NOTES ON TOPOLOGICAL QUANTUM FIELD THEORY

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ABSTRACT. These notes are meant to provide an introduction to topological quantum field theory for mathematicians. Topics include applications of Chern-Simons theory to the study of knots and Atiyah’s definition of a topological quantum field theory. We also include a self-contained introduction to Yang-Mills theory for physical motivation.

1. Introduction

1.1. Feynman integrals. Quantum field theory is the formalism developed by physicists for describing quantum mechanical systems that look classically like fields. It provides a unified framework for describing many different phenomena. For example, light is understood to be the result of excitations in a medium called the electromagnetic field, and there is a quantum field theory called quantum electrodynamics that describes the dynamics of this field and its interaction with matter. Other quantum field theories are used to describe the strong and weak nuclear forces and various condensed matter systems.

Mathematically, a quantum field theory can be understood as follows. To specify a quantum field theory, we first specify a smooth manifold \( M \) called spacetime (say a four-dimensional manifold with a Lorentzian metric \( g \)). We also specify a vector space \( \mathcal{F}(M) \) whose elements are called the fields on \( M \) (these could be smooth functions or vector fields on \( M \) for example). Finally, we specify a function \( S: \mathcal{F}(M) \to \mathbb{R} \) called the action and a measure \( D\varphi \) on \( \mathcal{F}(M) \). To perform calculations in quantum field theory, we have to compute certain integrals called Feynman integrals. In particular, we have to compute the integral

\[
Z = \int_{\mathcal{F}(M)} e^{iS(\varphi)} D\varphi
\]

which is known as the partition function. Once we know the value of this integral, we can use it to compute the expected values of various physical quantities. By physical quantity, we mean any attribute of a physical system to which we can assign a number by making the system interact with a measuring device. Such a quantity is represented by a function \( f: \mathcal{F}(M) \to \mathbb{R} \) whose value \( f(\varphi) \) on a field \( \varphi \) represents the value of the quantity if the field configuration happens to be \( \varphi \). If \( f \) is the function representing some physical quantity, then the average measured value of this quantity in the vacuum is

\[
\langle f \rangle = \frac{1}{Z} \int_{\mathcal{F}(M)} f(\varphi)e^{iS(\varphi)} D\varphi.
\]

Thus it would seem that performing calculations in quantum field theory is a simple matter in principle as it all boils down to computing integrals. The problem that makes calculations
extremely difficult in practice is that the measure $D\varphi$ usually cannot be defined rigorously. We are therefore forced to resort to nonrigorous and heuristic methods to extract physical predictions from quantum field theory.

Despite this shortcoming, it turns out that we can evaluate Feynman integrals in certain special cases. Interestingly, there are quantum field theories in which the values of these integrals are topological invariants of $M$, and in the case of Chern-Simons theory, knots embedded in $M$. Theories that compute topological invariants in this way are called topological quantum field theories (TQFTs). The invariants turn out to be easily computable once we have made sense of the integration, and for this reason, quantum field theory has recently become an important topic in pure mathematics.

1.2. Prerequisites. The purpose of these notes is to give a physically motivated introduction to TQFTs and their applications. They are primarily aimed at mathematicians who wish to understand recent developments in mathematical physics. The mathematical prerequisites therefore consist of standard graduate-level mathematics, including some understanding of basic differential geometry and category theory. We assume, for example, that the reader is familiar with the notion of a smooth manifold, tangent space, and vector bundle. We do not assume any familiarity with more advanced topics in differential geometry such as the theory of principal bundles or cobordisms. As far as category theory is concerned, these notes assume the reader is comfortable with the basic ideas of categories, functors, and natural transformations, but the prerequisites stop short of any advanced topics like monoidal or higher categories.

Since these notes are intended primarily for mathematicians, they do not presuppose any knowledge of physics and they do not discuss the physical ideas related to TQFTs in much detail. Instead, the physics is used to motivate the mathematical definition of a TQFT. The purpose of these notes is not to teach physics but to explain how the physicist’s idea of a Feynman integral can serve as a source of inspiration for mathematicians.

1.3. Organization. We begin with a discussion of the physical aspects of quantum field theory in Section 2. In this section, we explain how physical fields are represented mathematically and explain the notion of gauge symmetry. We then introduce principal bundles and principal bundle connections and use these mathematical tools to discuss classical Yang-Mills theory. To illustrate how Yang-Mills theory arises in physics, we briefly discuss its role in quantum electrodynamics. This discussion of Yang-Mills theory is necessary to motivate our later discussion of Chern-Simons theory and Dijkgraaf-Witten theory. The treatment in this section is based on the excellent notes [1].

In Section 3, we discuss Chern-Simons theory, a topological theory that gives rise to many topological invariants of knots. We begin by reviewing the basic invariants of knots and links, namely the writhe, Kauffman bracket, and Jones polynomial. Then we define the Chern-Simons action and introduce the notions of holonomy and Wilson loops. Finally, we sketch the remarkable connection between Chern-Simons theory and link invariants introduced previously. The treatment in this section borrows heavily from [2].

Finally, in Section 4, we give a physically motivated introduction to Atiyah’s axiomatization of a TQFT, and we explain how a theory satisfying the axioms can be used to compute
topological invariants. In the process, we have to introduce the theory of cobordisms and symmetric monoidal categories. Once we have defined TQFTs, we discuss Dijkgraaf-Witten theory as an example. The material in this section comes from [3], [4], and [5].

1.4. Convention. In these notes, we will follow the Einstein summation convention and implicitly sum over all indices which appear once as a superscript and once as a subscript. For example, when we write an expression \( y = c_\mu x^\mu \), we really mean the sum \( y = \sum_\mu c_\mu x^\mu \).

2. Yang-Mills Theory

2.1. The mathematical description of fields. Before we begin our discussion of topological quantum field theories, we will spend some time looking at how quantum field theory is used in physics. The main example that will motivate our discussion in this section is the theory of an \( n \)-component vector field. In other words, we will model physical fields as smooth functions \( \phi : M \to \mathbb{R}^n \) where for simplicity we take \( M = \mathbb{R}^4 \). We usually write a field \( \phi \) in terms of its components as \( \phi = (\phi_1, \ldots, \phi_n) \). To specify an action for the theory, we first form the expression

\[
\mathcal{L} = \sum_{i=1}^{n} \left( \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{2} m^2 \phi_i^2 \right)
\]

where \( m > 0 \) is a constant. Here we have defined \( \partial^\mu = g^{\mu\nu} \partial_\nu \) where \( (g_{\mu\nu}) \) is the inverse of the matrix \( (g^{\mu\nu}) \). An expression of this sort is called a Lagrangian density. Once we have a Lagrangian density, the action is obtained by integrating over all of spacetime:

\[
S(\phi) = \int_{\mathbb{R}^4} d^4x \mathcal{L}.
\]

Observe that in this theory, the fields are represented by sections of a certain fiber bundle, namely \( M \times \mathbb{R}^n \to M \). In general, physical fields are represented by smooth sections of some bundle over the spacetime manifold. This point of view will be useful later on when we begin our study of gauge fields.

2.2. Gauge invariance. An important feature of our \( n \)-component vector field theory is its invariance under the action of certain transformations. In this theory, the physical fields are functions \( M \to \mathbb{R}^n \). Now there is an action of the group \( O(n) \) on the \( n \)-dimensional vector space \( \mathbb{R}^n \) given by matrix multiplication, so there is an action of \( O(n) \) on functions \( \phi : M \to \mathbb{R}^n \) given by \( (g\phi)(p) = g\phi(p) \). Let \( g = (a_{ij}) \) be an orthogonal matrix. Then

\[
\partial_\mu (g\phi)_i = \partial_\mu \sum_{j=1}^{n} a_{ij} \phi_j = \sum_{j=1}^{n} a_{ij} \partial_\mu \phi_j = g(\partial_\mu \phi)_i
\]

and since orthogonal transformations preserve the inner product on \( \mathbb{R}^n \), it follows that

\[
\sum_i \partial_\mu (g\phi)_i \partial^\mu (g\phi)_i = \sum_i \partial_\mu \phi_i \partial^\mu \phi_i.
\]

We also have \( \sum_i (g\phi)_i^2 = \sum_i \phi_i^2 \), so the action of \( O(n) \)
preserves the full expression for the Lagrangian density. Since the Lagrangian density determines the partition function, it follows that all the physics is invariant under the action of $O(n)$. In general, whenever we have a group $G$ acting on the space of fields and preserving the Lagrangian density, we say that the theory is invariant under global gauge transformations. In this case, the group $G$ is called the gauge group of the theory.

The idea of Yang-Mills theory is to require the physics to be invariant under not only global gauge transformations but arbitrary local gauge transformations as well. In other words, we wish to consider a theory with gauge group $G$, and for any function $g : M \to G$, we wish to write down a Lagrangian density that is invariant under the transformation $\varphi(x) \to g(x)\varphi(x)$. Unfortunately, the Lagrangian density that we have been using is not invariant under these local gauge transformations because in general

$$\partial_\mu g(x)\varphi(x) \neq g(x)\partial_\mu \varphi(x).$$

We will therefore have to modify our Lagrangian density, replacing the partial derivatives $\partial_\mu$ by the more sophisticated notion of “covariant” derivative, which transforms in a nontrivial way when we apply a local gauge transformation.

2.3. Principal bundles. In order to write down a Lagrangian density that is invariant under local gauge transformations, we need to develop a theory of bundles with group actions.

**Definition 2.3.1.** Let $F$ be a smooth manifold. We say that a bundle $\pi : P \to M$ is locally trivial with standard fiber $F$ if there exists an open covering $\{U_\alpha\}$ of $M$ and for each $\alpha$ a homeomorphism $\varphi_\alpha$ called a local trivialization such that

$$\pi^{-1}(U_\alpha) \xrightarrow{\varphi_\alpha} U_\alpha \times F \xleftarrow{\text{proj}} U_\alpha$$

commutes where proj is the projection onto the first factor. Let $G$ be a Lie group and let $\pi : P \to M$ be a bundle with a continuous right action $P \times G \to P$ such that $G$ preserves the fibers of $\pi$ and acts freely and transitively on them. Such a bundle is called a principal $G$-bundle if it is locally trivial and the local trivializations are $G$-equivariant.

