Lattice Packings of Spheres

Nathan Kaplan

Harvard University

Cambridge, MA. December 2, 2010
How can we most efficiently pack spheres of fixed radius in \( n \)-dimensional space?

A packing is completely described by its set of centers. The radius of our spheres is one half of the minimum distance between two of these points.

Let \( C_N \) denote an \( n \)-dimensional cube of radius \( N \).

\[
\text{Density} = \lim_{N \to \infty} \frac{\text{Vol}(\text{Spheres} \cap C_N)}{N^n}.
\]

For a nonperiodic packing this limit is not guaranteed to exist, but we can get an upper limit.
How can we most efficiently pack spheres of fixed radius in $n$-dimensional space?

A packing is completely described by its set of centers. The radius of our spheres is one half of the minimum distance between two of these points.
Sphere Packings

How can we most efficiently pack spheres of fixed radius in $n$-dimensional space?

A packing is completely described by its set of centers. The radius of our spheres is one half of the minimum distance between two of these points.

Let $C_N$ denote an $n$-dimensional cube of radius $N$.

$$\text{Density} = \lim_{N \to \infty} \frac{\text{Vol}(\text{Spheres} \cap C_N)}{N^n}.$$ 

For a nonperiodic packing this limit is not guaranteed to exist, but we can get an upper limit.
Examples

For $n = 1$, packing spheres in $\mathbb{R}$, we can pack all of space with spheres of radius $1/2$ by taking our set of centers to be the integers.
Examples

For $n = 1$, packing spheres in $\mathbb{R}$, we can pack all of space with spheres of radius $1/2$ by taking our set of centers to be the integers.

For $n = 2$, a first guess is to put a circle of radius $1/2$ on every lattice point of a grid.

We can see that we are tiling space with $1 \times 1$ squares, each of which contains exactly one circle of radius $1/2$. This gives density $\frac{\pi}{4}$.
We do better by tiling space with regular hexagons of side length 1 and putting a circle of radius 1/2 on each vertex, and in the center.
Hexagonal Lattice

We do better by tiling space with regular hexagons of side length 1 and putting a circle of radius 1/2 on each vertex, and in the center.

We are tiling space with rhombuses, of side length 1 and angles 60° and 120°, each of which contains exactly one circle. This gives density \( \frac{\pi/4}{\sqrt{32}} = \frac{\pi}{\sqrt{12}} \).
Theorem (Thue, 1890/1910, Fejes-Tóth, 1940)

The hexagonal lattice gives the densest sphere packing in two dimensions.

Casselman’s Interactive proof (source for next two pictures):

http://math.sunysb.edu/∼tony/whatsnew/column/pennies-1200/cass1.html
**Thue’s Theorem**

**Theorem (Thue, 1890/1910, Fejes-Tóth, 1940)**

*The hexagonal lattice gives the densest sphere packing in two dimensions.*

Casselman’s Interactive proof (source for next two pictures):

http://math.sunysb.edu/~tony/whatsnew/column/pennies-1200/cass1.html

**Definition**

The Voronoi cell of a point $x_0$ in our collection of points $\mathcal{P}$ is

$$V_{x_0}(\mathcal{P}) = \{x \in \mathbb{R}^n \mid \forall y \in \mathcal{P}, \|x - x_0\| \leq \|x - y\|\}.$$
Thue’s Theorem

Theorem (Thue, 1890/1910, Fejes-Tóth, 1940)

The hexagonal lattice gives the densest sphere packing in two dimensions.

Casselman’s Interactive proof (source for next two pictures):
http://math.sunysb.edu/~tony/whatsnew/column/pennies-1200/cass1.html

Definition

The Voronoi cell of a point \( x_0 \) in our collection of points \( \mathcal{P} \) is

\[
V_{x_0}(\mathcal{P}) = \{ x \in \mathbb{R}^n \mid \forall y \in \mathcal{P}, \| x - x_0 \| \leq \| x - y \| \}.
\]

Voronoi cells partition space, and the density of a packing is at most the maximum density inside of a Voronoi cell of that packing.
Sketch of Thue’s Theorem

The Voronoi cell of each disc in the hexagonal packing is a regular hexagon around the circle.

