-sequence, there are points $x_i \in p \times \mathbb{R}$ that escape to $e^+(p)$ such that $\lim_{i \to \infty} g_i(x_i) = x_\infty \in p \times \mathbb{R}$. For each $i = 1, 2, \ldots$, let $x'_i = F(x_i)$.

Let $H_i$ be the hyperplane perpendicular to $x'_i$ for each $i$. The closure of $H_i$ in $\mathbb{H}^3$ separates $S^2_\infty$ into two discs, $D^+_i \ni e^+(p)$ and $D^-_i \ni e^-(p)$. When $i$ is large, $D^+_i$ is a small disc near $e^+(p)$, and $g_i(H_i)$ is close to $H_\infty$, so $g_i(D_i) \supset D_i$. It follows that $\rho_i \in D_i$ and $\lim_{i \to \infty} \rho_i = e^+(p)$ as desired.

On the other hand, when $i$ is large, $g_i(D^-_\infty)$ is a small ball around $g_i(e^-(p))$, which lies near $e^-(p)$. Then $g_i(D^-_\infty) \subset D^-_\infty$, so $\alpha_i \in g_i(D^-_\infty)$. Then $\lim_{i \to \infty} \alpha_i = e^-(p)$ as desired.

Our proof of the Homotopy Closing Lemma will use the following version of coarse contraction-expansion, which takes place entirely in the flowspace.

**Proposition 3.2** (Coarse contraction-expansion). Let $(g_i)$ be an $\omega$-sequence for $q \in \omega(p)$.
(1) If \( r \) is a point in \( P \) with \( e^+(r) = e^+(p) \), then the accumulation points of \( g_i(r) \) lie in the compact subset
\[
\{ e^-(q) \to e^+(q) \} \subset P.
\]

(2) If \( r \) is a point in \( P \) with \( e^-(r) = e^-(p) \) and \( e^+(r) \neq e^+(p) \), then the accumulation points of \( g_i(r) \) lie in
\[
\{ e^-(q) \} \subset S^1_u.
\]

Proof. Since \( (g_i) \) is an \( \omega \)-sequence, there are points \( x_i \in p \times \mathbb{R} \) that escape to \( e^+(p) \) such that \( \lim_{i \to \infty} g_i(x_i) = x_\infty \in q \times \mathbb{R} \).

Claim (1): For each \( i \), let \( y_i \in r \times \mathbb{R} \) be a point that lies in the same strong positive leaf as \( x_i \in p \times \mathbb{R} \). Then \( x'_i := F(x_i) \) and \( y'_i := F(y_i) \) are both contained in the horosphere \( S^+_i := S^+(x_i) = S^+(y_i) \). See Figure 3.

Notice that \( \lim_{i \to \infty} g_i(x'_i) = x'_\infty \). We will show that \( \lim_{i \to \infty} g_i(y'_i) = x'_\infty \) and the result follows.

![Figure 3. Coarse contraction.](image)

When \( i \) is large, \( d(x'_i, y'_i) \) is small, and \( g_i \) is an isometry, so \( d(g_i(x'_i), g_i(y'_i)) \) is small (here, \( d \) denotes the distance along the appropriate horosphere). Furthermore, \( g_i(S^+_i) \) is close to \( S^+_\infty \), and \( g_i(x'_i) \) is close to \( x'_\infty \) so
\[
\lim g_i(y'_i) = \lim g_i(x'_i) = x'_\infty
\]
as desired.

Claim (2): We need to show that \( \lim_{i \to \infty} e^\pm(g_i(p')) = e^-(q) \).

For each \( i \), let \( y_i \in r \times \mathbb{R} \) be a point in the same strong negative leaf as \( x_i \in p \times \mathbb{R} \). Then \( x'_i := -F(x_i) \) and \( y'_i := -F(y_i) \) are both contained in the horosphere \( S^-_i := S^-(x_i) = S^-(y_i) \).

When \( i \) is large, \( d(g_i(x'_i), g_i(y'_i)) \) is large. Furthermore, \( g_i(x'_i) \) is close to \( x'_\infty \), and \( S^-_i \) is close to \( S^-_\infty \), so
\[
\lim_{i \to \infty} e^\pm(g_i(r)) = e^-(q)
\]
as desired.

\( \square \)
4. Topology in the flowspace

Our search for closed orbits will use three major ingredients:

1. coarse contraction and expansion,
2. the "foliation-like" behavior of the positive and negative decompositions, and
3. the "pseudo-Anosov-like" character of the universal circle action.

The latter two points will require a considerable amount of topological work. For motivation, we will sketch a special case of the Homotopy Closing Lemma.

4.1. Homotopy Closing for 3-prongs. Let \( p \in P \) be an \( \omega \)-recurrent point, and let \( \hat{K} \) and \( \hat{L} \) be the positive and negative extended leaves through \( p \). Suppose that these are topological 3-prongs in standard position; see Figure 4(a). Let \( (g_i) \) be an \( \omega \)-sequence for \( p \in \omega(p) \). We will show that each \( g_i \) represents a closed orbit for \( i \) sufficiently large.

![Figure 4](image-url)

**Figure 4.** Sketch: homotopy closing for 3-prongs.

**Sketch.** Let \( (\hat{J}_i) \) be a Hausdorff convergent sequence of, say, positive extended leaves. In Section 4.2 we will see that \( \lim \hat{J}_i \) is contained in a single positive extended leaf.

If \( g_i(\hat{K}) = \hat{K} \) then \( g_i(p) = p \) and \( g_i \) represents a closed orbit. If only finitely many \( g_i \) represent closed orbits, then after taking a subsequence we can assume that \( g_i(\hat{K}) \neq \hat{K} \) for each \( i \).