2.4. Associated bundles. Next we are going to study a method of constructing bundles from a principal bundle. Let $\pi : P \to M$ be a principal $G$-bundle and $F$ a smooth manifold that admits a left $G$-action. Then there is a right action $(P \times F) \times G \to P \times F$ of $G$ on the product $P \times F$ given by

$$(u, f) \cdot g = (u \cdot g, g^{-1} \cdot f).$$

Define $P \times_G F = (P \times F)/G$ with the quotient topology. Since the composition $P \times F \xrightarrow{\text{proj}} P \xrightarrow{\pi} M$ is constant on $G$-orbits, there is an induced map $\tilde{\pi} : P \times_G F \to M$. This induced map has a special name.

**Definition 2.4.1.** The map $\tilde{\pi} : P \times_G F \to M$ is called the bundle associated to $P$ with fiber $F$. We use the notation $[p, f]$ for the class of $(p, f) \in P \times F$ in the total space of this bundle.
Proposition 2.4.2. The associated bundle $\tilde{\pi} : P \times_G F \to M$ is a locally trivial bundle with the same trivializing open cover as $\pi : P \to M$ and fibers diffeomorphic to $F$.

Proof. Let $U \subseteq M$ be a trivializing open set for the principal bundle $\pi : P \to M$, and let $\varphi : \pi^{-1}(U) \to U \times G$ be the associated local trivialization. For any point $[p, f] \in \pi^{-1}(U)$, we have $\varphi(p) = (x, g)$ for some $x \in U$ and $g \in G$, and we can define $\psi : \pi^{-1}(U) \to U \times F$ by

$$\psi([p, f]) = (x, g \cdot f).$$

It is straightforward to check that this map is well defined. On the other hand, there is a mapping $\psi' : U \times F \to \pi^{-1}(U)$ given by

$$\psi'(x, f)[p, f] = \varphi^{-1}(x, 1, f).$$

and one can show that $\psi$ is a local trivialization with inverse $\psi'$. This local trivialization takes the fiber over $x$ to $\{x\} \times F$ and therefore gives rise to a diffeomorphism between fibers of $\tilde{\pi}$ and the space $F$. This completes the proof. □

A special case of this construction is when $F$ is a vector space on which $G$ has a smooth representation $\rho : G \to GL(F)$. In this case the bundle associated to $P$ is called a vector bundle. The following examples and notations will appear over and over in our discussion.

Examples 2.4.3.

(1) If $G$ is a group of $n \times n$ matrices and $V$ is an $n$-dimensional vector space, then we can take $\rho$ to be the fundamental representation $G \hookrightarrow GL(V)$.

(2) Let $g$ be the Lie algebra of $G$ and take $\rho$ to be the adjoint representation $Ad : G \to GL(g)$. Then the associated bundle $P \times_G g \to M$ is called the adjoint bundle of $P$ and denoted $Ad(P)$.

2.5. Transition functions. If $U_\alpha$ and $U_\beta$ are two sets in our open cover having nonempty intersection $U_{\alpha \beta} = U_\alpha \cap U_\beta$, then there are two trivializations $\varphi_\alpha, \varphi_\beta : \pi^{-1}(U_{\alpha \beta}) \to U_{\alpha \beta} \times G$. These can be written $\varphi_\alpha(p) = (m, g_\alpha(p))$ and $\varphi_\beta(p) = (m, g_\beta(p))$ where $p \in \pi^{-1}(m)$, and we define $g_{\alpha \beta}(p)$ to be the element $G$ for which $g_\alpha(p) = g_{\alpha \beta}(p)g_\beta(p)$.

Lemma 2.5.1. There exist functions $g_{\alpha \beta} : U_{\alpha \beta} \to G$ such that $g_{\alpha \beta}(p) = g_{\alpha \beta}(\pi(p))$.

Proof. For any $g \in G$, we have

$$g_{\alpha \beta}(pg) = g_\alpha(pg)g_\beta(pg)^{-1} = g_\alpha(p)gg^{-1}g_\beta(p)^{-1} = g_{\alpha \beta}(p).$$

Since $G$ acts transitively on the fibers of $P$, it follows that $g_{\alpha \beta}$ is constant on each fiber. The result follows immediately. □

Definition 2.5.2. The functions $g_{\alpha \beta} : U_{\alpha \beta} \to G$ in the lemma are called transition functions.
2.6. \textit{g-valued forms}. Normally, when we talk about a differential \( p \)-form, we mean an operation that takes \( p \) vector fields as input and produces a number. Equivalently, one can define a \( p \)-form to be a smooth section of \( \Lambda^p T^* M \). Sometimes it is useful to generalize this notion and consider an operation that takes \( p \) vector fields and produces an element of some vector space \( X \).

\textbf{Definition 2.6.1.} If \( X \) is a vector space and \( M \) is a manifold, then an \textit{\( X \)-valued differential \( p \)-form} is a smooth section of \( X \otimes \Lambda^p T^* M \). The space of all \( p \)-forms with values in \( X \) is denoted \( \Omega^p(M,X) \).

Many operations and notations from the theory of ordinary real-valued differential forms can also be applied to vector-valued differential forms. For example, if \( \omega \) and \( \eta \) are vector-valued differential forms, then we define their wedge product by the usual formula with real multiplication replaced by tensor product:

\[
(\omega \wedge \eta)(v_1, \ldots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma) \omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \otimes \eta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)}).
\]

Thus the wedge product of \( p \)-form with values in \( X \) and a \( q \)-form with values in \( Y \) is a \((p + q)\)-form with values in the tensor product \( X \otimes Y \). In the special case where \( \omega \) and \( \eta \) take values in a Lie algebra \( g \), we can compose the wedge product operation with the Lie bracket \([−,−] : g \otimes g \rightarrow g \) to get a new element of \( g \) called the \textit{bracket}:

\[
[\omega, \eta] = [\omega \wedge \eta].
\]

We can also extend the notion of exterior derivative to \( X \)-valued differential forms. Once we choose a basis \( \{e_i\} \) for \( X \), such a form \( \omega \) can be written \( \omega = \omega^i \otimes e_i \) where \( \omega^i \) are real-valued differential forms, and we define the \textit{exterior derivative of} \( \omega \) by the formula \( d\omega = d\omega^i \otimes e_i \).

\textbf{Example 2.6.2.} For any Lie group \( G \), the \textit{Maurer-Cartan form} \( \theta \in \Omega^1(G, g) \) is defined by

\[
\theta_g = (L_{g^{-1}})_*: T_g G \rightarrow T_1 G = g
\]

where \( L_{g^{-1}} : G \rightarrow G, x \mapsto g^{-1}x \). One can show that in the special case where \( G \) is a matrix group, the Maurer-Cartan form is given by \( \theta_g = g^{-1}dg \). Other important properties of the Maurer-Cartan form include the transformation law

\[
R_g^{-1}\theta = \text{Ad}(g^{-1})\theta
\]

where \( R : G \rightarrow G \) is the function defined by \( R_g(x) = xg \) and the equation

\[
d\theta + \frac{1}{2}[\theta, \theta] = 0
\]

which is known as the \textit{Maurer-Cartan equation}.

2.7. \textbf{Connections}. The concept of a connection is central to differential geometry because it allows us to differentiate sections of bundles and transport vectors from one tangent space to another. To state the definition, we let \( \pi : P \rightarrow M \) be a principal \( G \)-bundle on \( M \) with trivializing open cover \( \{U_a\} \).
Definition 2.7.1. Let $R_g : P \rightarrow P$ denote right multiplication by $g \in G$, and let $\sigma_p : G \rightarrow P$ be the mapping $g \mapsto pg$ for $p \in P$. For any $X \in \mathfrak{g}$ let $\sigma X$ be the vector field on $P$ given by $(\sigma X)_p = (\sigma_p)_* X$. A connection 1-form on $P$ is an element $\omega$ of $\Omega^1(P, \mathfrak{g})$ such that

$$R_g^* \omega = \text{Ad}(g^{-1}) \omega$$

$$\omega(\sigma(X)) = X.$$

It is a fact, whose proof goes beyond the scope of these notes, that a connection 1-form exists on any principal bundle. Suppose $\omega$ is a connection 1-form on $P$. For each index $\alpha$, there is a canonical local section $s_\alpha : U_\alpha \rightarrow \pi^{-1}(U_\alpha)$ defined so that for each point $m \in U_\alpha$ we have $\varphi_\alpha(s_\alpha(m)) = (m, 1)$. Define $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ by

$$A_\alpha = s_\alpha^* \omega.$$

This 1-form is known in physics as a vector potential or gauge field. We will use the following result to see how the vector potential changes when we pass from one trivializing open set to another.

Proposition 2.7.2. Let $\omega$ be a connection 1-form, and let $\omega_\alpha$ be its restriction to $\pi^{-1}(U_\alpha)$. Then $\omega_\alpha$ can be written

$$\omega_\alpha = \text{Ad}(g_\alpha^{-1}) \pi^* A_\alpha + g_\alpha^* \theta$$

where $\theta$ is the Maurer-Cartan form defined above.