We prove a stronger statement, that this regular hexagon gives the densest possible Voronoi cell in two dimensions. We can clearly assume that there is no gap between circles large enough to add another circle (our packing is saturated).
Sketch of Thue’s Theorem

The Voronoi cell of each disc in the hexagonal packing is a regular hexagon around the circle.

We prove a stronger statement, that this regular hexagon gives the densest possible Voronoi cell in two dimensions. We can clearly assume that there is no gap between circles large enough to add another circle (our packing is saturated).

We consider when three circles come together. There is a unique triple point equidistant from all of them. We draw the pair of tangent lines to each circle, giving a kind of dunce cap. The entire cap will be contained in each Voronoi cell.
Sketch of Thue’s Theorem

We consider a circle circumscribed each of ours with radius \( \frac{2}{\sqrt{3}} \) times as large. We consider when two discs do not intersect, but their circumscribed discs do. These discs intersect at two points. We draw the radii of each disc to these points, forming a rhombus. Half of each rhombus will be contained in each Voronoi cell.
Sketch of Thue’s Theorem

We consider a circle circumscribed each of ours with radius $\frac{2}{\sqrt{3}}$ times as large. We consider when two discs do not intersect, but their circumscribed discs do. These discs intersect at two points. We draw the radii of each disc to these points, forming a rhombus. Half of each rhombus will be contained in each Voronoi cell.

We determine the maximum density of a Voronoi cell by partitioning it into three regions and giving the maximum density of each:

The points outside the circumscribed disc, the points in the rhombus associated to a neighboring cell, and the points in the disc but not in any rhombus.
The Kepler Conjecture, $n = 3$

For $n = 3$, the face centered cubic packing gives density $\frac{\pi}{\sqrt{18}}$.

This packing is built up by stacking layers of the hexagonal packing where centers of spheres lie above the deepest holes.
For \( n = 3 \), the face centered cubic packing gives density \( \frac{\pi}{\sqrt{18}} \).

This packing is built up by stacking layers of the hexagonal packing where centers of spheres lie above the deepest holes.

The Voronoi cells are all rhombic dodecahedra, but these are no longer the densest possible cells (for example regular dodecahedra are denser). However, one cannot tile \( \mathbb{R}^3 \) with these denser cells.
The Kepler Conjecture, \( n = 3 \)

Theorem (Hales, 1998)

*The face centered cubic lattice gives the densest sphere packing in \( \mathbb{R}^3 \).*
The Kepler Conjecture, $n = 3$

**Theorem (Hales, 1998)**

The face centered cubic lattice gives the densest sphere packing in $\mathbb{R}^3$.

Rogers: ‘Many mathematicians believe and all physicists know that the density cannot exceed $\frac{\pi}{\sqrt{18}}$. ’
The Kepler Conjecture, $n = 3$

Theorem (Hales, 1998)

*The face centered cubic lattice gives the densest sphere packing in $\mathbb{R}^3$.*

Rogers: ‘Many mathematicians believe and all physicists know that the density cannot exceed $\frac{\pi}{\sqrt{18}}$.’

This extremely difficult theorem requires a more difficult partition of space than just into Voronoi cells, and also the consideration of many locally dense configurations.
Definitions

Definition

A lattice $\Lambda$ is a free abelian group of rank $n$ with a positive definite symmetric pairing.
A lattice \( \Lambda \) is a free abelian group of rank \( n \) with a positive definite symmetric pairing. A lattice is a discrete subgroup \( \Lambda \subset \mathbb{R}^n \) such that \( \mathbb{R}^n/\Lambda \) is compact. \( \mathbb{R}^n/\Lambda \) is a fundamental domain of the lattice, and we can tile space with its translates.
Definitions

Definition

A lattice $\Lambda$ is a free abelian group of rank $n$ with a positive definite symmetric pairing.