After taking a further subsequence, we can assume that all of the \( g_i(\hat{K}) \) are contained on one side of \( \hat{K} \), \( \lim g_i(\hat{K}) \subset \hat{K} \), and \( \lim g_i(\hat{L}) \subset \hat{L} \). See Figure 4(b). In the limit, two ends of \( g_i(\hat{K}) \) must collapse to a single point. Some end of \( g_i(\hat{L}) \) is trapped between these two, so \( g_i(\hat{L}) \) must accumulate on this same point. Then \( \partial_n \hat{K} \) and \( \partial_n \hat{L} \) intersect at this point, a contradiction. Thus the \( g_i \) must eventually represent closed orbits. \( \square \)

Notice that we haven’t used coarse contraction-expansion. This only comes into play when \( \hat{K} \) and \( \hat{L} \) are 2-prongs, i.e. lines (see Proposition 5.7).
4.2. Hausdorff limits of extended leaves. Let \((A_i)_{i=1}^{\infty}\) be a sequence of subsets of a space \(X\). Then
\[
\lim A_i = \{ x \in X \mid \text{every neighborhood of } x \text{ intersects all but finitely many } A_i \}
\]
and
\[
\varprojlim A_i = \{ x \in X \mid \text{every neighborhood of } x \text{ intersects infinitely many } A_i \}.\]

If these limits agree then \((A_i)\) is Hausdorff convergent and
\[
\lim A_i = \varprojlim A_i = \varprojlim A_i
\]
is its Hausdorff limit. See [13].

If \(X\) is a compact metric space, then every sequence \((A_i)\) has a Hausdorff convergent subsequence. Furthermore, if each \(A_i\) is closed and connected then \(\lim A_i\) is closed and connected.

The following lemma is stated for positive extended leaves, but it holds as well for negative extended leaves.

**Lemma 4.1.** Let \((\hat{K}_i)_{i=1}^{\infty}\) be a sequence of positive extended leaves. If \(\lim \hat{K}_i\) intersects an positive extended leaf \(\hat{K}\), then \(\lim \hat{K}_i \subset \hat{K}\).

In particular, if \(p_i \in \hat{P}\) are points with \(\lim p_i = p_\infty\), then \(\varprojlim \hat{D}^\pm(p_i) \subset \hat{D}^\pm(p_\infty)\).

**Proof.** Let \(X\) be a compact metric space. A decomposition \(D\) of \(X\) is said to be upper semicontinuous if it satisfies any of the following equivalent conditions (see Section 3.6 in [13]):

1. If \(U \subset X\) is an open set containing a decomposition element \(K \in D\), then there is an open set \(V \supset K\) such that each decomposition element that intersects \(V\) is contained in \(U\).
2. If \(U \subset X\) is open, then the union of all decomposition elements that are contained in \(U\) is open.
3. If \(A \subset X\) is closed, then the union of all decomposition elements that intersect \(A\) is closed.
4. If \((K_i)\) is a sequence of decomposition elements, and \(\lim K_i\) intersects a decomposition element \(K_\infty\), then \(\varprojlim K_i \subset K_\infty\).

If \(f : X \to Y\) is a continuous, then the decomposition of \(X\) by point preimages is upper semicontinuous. If \(D\) is upper semicontinuous then the decomposition \(D'\) consisting of connected components of elements of \(D\) is upper semicontinuous. It follows that \(\hat{D}^+\) and \(\hat{D}^-\) are upper semicontinuous, and our lemma is condition (4).

If \((g_i)\) is an \(\omega\)-sequence for \(q \in \omega(p)\), and \(r \in P\) is contained in the positive extended leaf through \(p\), then by coarse contraction, together with this lemma,
\[
\varprojlim g_i(r) \subset \hat{D}^+(q) \cap \{ e^-(q) \to e^+(q) \}.
\]

If \(r \in \hat{P}\) is contained in the negative extended leaf through \(p\), and \(e^+(r) \neq e^+(p)\), then
\[
\varprojlim g_i(r) \subset \partial_{\omega} \hat{D}^-(q).
\]
4.3. Complementary components. If \( A \subset \hat{P} \), each component of \( \hat{P} \setminus A \) is called a complementary component. If \( B \subset S^1_u \), each component of \( S^1_u \setminus B \) is called a complementary interval.

**Lemma 4.2.** If \( U \) is a complementary component of an extended leaf \( \hat{K} \), then \( \partial_u U \) is a complementary interval of \( \partial_u \hat{K} \). Furthermore, the correspondence \( U \mapsto \partial_u U \) is a bijection between the complementary components of \( \hat{K} \) and complementary intervals of \( \partial_u \hat{K} \).

**Proof.** For concreteness, let \( \hat{K} \) be positive.

Let \( U \) be a complementary component of \( \hat{K} \). We will start by showing that \( \partial_u U \neq \emptyset \). Choose a point \( p \in U \) with \( e^+(p) \neq \hat{K} \), and let \( \hat{L} \) be the extended positive leaf through \( p \). Then \( \hat{L} \) is contained in \( U \), so \( \partial_u U \supset \partial_u L \neq \emptyset \). Clearly, \( \partial_u U \) is a union of complementary intervals of \( \hat{K} \).

If \( \partial_u U \) were disconnected, then we could find an arc \( \gamma \subset U \) with endpoints in distinct components of \( \partial_u U \). This would disconnect \( \hat{K} \), so \( \partial_u U \) is a single complementary interval.