Proof. We begin by proving that the two forms agree on the image of $s_\alpha$. Suppose we are given $m \in U_\alpha$ and $p = s_\alpha(m)$. Then we have a direct sum decomposition

$$T_p P = \text{im}(s_\alpha \circ \pi)_* \oplus V_p,$$

so we can write any tangent vector $v \in T_p P$ uniquely in the form $v = (s_\alpha)_* \pi_*(v) + \bar{v}$ for some $\bar{v} \in V_p$. Since $g_\alpha \circ s_\alpha$ is constant, we have $(g_\alpha \circ s_\alpha)_* = 0$ and therefore $(g_\alpha)_* v = (g_\alpha)_* \bar{v}$. It follows that

$$(\text{Ad}(g_\alpha(p)^{-1}) \pi^* A_\alpha + g_\alpha^* \theta)(v) = (\pi^* s_\alpha^* \omega)(v) + (g_\alpha^* \theta)(v)$$

$$= \omega((s_\alpha)_* \pi_* v) + \theta((g_\alpha)_* \bar{v})$$

$$= \omega((s_\alpha)_* \pi_* v) + \omega(\bar{v})$$

$$= \omega_\alpha(v),$$

so the two forms are agree on the image of $s_\alpha$. Next we will show that these two forms transform in the same way under the action of $G$. Indeed,

$$R_g^* (\text{Ad}(g_\alpha(pg)^{-1}) \pi^* A_\alpha + g_\alpha^* \theta) = \text{Ad}((g_\alpha(pg)^{-1}) R_g^* \pi^* A_\alpha + g_\alpha^* R_g^* \theta$$

$$= \text{Ad}(g^{-1} g_\alpha(p)^{-1}) \pi^* s_\alpha^* \omega + g_\alpha^* (\text{Ad}(g^{-1}) \theta)$$

$$= \text{Ad}(g^{-1}) (\text{Ad}(g_\alpha(p)^{-1}) \pi^* s_\alpha^* \omega + g_\alpha^* \theta)$$

$$= \text{Ad}(g^{-1}) \omega_\alpha.$$

It follows that equality holds everywhere on $\pi^{-1}(U_\alpha)$. 

□
Corollary 2.7.3. If $U_\alpha$ and $U_\beta$ overlap, then on the intersection $U_{\alpha\beta}$, we have

\[ A_\alpha = \text{Ad}(g_{\alpha\beta})A_\beta + g_{\beta\alpha}^*\theta. \]

Proof. Since $\omega_\alpha$ and $\omega_\beta$ are restrictions of a globally defined 1-form, we know that $\omega_\alpha = \omega_\beta$ on $\pi^{-1}(U_{\alpha\beta})$. Therefore on the intersection $U_{\alpha\beta}$, we have

\[ s_\alpha^*\omega_\alpha = s_\alpha^*\omega_\beta 
= s_\alpha^*(\text{Ad}(g_\beta(s_\alpha)^{-1})\pi^*A_\beta + g_\beta^*\theta) 
= \text{Ad}(g_{\alpha\beta})A_\beta + g_{\beta\alpha}^*\theta. \]

In the last step, we used the fact that $g_\beta \circ s_\alpha = g_{\beta\alpha}$. \qed

2.8. Gauge transformations. Like all geometric objects, principal bundles have a natural notion of automorphism. Since we ultimately would like to describe physical systems using the principal bundle formalism we have developed, we should make sure that all the physics is invariant under these automorphisms.

Definition 2.8.1. Let $\pi : P \to M$ be a principal $G$-bundle. A gauge transformation of $P$ is $G$-equivariant diffeomorphism $\Psi : P \to P$ such that the following diagram commutes.

\[
\begin{array}{ccc}
P & \xrightarrow{\Psi} & P \\
\downarrow{\pi} & & \downarrow{\pi} \\
M & \xrightarrow{\pi} & M
\end{array}
\]

If $\Psi$ is a gauge transformation of $P$, then $\Psi$ maps fibers to themselves and therefore restricts to a gauge transformation of the trivial bundle $\pi^{-1}(U_\alpha)$. Let $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ be the associated local trivialization. Then we can write $\varphi_\alpha(\Psi(p)) = (\pi(p), g_\alpha(\Psi(p)))$ and use this to define

\[ \tilde{\psi}_\alpha(p) = g_\alpha(\Psi(p))g_\alpha(p)^{-1}. \]

The fact that $g_\alpha$ and $\Psi$ are $G$-equivariant implies $\tilde{\psi}_\alpha(pg) = \tilde{\psi}_\alpha(p)$ for all $g \in G$. Since $G$ acts transitively on fibers of $P$, it follows that $\tilde{\psi}_\alpha$ is constant on fibers and hence there exists $\psi_\alpha : U_\alpha \to G$ such that $\tilde{\psi}_\alpha(p) = \psi(\pi(p))$. Let $m$ be a point of $U_{\alpha\beta}$ and $p$ a point in the fiber over $m$. Then we have

\[ \psi_\alpha(m) = g_\alpha(\Psi(p))g_\alpha(p)^{-1} 
= g_\alpha(\Psi(p))g_\beta(\Psi(p))^{-1}g_\beta(\Psi(p))g_\beta(p)^{-1}g_\beta(p)g_\alpha(p)^{-1} 
= g_{\alpha\beta}(m)\psi_\beta(m)g_{\alpha\beta}(m)^{-1} \]

since $\pi(\Psi(p)) = m$. This prove that $\psi_\alpha(m) = \text{Ad}(g_{\alpha\beta}(m))\psi_\beta(m)$ and therefore the maps $\psi_\alpha$ define a section of the adjoint bundle $P \times \text{Ad} G$. These maps correspond to the local gauge transformations that we considered earlier while discussing the physical motivation.

Proposition 2.8.2. Suppose we are given a connection 1-form $\omega$, and let $\omega' = (\Psi^{-1})^*\omega$ be the corresponding gauge transformed 1-form. Let $A_\alpha$ and $A'_\alpha$ be the gauge fields on $U_\alpha$ corresponding to $\omega$ and $\omega'$. Then

\[ A'_\alpha = \text{Ad}(\psi_\alpha)(A_\alpha - \psi_\alpha^*\theta). \]
Proof. We can write
\[ \omega_p = \text{Ad}(g_\alpha(p)^{-1})\pi^*A_\alpha + g_\alpha^*\theta \]
and
\[ \omega_p' = \text{Ad}(g_\alpha(p)^{-1})\pi^*A'_\alpha + g_\alpha^*\theta. \]
Letting \( q = \Psi^{-1}(p) \) and applying \((\Psi^{-1})^*\) to the first of these equations, we find
\[ \omega_p' = \text{Ad}(g_\alpha(q)^{-1})(\Psi^{-1})^*\pi^*A_\alpha + (\Psi^{-1})^*g_\alpha^*\theta \]
\[ = \text{Ad}(g_\alpha(q)^{-1})(\pi \circ \Psi^{-1})^*A_\alpha + (g_\alpha \circ \Psi^{-1})^*\theta \]
\[ = \text{Ad}(g_\alpha(q)^{-1})\pi^*A_\alpha + (g_\alpha \circ \Psi^{-1})^*\theta \]
\[ = \text{Ad}(g_\alpha(p)^{-1}\tilde{\psi}_\alpha(p))\pi^*A_\alpha + (g_\alpha \circ \Psi^{-1})^*\theta. \]
The fact that \((g_\alpha \circ \Psi^{-1})(p) = g_\alpha(q) = \tilde{\psi}_\alpha(p)^{-1}g_\alpha(p)\) implies
\[ (g_\alpha \circ \Psi^{-1})^*\theta = g_\alpha^*\theta - \text{Ad}(g_\alpha(p)^{-1}\tilde{\psi}_\alpha(p))\tilde{\psi}_\alpha^*\theta \]
and therefore
\[ \omega_p' = \text{Ad}(g_\alpha(p)^{-1}\tilde{\psi}_\alpha(m))\pi^*(A_\alpha - \psi_\alpha^*\theta) + g_\alpha^*\theta. \]
Combining this with our other formula for \( \omega_p' \) yields the desired result. \( \square \)

2.9. Covariant differentiation. Fix a trivializing open set \( U \subseteq M \), and let \( A \) a vector potential on this open set. If \( \tilde{\pi} : P \times G V \to M \) is the bundle defined in Example 2.4.3, then there is a local trivialization \( \tilde{\pi}^{-1}(U) \cong U \times V \), so we can think of any local section \( s : U \to \tilde{\pi}^{-1}(U) \) of \( \tilde{\pi} \) as being essentially the same thing as a smooth function \( s : U \to V \).

If we choose a basis \( \{e_i\} \) for \( V \), then we can write \( s : U \to V \) in terms of coordinates as \( s = s^i e_i \) with \( s^i \in C^\infty(M) \). Since the Lie algebra of \( GL(V) \) is \( \mathfrak{gl}(V) \), we can view the 1-form \( A \) as a matrix whose entries \( A_1^i \) are ordinary real-valued 1-forms. One then has the following derivative operation.

Definition 2.9.1. The covariant derivative of \( s \) is the section \( \nabla_v s \) of \( V \) defined by
\[ \nabla_v s = ds^i(v)e_i + s^i A_1^j(v)e_j. \]
We think of this new section \( \nabla_v s \) as the “derivative” of \( s \) in the “direction” \( v \). In the special case when \( v = \partial_\mu \), we write \( \nabla_v = \nabla_\mu \). Notice that gauge transformations act on covariant derivatives, taking \( \nabla_v \) to a new operation \( \nabla'_v \) in which the vector potential \( A \) is replaced by the gauge transformed vector potential \( A' \). The following proposition says that covariant derivatives behave in a particularly nice way under gauge transformations.

