Geodesic tracking for random walks on groups

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Summary

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2. Review on the mapping class group
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3. Main theorem
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4. Stable visibility and other groups
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$$\lim_{n \to \infty} \frac{X_1(\omega) + \cdots + X_n(\omega)}{n} = \ell$$
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The sequence

$$x_0, w_1 x_0, w_2 x_0, \ldots, w_n x_0, \ldots$$

is called a sample path.
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$$\text{Prob}(|w_{n+1}| > |w_n|) = \frac{3}{4}$$
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How about 3.?
Sublinear tracking

We say that a random walk on $G$ acting on the geodesic metric space $(X, d)$ has the **Sublinear Tracking property (ST)** if for almost every sample path $w_n$ there exists a geodesic ray $\gamma : [0, \infty) \to X$ such that

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2. Kaimanovich’s criterion: sublinear tracking allows one to identify the Poisson boundary of the walk

$$H^\infty(G, \mu) = L^\infty(\partial X, \nu)$$
History

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- Discrete groups of isometries of CAT(0) spaces [Karlsson-Margulis, ’99]
Mapping class groups

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E.g.: Dehn twist around a curve
Teichmüller space

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Thurston compactified $\mathcal{T}(S)$ with the space $\text{PMF}$ of projective measured foliations. $\text{PMF} \sim S^{6g-6}$ and $\text{Mod}(S)$ acts on $\text{PMF}$ by homeomorphisms.
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\( \mathcal{T}(S) \) is a geodesic metric space and the quotient

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Let $\mu$ a prob. measure on $G = \text{Mod}(S)$ and fix a basepoint $X \in \mathcal{T}(S)$. 

[9x252]Random walks on mapping class groups

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[2028x2140]Then almost every sample path converges to some foliation on $\partial \mathcal{T}(S) = \text{PMF}$. [Kaimanovich-Masur, '96]

[2028x2120]Question [Kaimanovich, '00]: does the sublinear tracking property holds for random walks on $\text{Mod}(S)$? 

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Main theorem: (ST) in the mapping class group

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\int_G d_T(x_0, gx_0) \, d\mu < \infty.
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Corollary
Poisson boundary = Thurston boundary [Kaimanovich-Masur]
Other groups: stable visibility

Let $X \subseteq \overline{X} = X \cup \partial X$ a compactification of $X$, on which $G$ acts by homeomorphisms.
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The compactification is **stably visible** if for each sequence $\gamma_n := [\eta_n, \xi_n] \subseteq X$ of geodesic segments such that $\eta_n \to \eta$, $\xi_n \to \xi$ there exists a bounded set $B$ which intersects all $\gamma_n$. 

E.g.:

▶ 4-valent tree is stably visible,
▶ hyperbolic plane $H_2$ is stably visible
▶ $\mathbb{Z}_2$ is NOT stably visible
▶ A Gromov-hyperbolic group is stably visible
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Let $X$ admit a stably visible compactification, and $\mu$ be a prob. measure on $G$, such that the group generated by its support is non-elementary, and with finite first moment.

**Corollary**

The sublinear tracking property holds for:
- Gromov-hyperbolic groups;
- groups with infinitely many ends;
- relatively hyperbolic groups.
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The end

Thank you!