A lattice is a discrete subgroup $\Lambda \subset \mathbb{R}^n$ such that $\mathbb{R}^n/\Lambda$ is compact. $\mathbb{R}^n/\Lambda$ is a fundamental domain of the lattice, and we can tile space with its translates.

It is a free abelian group of rank $n$, and is given by

$$\left\{ \sum_{j=1}^{n} c_j v_j, \text {such that } c_j \in \mathbb{Z}, \{v_1, \ldots, v_n\} \text{ generate } \mathbb{R}^n \right\}.$$
Definitions

**Definition**

The generator matrix $M$ has columns $v_1, \ldots, v_n$. The Gram matrix $A = MM^T$. Its $(i, j)$ entry is given by $\langle v_i, v_j \rangle$.

If we choose another set of generators given by $BM$, where $B \in \text{GL}_n(\mathbb{Z})$, our new Gram matrix is $B^T AB$ and is positive definite if and only if $A$ is.
Definitions

The generator matrix $M$ has columns $v_1, \ldots, v_n$. The Gram matrix $A = MM^T$. Its $(i, j)$ entry is given by $\langle v_i, v_j \rangle$.

If we choose another set of generators given by $BM$, where $B \in \text{GL}_n(\mathbb{Z})$, our new Gram matrix is $B^T A B$ and is positive definite if and only if $A$ is.

Suppose $v = \sum_{i=1}^{n} m_i v_i$, where $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Then

$$\langle v_i, v_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} m_i m_j,$$

which is the value of a quadratic form given by the Gram matrix $A$ at $m$. 
We define the determinant of a lattice,
\[\det(\Lambda) = \det(\Lambda) = \det(M)^2 = \text{Vol}(\mathbb{R}^n/\Lambda)^2.\]
We define the determinant of a lattice, 
\[ \det(\Lambda) = \det(A) = \det(M)^2 = \text{Vol}(\mathbb{R}^n/\Lambda)^2. \]

Let \( N(\Lambda) \) denote the minimal value of \( \langle v, v \rangle \) among all \( v \in \Lambda \). The density of a point set is the number of points per unit volume. For a lattice, it is \( \frac{1}{\text{Vol}(\mathbb{R}^n/\Lambda)}. \)
Density of a Lattice

We define the determinant of a lattice,
\[ \det(\Lambda) = \det(A) = \det(M)^2 = \text{Vol}(\mathbb{R}^n/\Lambda)^2. \]

Let \( N(\Lambda) \) denote the minimal value of \( \langle v, v \rangle \) among all \( v \in \Lambda \). The density of a point set is the number of points per unit volume. For a lattice, it is \( \frac{1}{\text{Vol}(\mathbb{R}^n/\Lambda)} \).

The density of a lattice is
\[ \Delta(\Lambda) = \frac{\text{Vol} \left( \text{Sphere of Radius} \frac{N(\Lambda)^{1/2}}{2} \right)}{\det(\Lambda)^{1/2}}. \]
Density of a Lattice

We define the determinant of a lattice,
\[ \det(\Lambda) = \det(A) = \det(M)^2 = \text{Vol}(\mathbb{R}^n/\Lambda)^2. \]

Let \( N(\Lambda) \) denote the minimal value of \( \langle \nu, \nu \rangle \) among all \( \nu \in \Lambda \).

The density of a point set is the number of points per unit volume. For a lattice, it is \( \frac{1}{\text{Vol}(\mathbb{R}^n/\Lambda)} \).