Each complementary interval of \( \partial_u \hat{K} \) is contained in some complementary component of \( \hat{K} \), so \( U \mapsto \partial_u U \) is surjective. If \( U \) and \( V \) are complementary components of \( \hat{K} \), and \( \partial_u U = \partial_u V \), then \( U = V \), so \( U \mapsto \partial_u U \) is injective.

Given \( a, b \in S^1_u \), the oriented interval between \( a \) and \( b \) is denoted \((a, b)\).

The following two lemmas are stated for positive leaves, but they work as well for negative leaves.

**Lemma 4.3.** Let \( U \) be a complementary component of a positive extended leaf \( \hat{K} \) with \( \partial_u U = (k, k') \). If \( (\hat{K}_i)_{i=1}^\infty \) is a sequence of extended positive leaves in \( U \), and \( \lim \hat{K}_i \) intersects \( \hat{K} \), then \( \lim \partial_u \hat{K}_i \subset (k, k') \).

**Proof.** The is an immediate consequence of Lemma 4.1.

If \( \hat{K} \) is an extended leaf, we can find extended leaves that lie arbitrarily close to \( \hat{K} \) in each complementary component:

**Lemma 4.4.** Let \( U \) be a complementary component of an extended positive leaf \( \hat{K} \). If \( C \subset U \) is compact, then there is a positive leaf \( L \subset U \) whose closure is a complementary component of \( \hat{K} \) from \( C \).

**Proof.** We will deal with the positive case for concreteness. Let us begin by finding an extended leaf \( \hat{K} \) that separates \( \hat{K} \) from \( C \).

Choose an arc \( A \subset U \) that separates \( \hat{K} \) from \( C \), and let \( B \) be the union of all positive extended leaves that intersect \( A \). This is obviously connected, and it is compact by property (3) of an upper semicontinuous decomposition. Let \( V \) be the component of \( \hat{P} \setminus (\hat{K} \cup B) \) that lies between \( \hat{K} \) and \( B \). Then \( B' = V \cap B \) is a connected set that separates \( A \) from \( \hat{K} \). We will show that \( B' \) is contained in a single extended leaf \( \hat{L} \).

Note that \( \partial_u V = (k, k') \cup (b, b') \), where \( k, k' \in \partial_u \hat{K} \) and \( b, b' \in \partial_u B' \).

Given a point \( p \in B' \), choose a sequence of points \( p_i \in V \) that converge to \( p \), and let \( \hat{L}_i = \hat{D}^+(p_i) \) be the corresponding positive extended leaves. Then \( \lim \hat{L}_i \subset \hat{D}^+(p) \). The ends of \( \hat{L}_i \) converge to a subset of \( \{b, b'\} \), so \( \hat{D}^+(p) = \hat{D}^+(b) \) or \( \hat{D}^+(b') \).
This works for every $p \in B'$, so
\[ B' \subset \tilde{D}^+(b) \cup \tilde{D}^+(b'). \]

In fact, $\tilde{D}^+(b) = \tilde{D}^+(b')$, since otherwise this would be a separation of $B'$. Therefore, $\hat{L} := \tilde{D}^+(b) = \tilde{D}^+(b')$ separates $K$ from $C$.

It remains to find a subleaf of $\hat{L}$ whose closure separates $\hat{K}$ from $C$. We use the following fact (see, e.g., [9] Lemma 2.7): if $A$ is a subset of the plane that separates points $x$ and $y$, then some component of $A$ separates $x$ and $y$.

Let $x$ and $y$ be points in $P$ that lie in the components of $\hat{P} \setminus \hat{L}$ that contain $\hat{K}$ and $C$ respectively. Since $\hat{L} \cap P$ separates $x$ and $y$, some component $L \subset \hat{L} \cap P$ does too. This is a subleaf of $\hat{L}$, and $\overline{L}$ separates $\hat{K}$ from $C$. \hfill \Box

4.4. Master leaves. Given a point $z \in S^2_\infty$, the master leaf corresponding to $z$ is
\[ Z := \{ z \to \} \cup \{ \to z \} \subset \hat{P}. \]

In other words, $Z$ is the union of all extended leaves, both positive and negative, that map to $z$.

**Lemma 4.5.** Each master leaf is connected.

**Proof.** Fix a master leaf $Z$ corresponding to a point $z \in S^2_\infty$. Let $B_1 \supset B_2 \supset \cdots$ be a nested sequence of closed balls around $z$ in $\mathbb{H}^3$ with $\bigcap_i B_i = z$. For each $i$, let $C_i = \pi_p(B_i \cap \mathbb{H}^3)$.

The $\overline{C}_i$ are compact and connected, so $\overline{C} := \bigcap_i \overline{C}_i$ is compact and connected. We will show that $Z = \overline{C}$.

Let $p \in Z \cap P$. This corresponds to a flowline with one end at $z$, which clearly intersects every $B_i$. Then $p \in C_i$ for each $i$, and hence $p \in \overline{C}$.

Let $p \in Z \setminus S^2_u$. For each $i$, let $U_i$ be a neighborhood of $z$ in $S^2_\infty$ that is contained in $B_i$. Then $\langle \to U_i \rangle$ is an open neighborhood of $p$, and $\{ \to U_i \} = \langle \to U_i \rangle \cap P$, so $p \in \{ \to U_i \}$. Furthermore, $\{ \to U_i \} \subset C_i$ for each $i$ by the preceding argument, so $p \in \overline{C}$.

Let $p \in \hat{P} \setminus Z$. For $i$ sufficiently large, we can find neighborhoods $V_{\pm}$ of $\hat{e}^\pm(p)$ in $S^2_\infty$ such no flowline in $\{ V_{-} \to V_{+} \}$ intersects $B_i$, and hence $\{ V_{-} \to V_{+} \}$ is disjoint from $C_i$. Here, we’re using the fact that each flowline is contained in a neighborhood of uniform size of the corresponding geodesic.