Proposition 2.9.2. Suppose \( G \) is a matrix group and \( g_\alpha : U_\alpha \to G \) are functions defining a gauge transformation. Under this gauge transformation,
\[ \nabla'_\mu g_\alpha s = g_\alpha \nabla_\mu s. \]
Proof. Let $g$ be the $g_\alpha$ defined on the open set $U$. Since $G$ is a matrix group, we can write $g$ in terms of its matrix coefficients, $g = (a^i_j)$. Then
\[ \partial_\mu (gs)^i = \partial_\mu (a^i_j s^j) \]
\[ = (\partial_\mu a^i_j) s^j + a^i_j (\partial_\mu s^j) \]
\[ = ((\partial_\mu g) s)^i + (g \partial_\mu s)^i, \]
and the gauge transformed vector potential can be written
\[ A'_\mu = g A_\mu g^{-1} - (\partial_\mu g) g^{-1} \]
where we denote $A_\mu = A(\partial_\mu)$ and $A'_\mu = A'(\partial_\mu).$ Therefore
\[ \nabla'_\mu gs = \partial_\mu (gs)^i e_i + (gs)^i (A'_\mu)^i_j e_j \]
\[ = ((\partial_\mu g) s)^i e_i + (g \partial_\mu s)^i e_i + (gs)^i (g A_\mu g^{-1})^i_j e_j - (\partial_\mu g g^{-1})^i_j (gs)^i e_j \]
\[ = (g \partial_\mu s)^i e_i + (gs)^i (g A_\mu g^{-1})^i_j e_j \]
\[ = g \nabla_\mu s \]
as desired. \qed

2.10. **An invariant Lagrangian density.** We have now developed the mathematics necessary to modify our $n$-component theory so that the Lagrangian density is invariant under local gauge transformations. Let $\tilde{\pi}$ be the vector bundle with fiber $\mathbb{R}^n$ associated to a principal $O(n)$-bundle $\pi : P \to M$. Fix a trivializing open set $U \subseteq M$ and let $\{e_i\}$ be the basis of local sections of $\tilde{\pi}$ over $U$ given by
\[ e_i(p) = (0, \ldots, 0, 1, 0, \ldots, 0) \]
with a 1 in the $i$th position and 0’s elsewhere. A physical field $\varphi$ is represented mathematically as a section of this vector bundle, so locally we can write $\varphi = \sum_i \varphi_i e_i$ for smooth functions $\varphi_i$ on $U$. Then we can rewrite our Lagrangian density using covariant instead of partial derivatives, and we obtain a new expression with the desired invariance property:
\[ \mathcal{L} = \sum_{i=1}^n \left( \frac{1}{2} (\nabla_\mu \varphi)_i (\nabla^\mu \varphi)_i + \frac{1}{2} m^2 \varphi_i^2 \right). \]

Notice that in order to write down a Lagrangian density which behaves nicely under local gauge transformations, we had to choose a $g$-valued 1-form $A$. In quantum field theory, this object is interpreted as a new physical field, and if we wish to describe this field quantum mechanically, we will need to specify an action.

2.11. **Curvature and field strength.** The first step in writing down an action for the gauge field $A$ is to construct an auxiliary field called the gauge field strength.

**Definition 2.11.1.** Let $\omega$ be a connection 1-form. Then the **curvature** of $\omega$ is the 2-form $\Omega = d\omega + \frac{1}{2} [\omega, \omega]$. If $s_\alpha : U_\alpha \to P$ is one of the canonical sections defined above, then the **gauge field strength** is the pullback $F_\alpha = s_\alpha^* \Omega$. 


Explicitly, the field strength $F_\alpha$ is given by the formula

$$F_\alpha = dA_\alpha + \frac{1}{2} [A_\alpha, A_\alpha].$$

Since the field strength $F_\alpha$ is defined locally, it is natural to ask how it changes when we pass from one trivializing open set to another. The answer is provided by the following proposition, which implies that the $F_\alpha$ patch together to form a globally defined 2-form with values in the adjoint bundle $\text{Ad}(P)$.

**Proposition 2.11.2.** Let $U_\alpha$ and $U_\beta$ be two trivializing open subsets of $M$. Then on their intersection we have

$$F_\alpha = \text{Ad}(g_{\alpha\beta})F_\beta$$

**Proof.** We have seen that on the intersection $U_{\alpha\beta}$ the gauge field $A_\alpha$ is given by

$$A_\alpha = \text{Ad}(g_{\alpha\beta})A_\beta + g^*_{\beta\alpha}\theta.$$

Bracketing this 1-form with itself and using graded commutativity of the bracket operation, we see that

$$[A_\alpha, A_\alpha] = [\text{Ad}(g_{\alpha\beta})A_\beta + g^*_{\beta\alpha}\theta, \text{Ad}(g_{\alpha\beta})A_\beta + g^*_{\beta\alpha}\theta]$$

$$= [\text{Ad}(g_{\alpha\beta})A_\beta, \text{Ad}(g_{\alpha\beta})A_\beta] + [\text{Ad}(g_{\alpha\beta})A_\beta, g^*_{\beta\alpha}\theta]$$

$$+ [g^*_{\beta\alpha}\theta, \text{Ad}(g_{\alpha\beta})A_\beta] + [g^*_{\beta\alpha}\theta, g^*_{\beta\alpha}\theta]$$

$$= [\text{Ad}(g_{\alpha\beta})A_\beta, \text{Ad}(g_{\alpha\beta})A_\beta] + 2[\text{Ad}(g_{\alpha\beta})A_\beta, g^*_{\beta\alpha}\theta] + [g^*_{\beta\alpha}\theta, g^*_{\beta\alpha}\theta].$$

Now this last expression contains

$$[\text{Ad}(g_{\alpha\beta})A_\beta, g^*_{\beta\alpha}\theta] = \text{Ad}(g_{\alpha\beta})[A_\beta, \text{Ad}(g_{\alpha\beta})^{-1}g^*_{\beta\alpha}\theta]$$

and we have

$$\text{Ad}(g_{\alpha\beta})^{-1}g^*_{\beta\alpha}\theta = g^*_{\beta\alpha}\text{Ad}(g_{\alpha\beta})^{-1}\theta$$

$$= g^*_{\beta\alpha}R^*_{g_{\beta\alpha}}\theta$$

$$= (R_{g_{\beta\alpha}} \circ g_{\beta\alpha})^*\theta.$$

The composition $R_{g_{\beta\alpha}}^{-1} \circ g_{\beta\alpha}$ equals the constant function 1, so its differential is zero, and therefore the pullback of the Maurer-Cartan form by this composition vanishes. It follows that

$$[A_\alpha, A_\alpha] = [\text{Ad}(g_{\alpha\beta})A_\beta, \text{Ad}(g_{\alpha\beta})A_\beta] + [g^*_{\beta\alpha}\theta, g^*_{\beta\alpha}\theta]$$

$$= \text{Ad}(g_{\alpha\beta})[A_\beta, A_\beta] + g^*_{\beta\alpha}[\theta, \theta]$$

$$= \text{Ad}(g_{\alpha\beta})[A_\beta, A_\beta] - 2g^*_{\beta\alpha}d\theta.$$

On the other hand, we have $dA_\alpha = \text{Ad}(g_{\alpha\beta})dA_\beta + g^*_{\beta\alpha}d\theta$, so the desired result is a consequence of the formula $F_\alpha = dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha]$. □
2.12. The Yang-Mills action. In order to write down an action for the field \( A \), we need to assume that our Lie algebra \( g \) admits an \( \text{Ad} \)-invariant inner product \( \langle -, - \rangle : g \otimes g \rightarrow \mathbb{R} \). Of course an inner product of this sort may not exist on an arbitrary Lie algebra, but from now on we will only consider gauge groups whose Lie algebras do admit such an inner product.

If we assume our manifold \( M \) comes equipped with a metric tensor \( g \), then there is also a natural inner product operation on differential forms. If \( \omega = \omega_\mu dx^\mu \) and \( \eta = \eta_\nu dx^\nu \) are 1-forms on \( M \), then this inner product is the function \( \langle \omega, \eta \rangle = g^{\mu \nu} \omega_\mu \eta_\nu \) on \( M \). More generally, for 1-forms \( e_1, \ldots, e_p \) and \( f_1, \ldots, f_p \), we define

\[
\langle e_1 \wedge \cdots \wedge e_p, f_1 \wedge \cdots \wedge f_p \rangle = \det(\langle e_i, f_j \rangle)
\]

and extend linearly to get an inner product on the space of \( p \)-forms.

Now, if \( S \otimes \omega \) and \( T \otimes \eta \) are \( g \)-valued differential forms written in terms of sections \( S \) and \( T \) of \( g \) and ordinary \( p \)-forms \( \omega \) and \( \eta \), then we can combine these notions of inner product and write

\[
\langle S \otimes \omega, T \otimes \eta \rangle = \langle S, T \rangle \langle \omega, \eta \rangle.
\]

In this way, we obtain an inner product operation on \( g \)-valued differential forms. We will be particularly interested in the squared norm \( |F_\alpha|^2 = \langle F_\alpha, F_\alpha \rangle \) where \( F_\alpha \) is the gauge field strength introduced above. By \( \text{Ad} \)-invariance of the inner product on \( g \), we know that this norm agrees with \( |F_\beta|^2 \) on the set \( U_{\alpha \beta} \) and therefore there is a globally well defined function \( |F|^2 \) on \( M \) where \( F \) is the field strength regarded as a section of \( \text{Ad}(P) \). With this notation, the Yang-Mills action can be written

\[
S(A) = \int_M |F|^2 \text{vol},
\]

provided this integral exists.

To rewrite this action in a form familiar to physicists, we need the following important operation.

**Definition 2.12.1.** The Hodge star operator is the unique linear map \( * : \Omega^p(M) \rightarrow \Omega^{n-p}(M) \) such that for all \( \omega, \eta \in \Omega^p(M) \) we have \( \omega \wedge *\eta = \langle \omega, \eta \rangle \text{vol} \). We also define \( * : \Omega^p(M, g) \rightarrow \Omega^{n-p}(M, g) \) by \( *(T \otimes \omega) = T \otimes *\omega \) where \( T \otimes \omega \) is a \( g \)-valued \( p \)-form written in terms of a section \( T \) of \( g \) and an ordinary \( p \)-form \( \omega \).

If \( \omega \) and \( \eta \) are \( g \)-valued 1-forms, it is easy to see that the expression \( \langle \omega, \eta \rangle \text{vol} \) is the same as the composition of \( \omega \wedge *\eta \) with the inner product \( \langle -, - \rangle : g \otimes g \rightarrow \mathbb{R} \). Abusing notation slightly, we write this composition as \( \text{Tr}(F \wedge *F) \). Then the Yang-Mills action can be written

\[
S(A) = \int_M \text{Tr}(F \wedge *F).
\]

2.13. Maxwell theory. To conclude this section, we briefly mention how Yang-Mills theory arises in quantum electrodynamics, the quantum theory describing electrons, positrons, and photons. According to this theory, electrons and positrons are the particle-like excitations in a field called the Dirac field. Although more complicated than the \( n \)-component vector field we have been discussing, the Dirac field has a similar Lagrangian density, which is invariant under global gauge transformations with gauge group \( U(1) \). As in the example above, we must use covariant derivatives to ensure that the expression is invariant under local gauge
transformations. As before, this requires introducing a gauge field $A$, and the associated field strength $F$ is called the electromagnetic or photon field. Excitations in this field are photons, the quantum mechanical particles that make up light.