The density of a lattice is
\[ \Delta(\Lambda) = \frac{\text{Vol} \left( \text{Sphere of Radius } \frac{N(\Lambda)^{1/2}}{2} \right)}{\det(\Lambda)^{1/2}}. \]

If we want to find dense lattice packings, we want to maximize
\[ \frac{N(\Lambda)^{n/2}}{\det(\Lambda)^{1/2}}. \]
We say that two lattices are isometric if we get one from another from an orthogonal linear transformation of $\mathbb{R}^n$. Two lattices are similar if there is some $\alpha > 0$ such that $\Lambda' = \alpha \Lambda$. 
We say that two lattices are isometric if we get one from another from an orthogonal linear transformation of $\mathbb{R}^n$. Two lattices are similar if there is some $\alpha > 0$ such that $\Lambda' = \alpha \Lambda$. Lattices are isometric if and only if

$$ M' = UMB, \ B \in \text{GL}_n(\mathbb{Z}), \ U \in \text{O}_n(\mathbb{R}) $$

and homothetic if and only if

$$ M' = \alpha UMB, \ \alpha > 0, \ B \in \text{GL}_n(\mathbb{Z}), \ U \in \text{O}_n(\mathbb{R}). $$

We can choose $\alpha$ such that $|\det(M)| = 1$, and we can choose $U, M$ with determinant 1.
Let \( \Lambda_n \) denote homothecy classes of lattices in \( \mathbb{R}^n \). We have identified it with

\[
SO_n(\mathbb{R}) \backslash SL_n(\mathbb{R}) / GL_n(\mathbb{Z}).
\]
Let $\Lambda_n$ denote homothecy classes of lattices in $\mathbb{R}^n$. We have identified it with

$$\text{SO}_n(\mathbb{R}) \backslash \text{SL}_n(\mathbb{R}) / \text{GL}_n(\mathbb{Z}).$$

We see that this space has dimension

$$(n^2 - 1) - \binom{n}{2} = (n - 1)(n + 2)/2.$$

It also has a natural metric, so we can ask if the homothecy class of $\Lambda$ has some property locally.
Examples

The Hexagonal lattice has generator matrix $\begin{pmatrix} 1 & -1 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$, and Gram matrix $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, which after scaling vector lengths by $\sqrt{2}$ gives $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.
Examples

The Hexagonal lattice has generator matrix \( \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \), and

Gram matrix \( \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \), which after scaling vector lengths by \( \sqrt{2} \) gives \( \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \).

In this case, \( N(\Lambda) = 2 \), and \( \det(\Lambda) = 3 \), giving \( \Delta(\Lambda) = \frac{\pi}{4\cdot\sqrt{3}} \).
The Hexagonal lattice has generator matrix \[
\begin{pmatrix}
1 & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2}
\end{pmatrix},
\]
and Gram matrix \[
\begin{pmatrix}
1 & -\frac{1}{2} \\
-\frac{1}{2} & 1
\end{pmatrix},
\]
which after scaling vector lengths by $\sqrt{2}$ gives \[
\begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}.
\]

In this case, $N(\Lambda) = 2$, and $\det(\Lambda) = 3$, giving $\Delta(\Lambda) = \frac{\pi}{4 \cdot \sqrt{3}}$.

The face centered cubic lattice has generator matrix \[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix},
\]
and Gram matrix \[
\begin{pmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{pmatrix}.
\]
These two examples are special cases of a more general construction. We identify $n$-dimensional Euclidean space with a hyperplane in $\mathbb{R}^{n+1}$,

$$\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \text{ such that } \sum x_i = 0\}.$$

Now let

$$A_n = \{(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} \text{ such that } \sum x_i = 0\}.$$
These two examples are special cases of a more general construction. We identify \( n \)-dimensional Euclidean space with a hyperplane in \( \mathbb{R}^{n+1} \),

\[
\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \text{ such that } \sum x_i = 0\}.
\]

Now let

\[
A_n = \{(x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} \text{ such that } \sum x_i = 0\}.
\]

The fcc lattice gives \( A_3 \), which in this special case is isometric with \( D_3 \).

\[
D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \text{ such that } \sum x_i \equiv 0 \pmod{2}\}.
\]
We say that two quadratic forms are equivalent if we get one from the other by multiplication by a matrix in $SL_n(\mathbb{Z})$. 

