1. Shifrin 5.4.6

2. Shifrin 5.4.15

3. Interpreting the Lagrange multiplier

Consider the problem of maximizing a smooth function $Z : \mathbb{R}^2 \to \mathbb{R}$ subject to some smooth constraint $F(x, y) = c$. The actual value of the Lagrange multiplier $\lambda$ at extrema usually has an interesting interpretation; in particular it tells us how sensitive $Z$ is to a change in the constraint $c$ at extrema.

(a) Define the Lagrangian function

$$L(x, y, \lambda, c) = Z(x, y) + \lambda(c - F(x, y))$$

Show that for any fixed $c$, the solutions $(x^*, y^*, \lambda^*)$ of

$$D_{(x, y, \lambda)}L = 0$$

correspond exactly to the constrained extrema of $Z$ subject to $F(x, y) = c$. (The notation $D_{(x, y, \lambda)}$ means the derivative of $L$ in the variables $(x, y, \lambda)$, regarding $c$ as a constant.)

(b) Let $G(x, y, \lambda, c) = D_{(x, y, \lambda)}L(x, y, \lambda, c)$. Apply the implicit function theorem to obtain $(x^*, y^*, \lambda^*)$ as functions of $c$. What assumptions are necessary on $G$ for this to work?

(c) Let $Z^*(c) = Z(x^*(c), y^*(c))$. Compute that

$$\frac{dZ^*}{dc} = \lambda^*(c)$$

That is, the value of the multiplier tells us how sensitive the extreme value $Z^*$ is to a small change in the constraint $c$.

Hint: Use the fact that $D_{(x, y, \lambda)}L = 0$ at constrained extrema, and the fact that $Z = L$ at extrema.

4. In class we discussed the fact that the derivative of a $C^1$ map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ induces a linear mapping of tangent spaces. Let $M$ be the zero level set of some smooth function $F : \mathbb{R}^n \to \mathbb{R}$ with $DF$ rank 1 near $p \in \mathbb{R}^n$.

(a) What is the dimension of $M$ near $p$?

(b) What is the dimension of $T_pM$ for $p \in M$?
(c) Assume $D\phi$ is invertible and $\text{rank}(DF) = 1$ in some neighborhood of $p$. Set $\tilde{M}$ to be the image of $M$ under $\phi$. Show that $\tilde{M}$ is the zero level set of the function $F \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{M}$ is locally a manifold of the same dimension as $M$.

(d) Show the claim from class:

$$v \in T_p M \iff (D\phi)|_p (v) \in T_{\phi(p)} \tilde{M}$$

5. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ and $\sigma : \mathbb{R} \to \mathbb{R}^n$ be a pair of parametrized curves in $\mathbb{R}^n$. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ conformal mapping. If we define the unit tangent vectors

$$T_\gamma(t) = \frac{\gamma'(t)}{|\gamma'(t)|} \quad \text{and} \quad T_\sigma(t) = \frac{\sigma'(t)}{|\sigma'(t)|}$$

and if the two curves intersect at some point $p = \gamma(t) = \sigma(s)$ for some $t, s$, then we can define the angle between the two curves by the formula

$$\cos \angle (\gamma(t), \sigma(s)) = T_\gamma(t) \cdot T_\sigma(s)$$

Show that the conformal map $\phi$ preserves the angle of intersection of curves, i.e., that

$$\angle (\gamma(t), \sigma(s)) = \angle ((\phi \circ \gamma)(t), (\phi \circ \sigma)(s))$$