3. Chern-Simons Theory

3.1. Basic knot theory. We begin this section with a review of some of the classical results of knot theory. As we will see, quantum field theory provides an elegant framework in which these results can be understood.

**Definition 3.1.1.** A knot in $M$ is a submanifold of $M$ diffeomorphic to a circle. A link is a submanifold diffeomorphic to a disjoint union of circles.

We will typically represent knots and links by drawing two dimensional diagrams. Formally, if $L$ is a link, we obtain a link diagram by taking a projection $p$ onto $\mathbb{R}^2$ in such a way that any point of $p(L)$ has a neighborhood that looks like a single segment, or two segments crossing at an angle. Below are diagrams of the trefoil, the figure-8 knot, and the Hopf link.

We will often consider knots and links equipped with a vector field called a framing. Precisely, if $L$ is a link and $p$ is any point on $L$, then we view the tangent space $T_pL$ as a subspace of $T_pM$. Let $v$ be a smooth function $L \to TM$ which assigns to each $p \in L$ a vector $v_p \in T_pM$.

**Definition 3.1.2.** We say that $v$ is a framing for the link $L$ if $v_p \notin T_pL$ for all $p \in L$. A link equipped with a framing is called a framed link.

In knot theory, one is interested in understanding when two knots or links are equivalent. The appropriate notion of equivalence in this case is the following.

**Definition 3.1.3.** We say that two links $L$ and $L'$ are ambient isotopic or simply isotopic if there exists a smooth map $F : M \times [0, 1] \to M$ such that $F(\cdot, 0)$ is the identity on $M$, $F(\cdot, 1)$ takes $L$ to $L'$, and $F(\cdot, t)$ is a diffeomorphism for each $t \in [0, 1]$. If in addition $L$ and $L'$ have orientation or framing, then we also insist that $F(\cdot, 1)$ take the orientation or framing of $L$ to the orientation or framing of $L'$.

When doing computations in knot theory, we can transform a link diagram into one representing an isotopic link by applying certain special operations at crossings.
**Definition 3.1.4.** The operations illustrated in the following three diagrams are called *Reidemeister moves* I, II, and III.

![Diagrams](image)

When dealing with links equipped with framing, we consider the following operation called Reidemeister move I’:

![Diagram](image)

**Proposition 3.1.5.** Two link diagrams of unframed links represent the same isotopy class if we can get from one to the other by applying Reidemeister moves I, II, and III at crossings. Two diagrams of framed links represent the same isotopy class if we can get from one to the other by applying moves I’, II, and III.

### 3.2. Invariants

Now that we have defined isotopy of links, we can start to look for isotopy invariants which will enable us to classify links up to isotopy. We begin by studying a simple invariant of links equipped with an orientation. If we look at a diagram of such a link, then any crossing looks like

![Crossings](image)

and we call these two types of crossings *right-handed* and *left-handed*, respectively.

**Definition 3.2.1.** The *writhe* $w(L)$ of a link $L$ is defined to be the number of right-handed crossings minus that number of left-handed crossings. That is,

$$w(L) = \#(\text{right-handed}) - \#(\text{left-handed})$$

It is easy to see that the writhe is invariant under Reidemeister moves I’, II, and III, and thus gives an invariant of framed links.
Definition 3.2.2. The Kauffman bracket of a link $L$ is the function $\langle L \rangle$ in the variables $A$, $B$, and $d$ determined by the following “skein” relations.

\[
\begin{align*}
\langle \emptyset \rangle &= 1 \\
\langle \bigcirc \bigcirc \rangle &= d \langle \bigcirc \bigcirc \rangle \\
\langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle &= A \langle \bigcirc \bigcirc \rangle + B \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle \\
\end{align*}
\]

Rather than attempt to explain in words the precise meaning of these relations, we illustrate the computation of the Kauffman bracket with an example.

Example 3.2.3. The Kauffman bracket of the Hopf link is

\[
\begin{align*}
\langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle &= A \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle + B \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle \\
&= A^2 \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle + AB \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle \\
&\quad + BA \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle + B^2 \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle \\
&= (A^2 + B^2)d^2 + 2ABd.
\end{align*}
\]

Theorem 3.2.4. The Kauffman bracket is an isotopy invariant of framed links for $B = A^{-1}$ and $d = -(A^2 + A^{-2})$.

Proof. By Proposition 3.1.5, it suffices to check that the Kauffman bracket is invariant under Reidemeister moves I’, II, and III. Indeed, we have invariance under move I’ since

\[
\begin{align*}
\langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle &= A \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle + B \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle \\
&= Ad \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle + B \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle \\
&= (Ad + B) \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle \\
&= A \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle + B \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle \\
&= \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle.
\end{align*}
\]
We also have $AB = 1$ and $(A^2 + B(Ad + B)) = 0$, and therefore
\[
\langle \begin{array}{c}
\circlearrowleft \\
\end{array} \rangle = A\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle + B\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle \\
= A\left(A\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle + B\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle \right) + B(Ad + B)\langle \begin{array}{c}
\circlearrowleft \\
\end{array} \rangle \\
= AB\langle \begin{array}{c}
\circlearrowleft \\
\end{array} \rangle + (A^2 + B(Ad + B))\langle \begin{array}{c}
\circlearrowleft \\
\end{array} \rangle \\
= \langle \begin{array}{c}
\circlearrowleft \\
\end{array} \rangle.
\]
Hence the Kauffman bracket is invariant under move II. Finally, it is invariant under move III since
\[
\langle \begin{array}{c}
\cap \\
\cap \\
\end{array} \rangle = A\langle \begin{array}{c}
\cap \\
\cap \\
\end{array} \rangle + B\langle \begin{array}{c}
\cap \\
\cap \\
\end{array} \rangle \\
= A\langle \begin{array}{c}
\cap \\
\cap \\
\end{array} \rangle + B\langle \begin{array}{c}
\cap \\
\cap \\
\end{array} \rangle \\
= \langle \begin{array}{c}
\cap \\
\cap \\
\end{array} \rangle.
\]
This completes the proof. □

The proof of Theorem 3.2.4 shows in particular that the Kauffman bracket is not invariant under Reidemeister move I. Indeed, our proof of invariance under the modified Reidemeister move I' shows that
\[
\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle = (Ad + B)\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle = -A^3\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle,
\]
and a nearly identical computation shows
\[
\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle = -A^{-3}\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle.
\]
Although the Kauffman bracket is not invariant under Reidemeister move I, these computations show that if we multiply it by the factor $(-A^{-3})^{w(L)}$, we will have a Laurent polynomial which is invariant under this move. This motivates the following definition.

**Definition 3.2.5.** Let $L$ be an oriented link. Then the *Jones polynomial* associated to $L$ is
\[
V_L(A) = (-A^{-3})^{w(L)}\langle \begin{array}{c}
\circlearrowright \\
\end{array} \rangle(A).
\]

It is immediate from the above discussion that the Jones polynomial is an invariant of unframed oriented links.
3.3. The Chern-Simons form. In the previous section, we formulated Yang-Mills theory in terms of the Yang-Mills action and indicated its use in quantum electrodynamics. We are now going to study a different field theory called Chern-Simons theory. We will see that this theory is similar to Yang-Mills theory, but instead of using it to describe a physical system, we will talk about its applications to topology. Specifically, we will discuss how Chern-Simons theory computes topological invariants of knots embedded in $M$.

To begin, recall that we defined the Yang-Mills action to be the integral over $M$ of the Lagrangian density $\text{Tr}(F \wedge \ast F)$. Likewise in Chern-Simons theory, the action is obtained by integrating a certain 3-form on an orientable 3-manifold $M$. To define this 3-form, we will assume that the gauge group $G$ is simply connected. In this case every principal $G$-bundle is trivial, and once we choose a trivialization, we can identify any connection on the bundle with a $g$-valued 1-form $A$ on $M$. Then the action is obtained by integrating the 3-form

$$\text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

which is known as the Chern-Simons form. Let us explain the notation in the above expression. As in Yang-Mills theory, we are assuming our Lie algebra $g$ admits an invariant inner product $\langle -, - \rangle$. The first term of the Chern-Simons form is obtained by wedging $A$ with $dA$ to get a $g \otimes g$-valued 3-form, and then composing with the inner product to get an ordinary real-valued 3-form which we can integrate over $M$. The second term should really be thought of as

$$\frac{1}{3} \text{Tr}([A, A] \wedge A)$$

where we first construct the $g \otimes g$-valued 3-form $[A, A] \wedge A$ and then compose with the inner product to get a real-valued 3-form.

3.4. Holonomy. Suppose that $\gamma : [0, T] \to M$ is a smooth path from $p$ to $q$ and for each $t \in [0, T]$ let $u(t)$ be a vector in the fiber of $P \times_G V$ over $\gamma(t)$. Recall that the covariant derivative of a section $s$ can be written as partial derivatives plus a vector potential:

$$\nabla_\mu s = \sum_{i=1}^n (\partial_\mu s_i) e_i + A_\mu s.$$

We would also like to be able to differentiate $u(t)$ in the direction $\gamma$ is going, namely $\gamma'(t)$. In analogy with the above formula, we define

$$\nabla_{\gamma'(t)} u(t) = \frac{d}{dt} u(t) + A(\gamma'(t)) u(t).$$

**Definition 3.4.1.** We will say that $u(t)$ is parallel transported along $\gamma$ if $\nabla_{\gamma'(t)} u(t) = 0$ for all $t \in [0, T]$.