Definition

Let $f$ be a positive definite $n$-dimensional quadratic form. We say that it is Minkowski reduced if it is expressed in terms of an integral basis $e_1, \ldots, e_n$ such that $f(e_i) \leq f(v)$ for all $v$ such that $e_1, \ldots, e_{i-1}, v$ can be extended to an integer basis of $\Lambda$.

Proposition

Every quadratic form is equivalent to a Minkowski reduced form.

Choose $v_i$ inductively to minimize $f(v_i)$. Then $v_1, \ldots, v_n$ form a basis. Let $f_1(x) = f(V \cdot x)$ and claim that $f_1$ is Minkowski reduced.
Minkowski Reduced Forms

We say that two quadratic forms are equivalent if we get one from the other by multiplication by a matrix in $\text{SL}_n(\mathbb{Z})$.

**Definition**

Let $f$ be a positive definite $n$ dimensional quadratic form. We say that it is Minkowski reduced if it is expressed in terms of an integral basis $e_1, \ldots, e_n$ such that $f(e_i) \leq f(v)$ for all $v$ such that $e_1, \ldots, e_{i-1}, v$ can be extended to an integer basis of $\Lambda$. 
We say that two quadratic forms are equivalent if we get one from the other by multiplication by a matrix in $\text{SL}_n(\mathbb{Z})$.

**Definition**

Let $f$ be a positive definite $n$ dimensional quadratic form. We say that it is Minkowski reduced if it is expressed in terms of an integral basis $e_1, \ldots, e_n$ such that $f(e_i) \leq f(v)$ for all $v$ such that $e_1, \ldots, e_{i-1}, v$ can be extended to an integer basis of $\Lambda$.

**Proposition**

Every quadratic form is equivalent to a Minkowski reduced form.
We say that two quadratic forms are equivalent if we get one from the other by multiplication by a matrix in SL$_n(\mathbb{Z})$.

**Definition**

Let $f$ be a positive definite $n$ dimensional quadratic form. We say that it is Minkowski reduced if it is expressed in terms of an integral basis $e_1, \ldots, e_n$ such that $f(e_i) \leq f(v)$ for all $v$ such that $e_1, \ldots, e_{i-1}, v$ can be extended to an integer basis of $\Lambda$.

**Proposition**

*Every quadratic form is equivalent to a Minkowski reduced form.*

Choose $v_i$ inductively to minimize $f(v_i)$. Then $v_1, \ldots, v_n$ form a basis. Let $f_1(x) = f(V \cdot x)$ and claim that $f_1$ is Minkowski reduced.
Lattice Packings $n = 2$

We want to maximize $\frac{N(\Lambda)^{n/2}}{\det(\Lambda)^{1/2}}$. We work with quadratic forms instead of lattices themselves.
Lattice Packings \( n = 2 \)

We want to maximize \( \frac{N(\Lambda)^{n/2}}{\det(\Lambda)^{1/2}} \). We work with quadratic forms instead of lattices themselves.

Let

\[
f(x_1, x_2) = f_{11}x_1^2 + 2f_{12}x_1x_2 + f_{22}x_2^2,
\]

and suppose that it is Minkowski reduced.

We have \( f_{11} \leq f_{22} \) and can suppose \( f_{12} \geq 0 \). Now \( f(-1, 1) \geq f(0, 1) \) implies \( 2f_{12} \leq f_{11} \).
Lattice Packings $n = 2$

We want to maximize $\frac{N(\Lambda)^{n/2}}{\det(\Lambda)^{1/2}}$. We work with quadratic forms instead of lattices themselves.

Let

$$f(x_1, x_2) = f_{11}x_1^2 + 2f_{12}x_1x_2 + f_{22}x_2^2,$$

and suppose that it is Minkowski reduced.

We have $f_{11} \leq f_{22}$ and can suppose $f_{12} \geq 0$. Now $f(-1, 1) \geq f(0, 1)$ implies $2f_{12} \leq f_{11}$.