If $p \in P$ then it is contained in $\{ V_{-} \to V_{+} \}$, so $p \notin \overline{C}$. If $p \in S^2_u$, then it is contained in the interior of $\{ V_{-} \to V_{+} \} \cap S^1_u$. Only the frontier of this set may be contained in $\overline{C}$, so $p \notin \overline{C}$. \hfill \Box

A master leaf is called trivial if it is contained in $S^1_u$. As with extended leaves, trivial master leaves are points.

4.5. The universal sphere. Take two copies of the compactified flowspace, and identify their boundaries to form a sphere.

That is, let $i^\pm : \hat{P} \to \hat{P}^\pm$ be homeomorphisms, and let
\[ S^2_u = \hat{P}^+ \cup \hat{P}^- / i^+(a) \sim i^-(a) \text{ for each } a \in S^1_u. \]
We will think of $\hat{P}^+$ and $\hat{P}^-$ as the northern and southern hemispheres of the *universal sphere* $S^2_u$. These meet at the equator, which is identified with $S^1_u$.

We can think of the endpoint maps as maps
\[ \hat{e}^\pm : \hat{P}^\pm \to S^2_\infty \]
defined on the hemispheres of the universal sphere. These agree on the equator, so they define a map
\[ \tilde{e} : S^2_u \to S^2_\infty. \]

The preimage $\tilde{e}^{-1}(z)$ of a point $z \in S^2_\infty$ is the union of all positive and negative extended leaves that map to $z$, where we think of positive leaves as lying in the northern hemisphere and negative leaves as lying in the southern hemisphere. The collection
\[ \tilde{D} = \{ \tilde{e}^{-1}(z) \mid z \in S^2_\infty \} \]
of such point preimages forms an upper semicontinuous decomposition of $S^2_u$.

Notice that the projection
\[ \pi : S^2_u \to \hat{P} \]
that identifies the two hemispheres takes $\tilde{Z} = \tilde{e}^{-1}(z)$ to a corresponding master leaf $Z$. We will call $\tilde{Z} \in \tilde{D}$ a lifted master leaf.

**Lemma 4.6.** *Lifted master leaves are compact, connected, and nonseparating.*

*Proof.* Let $\tilde{Z} = \tilde{e}^{-1}(z)$. This is compact since it is a closed subset of the compact space.

If $\tilde{Z}$ were disconnected, then we could write it as a disjoint union
\[ \tilde{Z} = A \sqcup B \]
of compact sets $A$ and $B$. Then $\pi(A)$ and $\pi(B)$ are disjoint. Indeed, $\pi(A)$ does not intersect $\pi(B)$ in $S^1_u$ since $\pi$ restricts to a homeomorphism on the equator. Also, $\pi(A)$ does not intersect $\pi(B)$ in $P$, since otherwise the corresponding master leaf $Z$ would have positive and negative subleaves that intersect in $P$. Therefore,
\[ Z = \pi(A) \sqcup \pi(B) \]
would be a separation of $Z$. But $Z$ is connected, so $\tilde{Z}$ must be connected.

It remains to show that $\tilde{Z}$ is nonseparating. Notice that $S^2_{\infty}$ is identified with the quotient
\[ S^2_u / \tilde{D} \]
obtained by collapsing each lifted master leaf to a point. If $\tilde{Z}$ were separating then it would map to a cut point in $S^2_\infty$, which is impossible. \qed

As an application, we have the following important lemma.

**Lemma 4.7** (No bigons). *Let $\hat{K}$ and $\hat{L}$ be extended leaves of any type (i.e. both positive, both negative, or one positive and one negative). Then $\partial_u \hat{K}$ intersects $\partial_u \hat{L}$ in at most one point.*

*Proof.* We will use following fact from classical analysis situs (see Theorem II.5.28a in [22]): If $A$ and $B$ are compact connected subsets of $S^2$, and $A \cap B$ is disconnected, then $A \cup B$ separates $S^2$. 

Think of $\hat{K}$ and $\hat{L}$ in the appropriate hemispheres of $S^2_u$. If they intersect in $S^1_u$, then they are contained in a lifted master leaf $\tilde{Z}$. If they intersect at more than one point in $S^1_u$, then $\tilde{Z}$ is separating, contradicting the preceding lemma. □

**Remark 4.8.** The universal sphere can be used to build a compactification of $\tilde{M}$ called the flow ideal compactification (cf. [5] and [6]).

Take $\hat{P} \times [-1, 1]$, and collapse each vertical interval in $S^1_u \times [-1, 1]$ to a point. This is a compactification of $\tilde{M} \cong \mathbb{H}^3$ whose boundary is $S^2_u$. It can be pictured as a “lens” foliated by the vertical segments connecting $i^-(p)$ to $i^+(p)$ for each $p \in P$. Collapsing the decomposition $\tilde{D}$ we recover the usual compactification $\mathbb{H}^3$ of $\mathbb{H}^3$, together with the foliation by flowlines.

4.6. **Linking.** Two disjoint subsets $A, B \subset S^1_u$ are $n$-linked if $B$ intersects exactly $n$ complementary components of $A$; equivalently, if $A$ intersects exactly $n$ complementary components of $B$.

The extended leaves through a point $p \in P$ are said to be $n$-linked if their ends are $n$-linked in $S^1_u$. They are $(\geq n)$-linked if they are $m$-linked for $m \geq n$, and linked if they are $(\geq 2)$-linked.