Intuitively, the statement that $u(t)$ is parallel transported along $\gamma$ means we can “drag” the vector $u(0)$ along $\gamma$, and $u(t)$ is the resulting vector at $\gamma(t)$. Suppose we are given a vector $u$ in the fiber over $p$. By the familiar existence result for ordinary differential equations, we can find a function $u(t)$ which satisfies

$$\frac{d}{dt} u(t) + A(\gamma'(t)) u(t) = 0.$$
with the initial condition \( u(0) = u \). This means we can parallel translate \( u \) along any smooth path. Let \( H(\gamma, \nabla)u = u(T) \) denote the result of parallel translating \( u \) along the path \( \gamma \) to the point \( q \). Since the differential equation defining parallel transport is linear, the map \( H(\gamma, \nabla) : V \to V \) is a linear transformation.

**Definition 3.4.2.** The map \( H(\gamma, \nabla) : V \to V \) is called the holonomy along the path \( \gamma \).

Let us investigate the effect of a local gauge transformation on the holonomy. Suppose \( u(t) \) satisfies the parallel transport equation

\[
\nabla'_{\gamma'(t)}u(t) = 0.
\]

In terms of a vector potential \( A \), this equation says

\[
\frac{d}{dt}u(t) = -A(\gamma'(t))u(t).
\]

or in components

\[
\frac{d}{dt}u(t) = -\gamma'^\mu(t)A_\mu u(t).
\]

Now a local gauge transformation \( g : M \to G \) takes \( u(t) \) to a function \( w(t) = g(\gamma(t))u(t) \). Differentiating this new function, we obtain

\[
\frac{d}{dt}w(t) = \left( \frac{d}{dt}g(\gamma(t)) \right) u(t) + g(\gamma(t)) \frac{d}{dt}u(t)
= \gamma'^\mu(t)(\partial_\mu g)u(t) - g\gamma'^\mu(t)A_\mu u(t)
= \gamma'^\mu(t)(\partial_\mu g)g^{-1}w(t) - \gamma'^\mu(t)gA_\mu g^{-1}w(t)
\]

We are assuming that the gauge group \( G \) is a group of matrices, so the gauge transformed vector potential \( A' \) can be written \( A'_\mu = gA_\mu g^{-1} - (\partial_\mu g)g^{-1} \) and therefore

\[
\frac{d}{dt}w(t) = -\gamma'^\mu(t)A'_\mu w(t).
\]

This proves that \( w \) satisfies the parallel transport equation \( \nabla'_{\gamma'(t)}w(t) = 0 \) where \( \nabla' \) is the connection obtained by applying the gauge transformation \( g \) to \( \nabla \). By definition then, the holonomy \( H(\gamma, \nabla') \) maps \( g(0)u(0) \) to \( g(T)u(T) \), and we have

\[
H(\gamma, \nabla') = g(\gamma(T))H(\gamma, \nabla)g(\gamma(0))^{-1}.
\]

### 3.5. Wilson loops.

Let \( \gamma \) be a loop in \( M \) based at \( p \). Then the transformation law that we derived above for holonomy becomes

\[
H(\gamma, \nabla') = g(p)H(\gamma, \nabla)g(p)^{-1}
\]

and therefore the trace of the holonomy is invariant under gauge transformations:

\[
\text{Tr} \; H(\gamma, \nabla') = \text{Tr}(g(p)H(\gamma, \nabla)g(p)^{-1}) = \text{Tr} \; H(\gamma, \nabla).
\]

**Definition 3.5.1.** The trace of the holonomy along \( \gamma \) is called a Wilson loop and is denoted

\[
W(\gamma, D) = \text{Tr}(H(\gamma, \nabla)).
\]
3.6. **Synthesis.** Finally, we are going to relate Chern-Simons theory to the link invariants that we introduced earlier in this section. Recall that in quantum field theory any physical quantity is represented by a function $f : \mathcal{F}(M) \rightarrow \mathbb{R}$ on the space of fields, and the vacuum expectation value of such a quantity is given by

$$\langle f \rangle = \frac{1}{Z} \int_{\mathcal{F}(M)} f(\varphi) e^{iS(\varphi)} D\varphi.$$ 

In a theory with gauge symmetry, two fields are physically equivalent if they are related by a gauge transformation, so in this setting we require the function $f$ to be gauge invariant. We have already seen that Wilson loops are invariant under gauge transformations, and indeed, they are among the simplest observables in gauge theory. Our main claim in this section is that the expectation values of Wilson loops in Chern-Simons theory are isotopy invariants of links. More precisely, if $L$ is a framed oriented link in $M$ with components $\gamma_1, \ldots, \gamma_n$, then the unnormalized expectation value

$$\int_{\mathcal{A}} W(\gamma_1, A) \cdots W(\gamma_n, A) e^{\frac{ik}{4\pi} S(A)} DA.$$ 

is an invariant of $L$. Here $k$ is an integer called the *level*, $S$ denotes the Chern-Simons action, and $\mathcal{A}$ denotes the space of all connections on $M$. As a special case, we can consider this integral for the empty link, and we find that the partition function

$$Z = \int_{\mathcal{A}} e^{\frac{ik}{4\pi} S(A)} DA$$

is an invariant of the 3-manifold $M$. In order to turn these claims into completely precise results, we would need to rigorously define the measure $DA$ appearing in the integrals and make a number of technical modifications to the statements. Although we will not attempt to do this here, we emphasize that these integrals have been made precise in certain special cases, and they are known to give rise to familiar invariants. For example, let $\mathcal{L}(L)$ denote our alleged knot invariant above, and consider the case where $M = S^3$ and $\rho$ is the fundamental representation of $U(1)$. Then we have

$$\mathcal{L}(L) = e^{i\pi w(L)/k}$$

where $w$ denotes the writhe of $L$. Alternatively, if we take $\rho$ to be the fundamental representation of $SU(2)$, then $\mathcal{L}(L)$ is simply the Kauffman bracket of $L$ evaluated at $A = q^{1/4}$ where $q = e^{\frac{2\pi i}{k+2}}$. By varying the gauge group of the theory, we obtain a wealth of other interesting invariants.

4. **Atiyah’s Definition**

4.1. **Cobordisms.** One of the major advantages of doing topology with TQFTs is the fact that fields are defined locally. This means that we can compute invariants by breaking our manifolds into simpler pieces and computing invariants on the pieces. In order to study TQFTs axiomatically, we should therefore talk about how to break a manifold into simple pieces, or equivalently, how to build up a closed $n$-manifold by gluing together simpler $n$-manifolds along their $(n - 1)$-dimensional boundaries.
The manifolds that we will use to build up more complicated manifolds by gluing are called “cobordisms”. Roughly speaking, a cobordism is an $n$-manifold that connects together two $(n - 1)$-dimensional manifolds.

**Definition 4.1.1.** Let $\Sigma_0$ and $\Sigma_1$ be closed $(n - 1)$-manifolds. A cobordism between $\Sigma_0$ and $\Sigma_1$ is an $n$-dimensional manifold with boundary $\partial M = \Sigma_0 \coprod \Sigma_1$.

**Example 4.1.2.** The illustrations below show two different ways to connect the manifold $\Sigma_0 = S^1 \coprod S^1$ to the manifold $\Sigma_1 = S^1 \coprod S^1 \coprod S^1$ by a two-dimensional cobordism. As illustrated in the picture on the right, neither the $\Sigma_i$ nor the cobordism $M$ is required to be a connected manifold.

Usually, we think of a cobordism as having a “direction” so that some of its boundary components can be viewed as the “source” and the others can be viewed as the “target” of the cobordism. To make this idea precise, consider an oriented manifold $M$ with boundary, and let $\Sigma$ be a connected component of the boundary of $M$. Given a point $x \in \Sigma$, let $[v_1, \ldots, v_{n-1}]$ be a positive basis for the tangent space $T_x \Sigma$. Then a vector $w \in T_x M$ is called a positive normal if $[v_1, \ldots, v_{n-1}, w]$ is a positive basis for $T_x M$. If a positive normal points inward, then $\Sigma$ is called an in-boundary, while if it points outward, then $\Sigma$ is called an out-boundary. (It is a nontrivial theorem in differential topology that these notions do not depend on the choice of the point $x$.)

**Definition 4.1.3.** Let $\Sigma_0$ and $\Sigma_1$ be closed oriented $(n - 1)$-manifolds. An oriented cobordism $M$ from $\Sigma_0$ to $\Sigma_1$ is a compact oriented manifold with a map $\Sigma_0 \to M$ taking $\Sigma_0$ diffeomorphically onto the in-boundary of $M$ and a map $\Sigma_1 \to M$ taking $\Sigma_1$ diffeomorphically onto the out-boundary of $M$.

For example, either of the manifolds illustrated in the above example can be regarded as an oriented cobordism once we choose an orientation. When drawing pictures of oriented cobordisms, we will always think of the cobordism as going from top to bottom so that the boundary components near the top of the page form the in-boundary, and the ones near the bottom of the page form the out-boundary.

**Definition 4.1.4.** We say that two cobordisms $\Sigma_0 \to M \leftrightarrow \Sigma_1$ and $\Sigma_0 \to M' \leftrightarrow \Sigma_1$ are equivalent if there exists an orientation-preserving diffeomorphism $\varphi : M \to M'$ making the following diagram commute.
4.2. Composition of cobordisms. Since every oriented cobordism has a source and target, it is natural to think of such cobordisms as the morphisms in a category whose objects are \((n - 1)\)-manifolds. Given a cobordism \(M_0\) from \(\Sigma_0\) to \(\Sigma_1\) and a cobordism \(M_1\) from \(\Sigma_1\) to \(\Sigma_2\), we would like to define their composite to be the cobordism obtained by gluing \(M_0\) to \(M_1\) along the manifold \(\Sigma_1\). The illustration below shows an example for \(n = 2\).

\[
g \circ f = g \circ f
\]

While the space \(M_0 \coprod_{\Sigma} M_1\) that we obtain by gluing two cobordisms does have a natural manifold structure, it is not immediately obvious that it can be given smooth structure. We will now define the smooth structure in two steps. The first step is to define the smooth structure when the cobordisms being composed are of a special type.

