Tuning and plateaux for the entropy of $\alpha$-continued fraction transformations

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Marseille, May 24, 2012
Joint work with C. Carminati (Pisa)
Summary

1. $\alpha$-continued fractions
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2. The entropy function $h(\alpha)$
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3. Quadratic intervals and matching
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4. Tuning operators
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5. Characterization of plateaux
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2. The entropy function $h(\alpha)$
3. Quadratic intervals and matching
4. Tuning operators
5. Characterization of plateaux
6. Local monotonicity of the entropy
Euclid’s algorithm and continued-fractions

\[ x = \frac{p}{q} \]

\[ p = a_0 q + r_0 \]
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\[ \frac{p}{q} = a_0 + \frac{1}{a_1 + \frac{r_1}{r_0}} = a_0 + \frac{1}{a_1 + \frac{1}{\frac{1}{a_{k-1} + \frac{1}{a_k}}}} \]
Continued-fraction expansion

\[ x \in \mathbb{R} \setminus \mathbb{Q} \]

\[ x = \lfloor x \rfloor + x_0 = \]

\[ x = \lfloor x \rfloor + x_1 = \]

\[ x = \lfloor x \rfloor + x_2 = \]

\[ \vdots \]
Continued-fraction expansion

$x \in \mathbb{R} \setminus \mathbb{Q}$

\[ x = \lfloor x \rfloor + x_0 = a_0 + x_0 \quad 0 \leq x_0 \leq 1 \]

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\[ x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}} \]

INFINITE EXPANSION
Dynamical interpretation: the Gauss map

\[
\frac{1}{x_n} = \left\lfloor \frac{1}{x_n} \right\rfloor + x_{n+1}
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Dynamical interpretation: the Gauss map

\[ \frac{1}{x_n} = \left\lfloor \frac{1}{x_n} \right\rfloor + x_{n+1} \]

\[ x_{n+1} = \left\{ \frac{1}{x_n} \right\} \]
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Nakada’s $\alpha$-continued fraction transformations

For each $\alpha \in [0, 1]$, we can define a $\alpha$-euclidean algorithm, where we take the remainder to be in $[\alpha - 1, \alpha]$. It is generated by $T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]$ as follows:
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$$T_\alpha(x) := \frac{1}{|x|} - c_\alpha(x),$$
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and associated to the $\alpha$-continued fraction expansion:

$$x = \frac{\epsilon_{1,\alpha}}{c_{1,\alpha} + \frac{\epsilon_{2,\alpha}}{c_{2,\alpha} + \cdots}}, \quad c_{n,\alpha} \in \mathbb{N}^+, \epsilon_{n,\alpha} \in \{\pm 1\}$$
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Entropy

What is the average speed of convergence of the $\alpha$-euclidean algorithm?
What is the average speed of convergence of the $\alpha$-euclidean algorithm? How does it vary with $\alpha$?
Entropy

For each $\alpha$, the topological entropy of $T_\alpha$ is infinite. However, every $T_\alpha$ has a unique invariant measure $\mu_\alpha$ in the Lebesgue measure class.
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$$h(\alpha) := \int \log |T'_\alpha| d\mu_\alpha$$
Entropy

\[ h(\alpha) := \int \log |T'_\alpha| d\mu_\alpha \]

It measures:

- the **speed of convergence** of the \( \alpha \)-euclidean algorithm
Entropy

\[ h(\alpha) := \int \log |T'_\alpha| d\mu_\alpha \]

It measures:

- the speed of convergence of the \( \alpha \)-euclidean algorithm: The average number of steps over all rationals of denominator less than \( N \) is

\[ P_N(\alpha) \approx \frac{2}{h(\alpha)} \log N \]

[Bourdon-Daireaux-Vallée]
Entropy

\[ h(\alpha) := \int \log |T'_\alpha| d\mu_\alpha \]

It measures:

- the speed of convergence of the \( \alpha \)-euclidean algorithm
- the growth rate of the denominators
Entropy

\[ h(\alpha) := \int \log |T'_{\alpha}| d\mu_{\alpha} \]

It measures:

- the speed of convergence of the \( \alpha \)-euclidean algorithm
- the growth rate of the denominators: For almost every \( x \in [0, 1] \)

\[ h(\alpha) = \lim_{n \to +\infty} \frac{2}{n} \log q_{n,\alpha}(x) \]

where \( p_{n,\alpha}(x)/q_{n,\alpha}(x) \) is the \( n \)-th convergent of the \( \alpha \)-expansion of \( x \)
Entropy

\[ h(\alpha) := \int \log |T'_\alpha| d\mu_\alpha \]