By the following lemma, the extended leaves through each point are $n$-linked for some finite $n$.

**Lemma 4.9.** Let $\hat{K}$ and $\hat{L}$ be the positive and negative extended leaves through a point $p \in P$. Then $\partial_u \hat{L}$ intersects only finitely many complementary components of $\partial_u \hat{K}$.

**Proof.** If $\partial_u \hat{L}$ intersects infinitely many complementary components of $\partial_u \hat{K}$, then we can find ends $k_i \in \partial_u \hat{K}$ and $l_i \in \partial_u \hat{L}$ so that $\lim_{i \to \infty} k_i = \lim_{i \to \infty} l_i$. The ends of extended leaves are closed, so this means that $\hat{K}$ intersects $\hat{L}$ in $S^1_u$, a contradiction. □

The idea of linking and $n$-linking works as well for leaves. If the leaves through a point are $n$-linked then the extended leaves through that point are $(\geq n)$-linked.

The following lemma is an immediate application of the Pigeonhole Principle.

**Lemma 4.10** (Linking Pigeonhole Principle). Suppose that $\hat{K}$ and $\hat{L}$ are $(\geq n)$-linked, and let $A$ be a subset of $S^1_u$ that contains the ends of $\hat{K}$. If $A$ has less than $n$ components then it also contains an end of $\hat{L}$.

5. **Closed orbits**

We now turn to the problem of finding closed orbits.

5.1. **Closed orbits and master leaves.**

**Lemma 5.1.** If $g \in \pi_1(M)$ fixes a nontrivial master leaf, then it represents a closed orbit.

**Proof.** Let $\alpha$ and $\rho$ be the attracting and repelling fixed points of $g$ in $S^2_\infty$, and let $A$ and $R$ be the corresponding master leaves. These are the only master leaves fixed by $g$.

Suppose that $R$ is nontrivial. Then $R \cap P$ is nonempty. A point $p \in R \cap P$ corresponds to a flowline with one endpoint at $\rho$. Applying $g$ takes the other
endpoint closer to \( \alpha \), so the forward orbit

\[ g(p), g^2(p), g^3(p), \ldots \]

remains in a bounded subset of \( P \). The Brouwer Plane Translation Theorem\(^2\) then implies that \( g \) has a fixed point in \( P \), and hence represents a closed orbit.

If \( A \) is nontrivial, then replace \( g \) by \( g^{-1} \). This interchanges \( A \) and \( R \), and we can use the same argument. \( \square \)

5.2. Closed orbits and the universal circle. A group \( \Gamma \) of orientation-preserving homeomorphisms of \( S^1 \) is said to be pA-like if for each \( g \in \Gamma \), some positive power \( g^n \) has an even number of fixed points, alternately attracting and repelling.

We will see that the action of \( \pi_1(M) \) on \( S^1_u \) is pA-like.

**Lemma 5.2.** Let \( g \) be an element of \( \pi_1(M) \). If \( g \) fixes some point in \( S^1_u \), then it has an even number of fixed points, alternately attracting and repelling.

**Proof.** Let \( \alpha \) and \( \rho \) be the attracting and repelling fixed points of \( g \) in \( S^2_{u} \). Let \( F \) be the set of fixed points of \( g \) in \( S^1_u \). We will show that \( F = \langle \alpha \rangle \cup \langle \rho \rangle \), where each point in \( \langle \alpha \rangle \) is attracting, and each point in \( \langle \rho \rangle \) is repelling.

If \( x \in S^1_u \) is not in \( \langle \alpha \rangle \) or \( \langle \rho \rangle \), then \( \lim_{i \to \infty} e_u(g^i(x)) = \alpha \), so \( \lim_{i \to \infty} g^i(x) \in \langle \alpha \rangle \). Similarly, \( \lim_{i \to -\infty} g^i(x) \in \langle \rho \rangle \).

In particular, if \( x \in S^1_u \setminus (\langle \alpha \rangle \cup \langle \rho \rangle) \), then \( g(x) \neq x \). Therefore, \( F \subset (\langle \alpha \rangle \cup \langle \rho \rangle) \).

Let \( I \) be a complementary interval of \( F \). Then \( g \) acts as a translation on \( I \), fixing its endpoints. One of these endpoints, \( a_I \), is attracting, while the other, \( r_I \), is repelling (with respect to points in \( I \)). Then \( a_I \in \langle \alpha \rangle \) and \( r_I \in \langle \rho \rangle \). Indeed, take \( x \in I \setminus (\langle \alpha \rangle \cup \langle \rho \rangle) \). Then \( a_I = \lim_{i \to \infty} g^i(x) \in \langle \alpha \rangle \), and \( r_I = \lim_{i \to -\infty} g^i(x) \in \langle \rho \rangle \).

Let \( a \in \langle \alpha \rangle \). If \( g(a) \neq a \), then \( a \) is contained in some complementary interval \( J \) of \( F \). Then \( \lim_{i \to -\infty} g^i(a) = r_J \). But \( \langle \alpha \rangle \) is \( g \)-invariant, so this means that \( r_J \in \langle \alpha \rangle \), a contradiction. Hence \( \langle \alpha \rangle \subset F \). Similarly, \( \langle \rho \rangle \subset F \), so \( F = \langle \alpha \rangle \cup \langle \rho \rangle \).