**Definition 4.2.1.** A cylinder is a manifold of the form \(\Sigma \times [a, b]\) where \(\Sigma\) is a closed manifold of dimension \(n - 1\).

We will always regard such a cylinder as an oriented cobordism from \(\Sigma\) to itself. Let \(M_0\) and \(M_1\) be cobordisms that are equivalent (in the sense of Definition 4.1.4) to cylinders \(\Sigma_0 \times [0, 1]\) and \(\Sigma_1 \times [1, 2]\), the equivalences being given by \(\varphi_0 : M_0 \to \Sigma \times [0, 1]\) and \(\varphi_1 : M_1 \to \Sigma \times [1, 2]\). Then there is a homeomorphism

\[
\varphi_1 \coprod_{\Sigma} \varphi_2 : M_0 \coprod_{\Sigma} M_1 \to \Sigma \times [0, 2]
\]

The manifold \(\Sigma \times [0, 2]\) has smooth structure which agrees with that of \(\Sigma_0 \times [0, 1]\) and \(\Sigma_1 \times [1, 2]\), so we can put a smooth structure on \(M_0 \coprod_{\Sigma} M_1\) by pulling back the atlas of \(\Sigma \times [0, 2]\) along this homeomorphism.

The next step is to use the idea of the last paragraph to define the smooth structure on an arbitrary composition of oriented cobordisms. Here we require a technical notion from the subject known as Morse theory.

**Definition 4.2.2.** Let \(M\) be a compact manifold and \(f : M \to [0, 1]\) a smooth map to the closed unit interval. A critical point \(x\) of this function \(f\) is said to be nondegenerate if the matrix

\[
\frac{\partial^2 f}{\partial x^j \partial x^j}
\]

is nonsingular in any coordinate system, and \(f\) is called a Morse function if all of its critical points are nondegenerate and \(f^{-1}(\{0, 1\}) = \partial M\).

A theorem of differential topology states that Morse functions always exist, so if \(M_0\) and \(M_1\) are the oriented cobordisms we wish to compose, we can take Morse functions \(f_0 : M_0 \to [0, 1]\) and \(f_1 : M_1 \to [1, 2]\). By choosing \(\varepsilon > 0\) to be small, we can ensure that \(f_0\) and \(f_1\) are regular on \([1 - \varepsilon, 1]\) and \([1, 1 + \varepsilon]\). Then the preimages of these two intervals are diffeomorphic to cylinders, and we know how to get a smooth structure on the composition.
4.3. Composition of cobordism classes. Finally, we show that the composition of two cobordisms does not depend on the actual cobordisms chosen, but only on their equivalence classes. For this we need the following result from differential topology.

**Proposition 4.3.1.** Let $M_0$ and $M_1$ be composable cobordisms with common boundary component $\Sigma$, and let $M_1 \circ M_0 = M_0 \bigsqcup \Sigma M_1$ be their composition. If $\alpha$ and $\beta$ are two smooth structures on $M_1 \circ M_0$ which both induce the original smooth structure on $M_0$ and $M_1$, then there is a diffeomorphism $M_1 \circ M_0 \to M_1 \circ M_0$ taking $\alpha$ to $\beta$.

Suppose we are given composable cobordisms $M_0$ and $M_1$ which are equivalent (rel the boundary) to cobordisms $M'_0$ and $M'_1$:

$$
\begin{array}{cccc}
\Sigma_0 & \rightarrow & M_0 & \leftarrow & \Sigma_1 \\
& \downarrow^\varphi_0 & & \downarrow^\varphi_1 & \\
M'_0 & & M_1 & & \Sigma_2 \\
& \downarrow & & \downarrow & \\
& M'_0 & & M'_1 \\
\end{array}
$$

Then we have the composition $M_1 \circ M_0$ and the composition $M'_1 \circ M'_0$, and the maps $\varphi_0$ and $\varphi_1$ glue to give a homeomorphism $\varphi : M_1 \circ M_0 \to M'_1 \circ M'_0$ which restricts to a diffeomorphism on each of the original cobordisms:

$$
\begin{array}{cccc}
\Sigma_0 & \rightarrow & M_1 \circ M_0 & \leftarrow & \Sigma_2 \\
& \downarrow^\varphi & & \\
M'_1 \circ M'_0 & \\
\end{array}
$$

We can use this homeomorphism to define a smooth structure on $M'_1 \circ M'_0$. Although this smooth structure may not coincide with the original, Proposition 4.3.1 implies that the two structures are diffeomorphic rel the boundary. Thus we see that there is a well defined way to compose equivalence classes of oriented cobordisms.

4.4. The category of cobordisms. The notions introduced so far allow us to define a category $\text{Cob}(n)$ which is an important ingredient in Atiyah’s definition of a TQFT. The objects of this category are closed oriented $(n-1)$-dimensional manifolds. Given two such objects $\Sigma_0$ and $\Sigma_1$, a morphism $\Sigma_0 \to \Sigma_1$ is a diffeomorphism class of oriented cobordisms from $\Sigma_0$ to $\Sigma_1$. It is simple to check that the composition law defined above is associative. If $M$ is a cobordism from $\Sigma_0$ to $\Sigma_1$ and $C$ is the cylinder on $\Sigma_0$, then one can check that $M \circ C = M$ up to equivalence, and similarly $C \circ M = M$ if $C$ is the cylinder on $\Sigma_1$. This proves that $\text{Cob}(n)$ is in fact a category. This category is called the category of $n$-dimensional cobordisms.

4.5. Some terminology. Before we get to the definition of a TQFT, we have to discuss some ideas from category theory. In the discussion that follows, we will encounter definitions involving many commutative diagrams, and we would like to use the notion of a natural transformation to simplify these definitions. Recall that a natural transformation $\alpha$ between a functor $F : \mathcal{C} \to \mathcal{D}$ and a functor $G : \mathcal{C} \to \mathcal{D}$ consists of a morphism $\alpha_X : F(X) \to G(X)$ for each object $X$ of $\mathcal{C}$. These morphisms are called the components of $\alpha$. In the following, if $F, G : \mathcal{C}_1 \times \cdots \times \mathcal{C}_n \to \mathcal{D}$ are functors and $\alpha_{X_1, \ldots, X_n} : F(X_1, \ldots, X_n) \to G(X_1, \ldots, X_n)$ are
morphisms, then these $\alpha$’s are called “natural isomorphisms” or a “natural transformation” if they are the components of a natural isomorphism or natural transformation $F \to G$.

4.6. **Monoidal categories.** The main reason for the present categorical digression is that we need to equip our categories with an operation $\boxtimes$ so that for any two objects $X$ and $Y$ we can form the “product” $X \boxtimes Y$. The example to keep in mind is the disjoint union operation in the category of $n$-dimensional cobordisms. We want this operation $\boxtimes$ to be unital and associative so that our category is in some ways like a monoid. It is unnatural, however, to require the operation to be strictly unital and associative. Instead, we would like for $\boxtimes$ to be unital and associative up to isomorphism, and the isomorphisms $(X \boxtimes Y) \boxtimes Z \cong X \boxtimes (Y \boxtimes Z)$ and $1 \boxtimes X \cong X \cong X \boxtimes 1$ should be part of the structure, satisfying certain axioms called coherence conditions.

**Definition 4.6.1.** A monoidal category is a category $\mathcal{C}$ together with a functor $\boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a distinguished object $1 \in \mathcal{C}$, and natural isomorphisms $\alpha_{X,Y,Z} : (X \boxtimes Y) \boxtimes Z \to X \boxtimes (Y \boxtimes Z)$, $\lambda_X : 1 \boxtimes X \to X$, and $\rho_X : X \boxtimes 1 \to X$. These isomorphisms are required to make the diagram

![Diagram](attachment:diagram.png)

and the diagram

![Diagram](attachment:diagram2.png)

commute for all objects $X$, $Y$, and $Z$.

It is easy to see that the above definition gives a categorical analog of the notion of a monoid. In these notes, what we really need is a categorical analog of the notion of a commutative monoid. We can get this by requiring our category to include isomorphisms $X \boxtimes Y \to Y \boxtimes X$ that reverse the order of factors. These isomorphisms are part of the structure and are required to satisfy additional coherence conditions.
Definition 4.6.2. A **braided monoidal category** is a monoidal category $\mathcal{C}$ together with natural isomorphisms $\sigma_{X,Y} : X \square Y \to Y \square X$, such that the diagrams

\[
\begin{array}{ccc}
X \square (Y \square Z) \xrightarrow{\alpha_{X,Y,Z}} (Y \square Z) \square X & \quad & (X \square Y) \square Z \xrightarrow{\alpha_{X,Y,Z}^{-1}} Z \square (X \square Y) \\
\sigma_{X,Y,Z} & \quad & \sigma_{X,Y,Z}^{-1}
\end{array}
\]

\[
\begin{array}{ccc}
X \square (Z \square Y) & \quad & (Z \square X) \square Y \\
\sigma_{X,Y,Z} \quad & \quad & \sigma_{X,Y,Z}^{-1}
\end{array}
\]

\[
\begin{array}{ccc}
(1) & \quad & \\
Y \square (Z \square X) & \quad & X \square (Y \square Z) & \quad & (Z \square X) \square Y
\end{array}
\]

commute for all $X$, $Y$, and $Z$. A braided monoidal category is called a **symmetric monoidal category** if in addition we have $\sigma_{Y,X} \circ \sigma_{X,Y} = 1_{X \square Y}$.

Examples 4.6.3. The following symmetric monoidal categories arise frequently in category theory and the study of TQFTs.

1. The category $\bf{Cob}(n)$ or $n$-dimensional cobordisms with $\square = \coprod$.
2. The category $\bf{Vect}(k)$ of vector spaces over a field $k$ with $\square = \otimes_k$.
3. The category $\bf{Set}$ of sets with $\square = \coprod$.
4. The category $\bf{Set}$ of sets with $\square = \times$.