It measures:

- the speed of convergence of the \( \alpha \)-euclidean algorithm
- the growth rate of the denominators
- how chaotic the map \( T_\alpha \) is
The entropy function $\alpha \mapsto h(T_\alpha)$
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Is entropy monotone increasing for $\alpha < \frac{1}{2}$?
Zooming in
No, it is not monotone!
Zooming in
It seems like entropy displays a **fractal** structure
Global behaviour of $h(\alpha)$

$h(\alpha)$ is:
- non-monotone [Nakada-Natsui]
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- Hölder-continuous with exponent $\left(\frac{1}{2} - \epsilon\right)$ [T.]
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- Hölder-continuous with exponent $(1/2 - \epsilon)$ [T.]

How to describe and explain the fractal structure?
Matching, a dynamical source of monotonicity

Nakada and Natsui defined *matching intervals* as intervals on which the orbits of the two endpoints collide:

\[ T^{N+1}_\alpha(\alpha) = T^{M+1}_\alpha(\alpha - 1) \quad M, N \in \mathbb{N} \]
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They proved that, whenever this happens, the entropy \( h(\alpha) \) is monotone near the parameter \( \alpha \);
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Indeed, they found examples of

- matching intervals where $h(\alpha)$ is increasing;
- matching intervals where $h(\alpha)$ is decreasing;
- matching intervals where $h(\alpha)$ is constant.

Conjecture: The union of all matching intervals is dense and has full measure in parameter space.
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Quadratic intervals

FACT:

Every rational value admits exactly two C.F. expansions.

\[ \frac{3}{10} = \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} \]

\[ \frac{3}{10} = \left[ 0; 3, 3 \right] \]

So any \( a \in \mathbb{Q} \cap (0, 1) \) will have two C.F. expansions of the type

\[ a = \left[ 0; A^- \right] = \left[ 0; A^+ \right] \]

Using such strings we can construct the two quadratic irrationals \( \alpha^- := \left[ 0; A^- \right] \) (E.g. \( \alpha^- = \left[ 0; 3, 2, 1 \right] = \sqrt{37} - \frac{4}{7} \)) \( \alpha^+ := \left[ 0; A^+ \right] \) (E.g. \( \alpha^+ = \left[ 0; 3, 3 \right] = \sqrt{13} - \frac{3}{2} \))
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\[\alpha^+ := [0; A^-]\]
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For each $a \in \mathbb{Q} \cap (0, 1)$
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a = [0; A^\pm] \mapsto I_a := (\alpha^-, \alpha^+), \quad \alpha^\pm := [0; A^\pm].
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The interval \(I_a := (\alpha^-, \alpha^+)\) will be called
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The interval \( I_a := (\alpha^-, \alpha^+) \) will be called the *quadratic interval* generated by \( a \in \mathbb{Q} \cap (0, 1) \).
Quadratic intervals are matching intervals

Theorem (Carminati-T., 2010)

Let $I_r$ be a maximal quadratic interval, and $r = [0; a_1, \ldots, a_n]$ with $n$ even. Let

\[
N = \sum_{i \text{ even}} a_i \quad M = \sum_{i \text{ odd}} a_i \quad (1)
\]

Then for all $\alpha \in I_r$,

\[
T_{\alpha}^{N+1}(\alpha) = T_{\alpha}^{M+1}(\alpha - 1) \quad (2)
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$$T_{\alpha}^{N+1}(\alpha) = T_{\alpha}^{M+1}(\alpha - 1)$$ \hfill (2)

Corollary
The union of all matching intervals is dense of full measure.
The story so far

- Parameter space splits into countably many open intervals, each one of them labelled by a rational number \( r \).
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- $h$ is monotone on $I_r$, and its monotonicity type is determined by the continued fraction expansion of $r$. 

How about the fractal structure?
The story so far

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- The complement is a set of parameters $\mathcal{E}$ which will be called the **bifurcation set**.
The story so far

- Parameter space splits into countably many open intervals, each one of them labelled by a rational number $r$.
- $h$ is monotone on $I_r$, and its monotonicity type is determined by the continued fraction expansion of $r$.
- The complement is a set of parameters $\mathcal{E}$ which will be called the bifurcation set.