Each \( a \in \langle \alpha \rangle \) is the boundary of two complementary intervals of \( F \). It is attracting with respect to each of these intervals, and hence attracting overall. Similarly, each point in \( \langle \rho \rangle \) is repelling. Clearly, the attracting fixed points must alternate with the repelling fixed points. \( \square \)

**Lemma 5.3.** Let \( g \) be an element of \( \pi_1(M) \). Then some positive power \( g^n \) has fixed points in \( S^1_u \)

**Proof.** Suppose that \( F = \emptyset \).

Let \( J = (a, r) \) be a complementary interval of \( \langle \alpha \rangle \cup \langle \rho \rangle \) bounded by \( a \in \langle \alpha \rangle \) and \( r \in \langle \rho \rangle \). If \( g^i(J) = g^j(J) \) for some \( i > j > 0 \) then \( g^{i-j}(J) = J \), so \( g^{-j}(a) = a \) and we are done.

Otherwise, the intervals \( J, g^1(J), g^2(J), \ldots \) are disjoint, hence their diameters must go to zero. After taking a subsequence, \( g^i(J) \) converges to a point, and we have \( \lim_{i \to \infty} g^i(a) = \lim_{i \to -\infty} g^i(b) \). This contradicts the fact that \( \langle \alpha \rangle \cap \langle \rho \rangle = \emptyset \). \( \square \)

**Lemma 5.4.** Suppose that \( g \in \pi_1(M) \) does not represent a closed orbit. Then it acts on \( S^1_u \) with exactly two fixed points in an attracting-repelling pair.

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\(^2\)If \( f \) is a homeomorphism of the plane with a bounded forward orbit, then \( f \) has a fixed point. See, e.g., [10]
Proof. Note that \( F \neq \emptyset \). Indeed, if \( F = \emptyset \) then \( g \) must fix a point in \( P \) by Brouwer’s fixed point theorem, and hence represent a closed orbit.

Let \( \langle \alpha \rangle \) and \( \langle \rho \rangle \) be the attracting and repelling fixed point sets as in the preceding lemmas. Notice that \( \langle \alpha \rangle = \partial_\alpha A \) and \( \langle \rho \rangle = \partial_\rho R \), where \( A \) and \( R \) are the master leaves corresponding to \( \alpha \) and \( \rho \).

If \( \langle \alpha \rangle \) were disconnected then \( A \) would be nontrivial, and if \( \langle \rho \rangle \) were disconnected then \( R \) would be nontrivial. Either way, \( g \) would represent a closed orbit by Lemma 5.1. \( \square \)

5.3. The closing lemma. We will now prove our first major result.

**Homotopy Closing Lemma.** Let \( (g_i)_{i=1}^\infty \) be an \( \omega \)-sequence for a recurrent point \( p \in P \). If the extended leaves through \( p \) are linked, then each \( g_i \) represents a closed orbit for \( i \) sufficiently large.

We begin with the \((\geq 3)\)-linked case.

**Proposition 5.5** (Homotopy closing for \((\geq 3)\)-links). Let \( (g_i) \) be an \( \omega \)-sequence for a recurrent point \( p \in P \). If the extended leaves through \( p \) are \((\geq 3)\)-linked, then each \( g_i \) represents a closed orbit for \( i \) sufficiently large.

Proof. Let \( \hat{K} = \hat{\mathcal{D}}^+(p) \) and \( \hat{L} = \hat{\mathcal{D}}^-(p) \) be the extended leaves through \( p \). For each \( i \), let \( \hat{K}_i = g_i(\hat{K}) \) and \( \hat{L}_i = g_i(\hat{L}) \).

Suppose that there are infinitely many \( g_i \) that do not represent closed orbits, and take a subsequence consisting only of these. By Lemma 5.1, \( \hat{K}_i \neq \hat{K} \) and \( \hat{L}_i \neq \hat{L} \) for each \( i \).

The \( \hat{K}_i \) can visit at most finitely many complementary components of \( \hat{K} \). Indeed, let \( U_i \) be the complementary component that contains \( \hat{K}_i \) for each \( i \). Then \( \partial_\alpha U_i \) contains the ends of \( \hat{K}_i \), hence it must also contain an end of \( \hat{L}_i \). If infinitely many of the \( U_i \) were distinct, the diameters of the corresponding complementary intervals would go to zero. Then \( \hat{K}_i \) and \( \hat{L}_i \) would both accumulate on the same point in \( S^1 \), and \( \partial_\alpha \hat{K} \cap \partial_\alpha \hat{L} \neq \emptyset \), a contradiction.

Therefore, infinitely many \( \hat{K}_i \) lie in a single complementary interval \( U = U_i \). After taking a subsequence, \( \partial_\alpha \hat{K}_i \) accumulates on at most two points, the boundary points of \( \partial_\alpha U \). Since \( \hat{L}_i \) is \((\geq 3)\)-linked with \( \hat{K}_i \), the Linking Pigeonhole Principle says that \( \hat{L}_i \) also accumulates on one of these points. Again, this means that \( \hat{K} \) and \( \hat{L} \) intersect at this point, a contradiction. \( \square \)

In fact, any \((\geq 3)\)-linked pair of extended leaves \( \hat{K}, \hat{L} \) that meet in \( P \) are fixed by some \( g \in \pi_1(M) \). Simply take a point \( p \in \hat{K} \cap \hat{L} \), and an \( \omega \)-limit point \( q \in \omega(p) \). The elements \( g_i \) of an \( \omega \)-sequence eventually take \( \hat{K} \) and \( \hat{L} \) to the extended leaves through \( q \), so \( g_i^{-1} g_j \) fixes \( \hat{K} \) and \( \hat{L} \) for large \( i \neq j \).

**Remark 5.6.** The reader may notice that our proof works with the extended leaves through \( p \), quickly forgetting about \( p \) itself. In fact, \( p \) may not correspond to a closed orbit.