Now that we have defined symmetric monoidal categories and given some examples, we can define functors that preserve the structure of these categories.

Definition 4.6.4. Let $(\mathcal{C}, \square, 1_\mathcal{C})$ and $(\mathcal{D}, \square, 1_\mathcal{D})$ be two monoidal categories. A **monoidal functor** between these categories is a functor $F : \mathcal{C} \to \mathcal{D}$, together with a natural transformation $\varphi_{X,Y} : F(X \square Y) \to F(X \square Y)$ and a morphism $\varphi : 1_\mathcal{D} \to 1_\mathcal{C}$ such that the diagram

\[
\begin{array}{ccc}
(FX \square FY) \square FZ & \xrightarrow{\alpha_{F,X,Y,Z}} & FX \square (FY \square FZ) \\
\varphi_{X,Y} \quad & \quad & 1_{FY} \square \varphi_{Y,Z} \\
FX \square (Y \square Z) & \xrightarrow{\varphi_{X,Y,1_\mathcal{D}}} & F(X \square (Y \square Z))
\end{array}
\]

and the diagrams

\[
\begin{array}{ccc}
FX \square 1_\mathcal{D} & \xrightarrow{1_{FX} \square \varphi} & FX \square 1_\mathcal{C} & \quad & 1_\mathcal{D} \square FY & \xrightarrow{\varphi \square 1_{FY}} & F1_\mathcal{C} \square FY \\
\rho_{FX} & \quad & \varphi_{X,1_\mathcal{C}} & \quad & \lambda_{FY} & \quad & \varphi_{1_\mathcal{C},Y}
\end{array}
\]

commute for all $X$, $Y$, $Z \in \mathcal{C}$. A monoidal functor between two symmetric monoidal categories is called a **symmetric monoidal functor** if, in addition to satisfying the axioms above,
the following diagram is commutative for all $X$ and $Y$ in $C$.

$$
\begin{array}{c}
F(X \Box Y) F(Y \Box X) \\
\phi_{X,Y} \\
\end{array}
\quad
\begin{array}{c}
FX \Box FY \\
\phi_{X,Y} \\
\end{array}
\quad
\begin{array}{c}
FY \Box FX \\
\phi_{Y,X} \\
\end{array}
$$

4.7. **Topological quantum field theories.** We have now introduced the language of symmetric monoidal categories, and we are ready to talk about TQFTs. Recall that in order to specify a quantum field theory, we must specify a spacetime manifold $M$, a vector space $\mathcal{F}(M)$ of fields on $M$, an action $S : \mathcal{F}(M) \to \mathbb{R}$, and a measure $D\varphi$. The goal of topological field theory is to construct a topological invariant of a closed manifold $M$ from the action $S$ and measure $D\varphi$.

In fact, we will associate an invariant not only to closed manifolds, but to any $n$-dimensional cobordism as well. Let $\Sigma_1$ and $\Sigma_2$ be closed $(n-1)$-dimensional manifolds, and let $M$ be a cobordism from $\Sigma_1$ to $\Sigma_2$. Write $\text{Fun}(\mathcal{F}(\Sigma_i))$ for the space of functions on $\mathcal{F}(\Sigma_i)$. Then the invariant that we associate to $M$ is the integral operator $ZM : \text{Fun}(\mathcal{F}(\Sigma_1)) \to \text{Fun}(\mathcal{F}(\Sigma_2))$ with kernel

$$
K_M(\varphi_1, \varphi_2) = \int_{\varphi \in \mathcal{F}(M)} e^{iS(\varphi)} D\varphi.
$$

As a special case, we can take $M$ to be a closed $n$-manifold, regarded as a cobordism from the empty $(n-1)$-manifold to itself. By convention, the vector space $\mathcal{F}(\emptyset)$ is the zero space so that $\text{Fun}(\mathcal{F}(\emptyset)) \cong \mathbb{R}$. Thus $ZM$ is a linear map $\mathbb{R} \to \mathbb{R}$, which is the same thing as an element of $\mathbb{R}$. In this way, our construction assigns a numerical invariant to any closed oriented $n$-manifold.

To describe this invariant more explicitly, let $M$ be a closed $n$-manifold, regarded as a cobordism from $\Sigma_1 = \emptyset$ to $\Sigma_2 = \emptyset$. We write $0_-$ for the unique field in $\mathcal{F}(\Sigma_1)$ and $0_+$ for the unique field in $\mathcal{F}(\Sigma_2)$. For any $f \in \text{Fun}(\mathcal{F}(\Sigma_1))$, we have

$$
ZM(f)(0_+) = \int_{\psi_- \in \mathcal{F}(\Sigma_1)} f(\psi_-) \left( \int_{\varphi \in \mathcal{F}(M)} e^{iS(\varphi)} D\varphi \right) D\psi_-
$$

$$
= \left( \int_{\mathcal{F}(M)} e^{iS(\varphi)} D\varphi \right) f(0_-)
$$

since the total measure of $\mathcal{F}(\Sigma_1)$ equals 1. The invariant that we are assigning to $M$ is therefore just the partition function $\int_{\mathcal{F}(M)} e^{iS(\varphi)} D\varphi$ of the field theory.

The reason for making our field theory assign invariants to all $n$-dimensional cobordisms and not just closed $n$-manifolds is that this enables us to compute invariants by breaking our manifolds into simpler pieces and computing invariants on the pieces. For example, let $\Sigma_1$, $\Sigma_2$, and $\Sigma_3$ be closed $(n-1)$-manifolds, and let $M_1$ be a cobordism from $\Sigma_1$ to $\Sigma_2$, $M_2$ a cobordism from $\Sigma_2$ to $\Sigma_3$. Then

$$
(ZM_2 \circ ZM_1)(f)(\psi_+) = \int_{\psi \in \mathcal{F}(\Sigma_3)} \int_{\psi_- \in \mathcal{F}(\Sigma_1)} f(\psi_-) K(\psi_-, \psi) K(\psi, \psi_+) D\psi_- D\psi
$$
Let $M$ be the cobordism obtained by gluing $M_1$ and $M_2$ along $\Sigma_2$. We know that if $\varphi$ is a field on $M$ and $\varphi_1 = \varphi|_{M_1}$, $\varphi_2 = \varphi|_{M_2}$, then the action satisfies $S(\varphi) = S(\varphi_1) + S(\varphi_2)$. Moreover, since fields are defined locally, we know that such a field $\varphi$ is completely determined by its restriction to the manifolds $M_i$ and the $\Sigma_i$. Therefore

$$
\int_{\psi \in F(\Sigma_2)} K(\psi_-, \psi) K(\psi, \psi_+) D\psi \\
= \int_{\psi \in F(\Sigma_2)} \left( \int_{\varphi_1 \in F(M_1)} e^{iS(\varphi_1)} D\varphi_1 \right) \left( \int_{\varphi_2 \in F(M_2)} e^{iS(\varphi_2)} D\varphi_2 \right) D\psi \\
= \int_{\psi \in F(M)} e^{i(S(\varphi|M_1) + S(\varphi|M_2))} D\varphi \\
= \int_{\psi \in F(M)} e^{iS(\varphi)} D\varphi \\
= K(\psi_-, \psi_+).
$$

Plugging this result back into the previous equation gives $Z(M_2 \circ M_1) = Z M_2 \circ Z M_1$. We can express this fact more abstractly by saying that a topological quantum field theory is a functor from $\text{Cob}(n)$ to $\text{Vect}(k)$ which assigns the vector space $\text{Fun}(F(\Sigma))$ to an object $\Sigma$ and assigns the linear map $Z M$ to a cobordism $M$.

In addition, it is possible to convince oneself that a topological quantum field theory should preserve the monoidal structure of these categories. Locality of fields implies that $F(\Sigma_1 \coprod \Sigma_2) = F(\Sigma_1) \times F(\Sigma_2)$, and it follows that $\text{Fun}(F(\Sigma_1 \coprod \Sigma_2)) \cong \text{Fun}(F(\Sigma_1)) \otimes \text{Fun}(F(\Sigma_2))$. One can argue, using the sort of heuristic arguments involving Feynman integrals that we have been using, that $Z(M_1 \coprod M_2) = Z M_1 \otimes Z M_2$. We are thus led to make the following definition.

**Definition 4.7.1.** An $n$-dimensional topological quantum field theory is a symmetric monoidal functor $\text{Cob}(n) \rightarrow \text{Vect}(k)$.

We should emphasize at this point that the above “computations” involving Feynman integrals are not really well defined because we do not know how to define the measure $D\varphi$. The only reason for including them in the discussion is that they provide some physical motivation for the definition of a TQFT. On the other hand, Definition 4.7.1 is completely precise, and it provides a rigorous framework for constructing quantum invariants of spaces.

4.8. Dijkgraaf-Witten theory. A simple example of a TQFT in the sense of Atiyah is the theory introduced by Dijkgraaf and Witten. This example is similar in some ways to a gauge theory but has a well defined partition function. Precisely, if $G$ is a finite group and $\Sigma$ a manifold, then we write $F(\Sigma)$ for the set of all isomorphism classes of principal $G$-bundles on $\Sigma$. If $M$ is a surface with boundary $\partial M = \Sigma_0 \coprod \Sigma_1$, then the kernel defining $Z M : \text{Fun}(F(\Sigma_1)) \rightarrow \text{Fun}(F(\Sigma_2))$ is

$$K_M(\pi_1, \pi_2) = \sum_{\pi \in F(M) \atop \pi|\Sigma_1 = \pi_1, \pi|\Sigma_2 = \pi_2} \frac{1}{|\text{Aut}(\pi)|}.$$
where $\text{Aut}(\pi)$ is the finite group of automorphisms of $\pi$. Then

$$ZM(f)(\pi_2) = \sum_{\pi_1 \in \mathcal{F}(\Sigma_1)} K_M(\pi_1, \pi_2) f(\pi_1).$$

Since the group $G$ is finite, it follows that the sums in these expressions are finite and there are no problems of convergence. One can easily check that these data define a TQFT in the sense of Definition 4.7.1.

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**References**