How about the fractal structure?
Tuning operators

The self-similarity of $h(\alpha)$ can be explained in terms of tuning operators.
Tuning operators

The self-similarity of $h(\alpha)$ can be explained in terms of tuning operators. Each $r \in \mathbb{Q}$ determines a map

$$\tau_r : [0, 1] \mapsto [0, 1]$$

of parameter space into itself.
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Idea: $\tau_r$ maps the large scale structure to a smaller scale structure, thus creating the fractal self-similarity.
Results: self-similarity of parameter space

Theorem
If $h$ is increasing on a maximal interval $I_r$, then the monotonicity of $h$ on the tuning window $W_r$ reproduces the behaviour on the interval $[0, 1]$, but with reversed sign.
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If $h$ is increasing on a maximal interval $I_r$, then the monotonicity of $h$ on the tuning window $W_r$ reproduces the behaviour on the interval $[0, 1]$, but with reversed sign. More precisely, if $I_p$ is another maximal interval, then

1. $h$ is increasing on $I_{\tau_r(p)}$ iff it is decreasing on $I_p$;
2. $h$ is decreasing on $I_{\tau_r(p)}$ iff it is increasing on $I_p$;
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\[ \begin{array}{|c|}
\hline
0 & 0.2 & 0.4 & g & 0.8 & 1 \\
\hline
\end{array} \]

\[ \begin{array}{|c|}
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0.298 & 0.299 & 0.3 & 0.301 & 0.302 & 0.303 & 0.304 \\
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If, instead, $h$ is decreasing on $I_r$, then the monotonicity of $I_p$ and $I_{\tau r}(p)$ is the same.
A plateau of a real-valued function is a maximal open interval on which the function is constant.

**Theorem (Kraaikamp-Schmidt-Steiner)**

*The interval \((g^2, g)\) is a plateau for \(h(\alpha)\).*
A **plateau** of a real-valued function is a maximal open interval on which the function is constant.

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The interval \((g^2, g)\) is a plateau for \(h(\alpha)\).

**Definition**

A tuning window \(W_r\) is **neutral** if, given \(r = [0; a_1, \ldots, a_n]\) the expansion of \(r\) of even length,

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a_1 - a_2 + \cdots + a_{n-1} - a_n = 0
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Every plateau of \(h\) is the interior of a neutral tuning window \(W_r\).
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**Theorem**

Every plateau of \(h\) is the interior of a neutral tuning window \(W_r\).
Theorem
Let $\alpha$ be a parameter in the parameter space of $\alpha$-continued fractions. Then:

1. if $\alpha \not\in E$, then $h$ is monotone on a neighbourhood of $\alpha$;
2. if $\alpha \in E$, then either
   (i) $\alpha$ is a phase transition: $h$ is constant on the left of $\alpha$ and strictly monotone (increasing or decreasing) on the right of $\alpha$;
   (ii) $\alpha$ lies in the interior of a neutral tuning window: then $h$ is constant on a neighbourhood of $\alpha$;
   (iii) otherwise, $h$ has mixed monotonic behaviour at $\alpha$, i.e. in every neighbourhood of $\alpha$ there are infinitely many intervals on which $h$ is increasing, infinitely many on which it is decreasing and infinitely many on which it is constant.
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Classification of local monotonic behaviour: remarks

Note:

- all cases occur for infinitely many parameters;
- 1. occurs for a set of parameters of full Lebesgue measure;
- 2. there are countably many phase transitions, and they all are tuned images of the phase transition at $\alpha$;
- 2.(iii) for a set of parameters of Hausdorff dimension 1 there is an explicit algorithm to decide which case occurs, given the usual continued fraction expansion of $\alpha$. 
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The end

Thank you!
Bonus level: tuning from complex dynamics

Let $f_c(z) := z^2 + c$. 
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Let $f_c(z) := z^2 + c$. The \textit{Mandelbrot set} $\mathcal{M}$ is the set of $c \in \mathbb{C}$ for which the orbit of 0 is bounded:

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The Mandelbrot set has a self-similar structure. More precisely, there are baby copies of $\mathcal{M}$ everywhere near its boundary.
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Baby copies are images of $\mathcal{M}$ via the Douady-Hubbard tuning maps $\tau_W$. 
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The boundary of $\mathcal{M}$ can be described combinatorially in terms of the doubling map.

E.g.: Feigenbaum parameter $\Longleftrightarrow$ Thue-Morse sequence!
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The set of rays landing on the real slice of the Mandelbrot set is isomorphic to the bifurcation set $\mathcal{E}$ for $\alpha$-c.f. [Bonanno, Carminati, Isola, T., 2011]
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Thank you!