For example, start with the suspension flow of a pseudo-Anosov diffeomorphism, which is both quasigeodesic and pseudo-Anosov. Blow up some singular orbit \( \gamma \) to a solid torus foliated by parallel closed orbits. One may perturb the flow on this solid torus, breaking some of the closed orbits while keeping the flow quasigeodesic.

Using coarse contraction-expansion we can extend this to the 2-linked case.
Proposition 5.7 (Homotopy closing for 2-links). Let \((g_i)\) be an \(\omega\)-sequence for a recurrent point \(p \in P\). If the extended leaves through \(p\) are 2-linked, then each \(g_i\) represents a closed orbit for \(i\) sufficiently large.

Proof. Let \(\hat{K} = \hat{D}^+(p)\) and \(\hat{L} = \hat{D}^-(p)\) be the extended leaves through \(p\). For each \(i\), let \(\hat{K}_i = g_i(\hat{K})\), and \(\hat{L}_i = g_i(\hat{L})\).

Suppose that infinitely many \(g_i\) do not represent closed orbits, and take a subsequence consisting only of these. In particular, this means that \(\hat{K}_i \neq \hat{K}\) and \(\hat{L}_i \neq \hat{L}\) for each \(i\). By Lemma 5.4, each \(g_i\) acts on \(S_u^1\) with exactly two fixed points, \(a_i\) and \(r_i\), in an attracting-repelling pair.

As in the preceding lemma, we can assume that each \(\hat{K}_i\) is contained in a single complementary component \(U\) of \(\hat{K}\). The corresponding complementary interval is of the form \(\partial_u U = (k, k')\), where \(k, k' \in \partial_u \hat{K}\) (Figure 5). Taking a further subsequence, the points \(k_i := g_i(k)\) and \(k'_i := g_i(k')\) converge to either \(k\) or \(k'\).

Furthermore, \(\lim k_i \neq \lim k'_i\), since otherwise \(\lim k_i \in \lim \hat{L}_i\).

Step 1:

Suppose that \(\lim k_i = k'\) and \(\lim k'_i = k\).

For \(i\) sufficiently large, \(g_i\) takes the interval \((k', k)\) to a disjoint interval \((k'_i, k_i)\). Then \(g_i\) no fixed points in either of these two intervals, so \(a_i\) and \(r_i\) are contained in either \((k, k'_i)\) or \((k_i, k')\). In fact, \(a_i\) and \(r_i\) must lie in the same one of these intervals as illustrated in Figure 5(b). Then \(a_i\) and \(r_i\) both accumulate on \(\hat{K}\), so

\[
\lim_{i \to \infty} \hat{e}(a_i) = \hat{e}^+(\hat{K}) = \lim_{i \to \infty} \hat{e}(r_i),
\]

which contradicts Lemma 3.1.

Step 2:

Suppose that \(\lim k_i = k\) and \(\lim k'_i = k'\). We will find negative leaves \(N\) and \(N'\) that cross \(\hat{K}\) near \(k\) and \(k'\) respectively, and are pulled inward by the \(g_i\). This will produce two repelling fixed points, contradicting our assumptions.

Special case: Suppose that \(\hat{K}\) contains a single subleaf \(K\), so that \(\hat{K} = \overline{K}\). Let \(\hat{N}_k\) and \(\hat{N}_{k'}\) be the negative extended leaves through \(k\) and \(k'\). Then \(K \subset\)
$V \cap V'$, where $V$ is a complementary component of $\hat{N}_k$, and $V'$ is a complementary component of $\hat{N}_{k'}$.

By Lemma 4.7, $k' \notin \hat{N}_k$, so $k' \subset V$. Similarly, $k \subset V'$.

(a) Choose negative leaves $N$ and $N'$ that separates $A$ from $\hat{N}_k$ and $\hat{N}_{k'}$ respectively, and label their ends as in Figure 6(a).

(b) Choose points $q \in N \cap K$ and $q' \in N' \cap K$. When $i$ be large, $g_i(q)$ and $g_i(q')$ are close to $A$, and the ends of $g_i(N)$ and $g_i(N')$ are arranged as in Figure 6(b). In particular, $g_i(n_1, n_2) \supseteq (n_1, n_2)$ and $g_i(n'_1, n'_2) \supseteq (n'_1, n'_2)$, so $g_i$ must have at least two repelling fixed points, $r_i \in (n_1, n_2)$ and $r'_i \in (n'_1, n'_2)$. This is a contradiction.

**General case:** We return to the general case, where $K$ may contain more than one subleaf.

Again, let $\hat{N}_k$ be the negative extended leaf through $k$. By Lemma 4.7, $k' \notin \hat{N}_k$, so $k'$ is contained in some complementary component $V$ of $\hat{N}_k$. The generic picture is illustrated in Figure 7(a).

Notice that $\hat{K}$ may have some "bad subleaves" that are not contained in $V$. If

$$A = \hat{K} \cap \{ e^-(p) \rightarrow e^+(p) \}$$

intersects one of these bad subleaves, then $A \notin V$, and it doesn’t make sense to ask for a leaf that separates $A$ from $\hat{N}_k$. Instead, we will use

$$A^* = A \cap V.$$  

Take a negative leaf $N \subset V$ that links $\hat{K}$, and let $q \in N \cap \hat{K}$. We will show that $\lim g_i(q) \subset A^*$.

Indeed, suppose that $g_i(q)$ accumulates on a point $q_\infty$ in a bad subleaf $K'$. This subleaf must intersect $U$, since $K_i \subset U$ for all $i$. Using Lemma 4.7 we see that $\partial_t \hat{R}' = k$. The complementary component $W$ of $\hat{N}_k$ that contains $K'$ corresponds to a complementary interval of the form $\partial_u W = (k, l) \subset (k, k')$. 

\[ \text{Figure 6. Homotopy closing for 2-links: special case.} \]
Take a subsequence so that $\lim g_i(q) = q_\infty$. Then $\varprojlim g_i(N)$ is contained in the negative extended leaf $\hat{N}_\infty$ through $q_\infty$, which lies in $W$. But $N$ and $\hat{K}$ are linked, so $\hat{K}_i$ must accumulate on some point in $(l, k)$. This is impossible, since $\varprojlim g_i(\hat{K}) \subset \hat{K}$, which has no ends in $(l, k)$. See Figure 7(b). Thus $\varprojlim g_i(q) \subset A^*$.

Finally, let $N$ be a negative leaf that separates $A^*$ from $\hat{N}_k$, and choose a point $q \in N \cap \hat{K}$. Take $i$ to be large, so that $k_i$ is close to $k$, $k_i'$ is close to $k'$, and $g_i(q)$ is close to $A^*$. See Figure 8. As in the special case, $g_i$ must have a repelling fixed point near $k$.

We find another repelling fixed point near $k'$ in the same manner. We had assumed that $g_i$ has only one repelling fixed point, so this is a contradiction.

5.4. Recurrent links.
Recurrent Links Lemma. If $\Phi$ is a quasigeodesic flow on a closed hyperbolic 3-manifold $M$, then some point $p \in P$ is recurrent and has linked extended leaves.

Together with the Homotopy Closing Lemma, this completes the proof of the Closed Orbits Theorem.

To start,

**Lemma 5.8.** Some positive leaf $K$ has least two ends.

**Proof.** Suppose that each positive leaf has exactly one end. Define a map $r: \hat{P} \to S^1_u$ by

$$r(p) = \partial_u \hat{D}^+(p).$$

This is continuous because $\hat{D}^+$ is an upper semicontinuous decomposition. Furthermore, $r$ restricts to the identity on $S^1_u$, so we have produced a retraction of a closed disc onto its boundary, which is impossible. \hspace{1cm} \Box

**Lemma 5.9.** The leaves through some $p \in P$ are linked.

**Proof.** Let $K$ be a positive leaf with at least two ends, and let $k, k' \in \partial_u K$. Let $N$ be the negative extended leaf through $k$. Then $k' \notin N$, so $k'$ is contained in a complementary component $U$ of $N$. Let $L$ be a negative leaf whose closure separates $k$ from $k'$, and take $p \in K \cap L$. \hspace{1cm} \Box

**Proof of Recurrent Links Lemma.** Let $p \in P$ be a point whose leaves are linked, and let $x \in M$ be a point in the corresponding orbit. The $\omega$-limit set $\omega(x)$ is invariant under $\Phi$, so it contains some minimal set. Each minimal set is the closure of an almost-periodic orbit ([1], Theorem 1.7), which is a fortiori recurrent. Therefore, we have some $q \in \omega(p)$ that is $\omega$-recurrent.

An $\omega$-sequence for $q \in \omega(p)$ takes the leaves through $p$ to the extended leaves through $q$, which are therefore linked. \hspace{1cm} \Box

6. Questions

6.1. **Coarse transverse hyperbolicity.** Our proof of the Homotopy Closing Lemma holds just as well for pseudo-Anosov flows, even ones that are not quasigeodesic. In fact, it should hold for a larger class of coarsely hyperbolic flows, defined by the existence of a pair of decompositions that are coarsely contracted/expanding.

It’s easy to construct coarsely hyperbolic flows that are neither quasigeodesic nor pseudo-Anosov. For example, start with a pseudo-Anosov flow that is not quasigeodesic and blow up a closed orbit. These examples are quite trivial since the Homotopy Closing Lemma follows easily from the Anosov Closing Lemma for the original flow. It would be interesting to construct a less trivial class of examples.

A flow on a closed 3-manifold $M$ is called **product covered** if the lifted flow on the universal cover $\hat{M}$ is conjugate to the vertical flow on $\mathbb{R}^3$.

**Question.** Let $\Phi$ be a product-covered flow on a closed hyperbolic 3-manifold $M$. Is $\Phi$ coarsely hyperbolic?

More generally:

**Question.** Let $\Phi$ be a product-covered flow on a closed hyperbolic 3-manifold $M$. Does $\Phi$ have a closed orbit?
This leads us to the following question.

**Question.** Can the fundamental group of a closed hyperbolic 3-manifold act freely on the plane?

If the answer is no, then each product-covered flow on a closed hyperbolic 3-manifold must have closed orbits.

### 6.2. Möbius-like groups.

In [8] we proposed a very different method for proving the Closed Orbits Theorem.

Let $\Gamma$ be a group. An action of $\Gamma$ on a circle $S^1$ is called Möbius-like if each $g \in \Gamma$ is conjugate to a Möbius transformation. It is called hyperbolic Möbius-like if each $g \in \Gamma$ is conjugate to a hyperbolic Möbius transformation. A Möbius-like or hyperbolic Möbius-like action is called Möbius or hyperbolic Möbius, respectively, if it is conjugate to an action by Möbius transformations.

The fundamental group of a closed hyperbolic 3-manifold can never act as a hyperbolic Möbius group (see [8]). The only known examples of Möbius-like actions that are not Möbius are found in [15]. We propose the following conjecture.

**Conjecture 6.1.** The fundamental group of a closed hyperbolic 3-manifold can not act as a hyperbolic Möbius-like group.

By Lemma 5.4, this conjecture implies the Closed Orbits Theorem.

**References**


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