Let $D = f_{11}f_{22} - f_{12}^2$ and consider

$$4D - 3f_{11}f_{22} = f_{11}f_{22} - 4f_{12}^2 \geq f_{11}^2 - 4f_{12}^2 \geq 0.$$
Lattice Packings $n = 2$

We want to maximize $\frac{N(\Lambda)^{n/2}}{\det(\Lambda)^{1/2}}$. We work with quadratic forms instead of lattices themselves.

Let

$$f(x_1, x_2) = f_{11}x_1^2 + 2f_{12}x_1x_2 + f_{22}x_2^2,$$

and suppose that it is Minkowski reduced. We have $f_{11} \leq f_{22}$ and can suppose $f_{12} \geq 0$. Now $f(-1, 1) \geq f(0, 1)$ implies $2f_{12} \leq f_{11}$.

Let $D = f_{11}f_{22} - f_{12}^2$ and consider

$$4D - 3f_{11}f_{22} = f_{11}f_{22} - 4f_{12}^2 \geq f_{11}^2 - 4f_{12}^2 \geq 0.$$

Equality holds if and only if $f(x_1, x, 2) = f_{11}(x_1^2 + x_1x_2 + x_2^2)$. This is the Gram matrix of the Hexagonal lattice.
Gauss proved that the face centered cubic lattice gives the dense lattice packing in $\mathbb{R}^3$.

**Proposition**

Let $f(x) = \sum_{1 \leq i,j \leq 3} f_{ij}x_i x_j$.

1. There exists a nonzero $u \in \mathbb{Z}^2$ such that $f(u) \leq (2D)^{1/3}$.
2. $f(x)$ is Minkowski reduced if and only if $f_{11}f_{22}f_{33} \leq 2D$.
3. Equality holds if and only if the form is equivalent to the one given by fcc.
Gauss proved that the face centered cubic lattice gives the dense lattice packing in $\mathbb{R}^3$.

**Proposition**

Let $f(x) = \sum_{1 \leq i, j \leq 3} f_{ij}x_i x_j$.

1. There exists a nonzero $u \in \mathbb{Z}^2$ such that $f(u) \leq (2D)^{1/3}$.
2. $f(x)$ is Minkowski reduced if and only if $f_{11}f_{22}f_{33} \leq 2D$.
3. Equality holds if and only if the form is equivalent to the one given by fcc.

We note that the first statement follows from the others. Consider two cases based on the sign of $f_{12}f_{23}f_{31}$. If the product is positive, we can suppose each is positive. Let $\Theta_{ij} = f_{ii} - 2f_{ij}$. If $f$ is reduced, this is nonnegative. Next, consider $2D - f_{11}f_{22}f_{33}$. Using the $\Theta_{ij}$, this is nonnegative.
Hermite’s Theorem

**Theorem (Hermite)**

Every lattice $\Lambda$ has a basis $e_1, \ldots, e_n$ such that

$$N(e_1) \cdots N(e_n) \leq \left(\frac{4}{3}\right)^{n(n-1)/2} \det(\Lambda).$$

**Corollary**

$$\frac{N(\Lambda)^{n/2}}{\det(\Lambda)^{1/2}} \leq \left(\frac{4}{3}\right)^{n(n-1)/4}.$$
Hermite’s Theorem

**Theorem (Hermite)**

Every lattice \( \Lambda \) has a basis \( e_1, \ldots, e_n \) such that

\[
N(e_1) \cdots N(e_n) \leq \left( \frac{4}{3} \right)^{n(n-1)/2} \det(\Lambda).
\]

**Corollary**

\[
\frac{N(\Lambda)^{n/2}}{\det(\Lambda)^{1/2}} \leq \left( \frac{4}{3} \right)^{n(n-1)/4}.
\]

Pick \( e_1 \) minimal and project onto the intersection of \( \Lambda \) with the orthogonal complement of \( \langle e_1 \rangle \). Induct on \( n \).
Hermite’s Theorem

Theorem (Hermite)

*Every lattice* $\Lambda$ *has a basis* $e_1, \ldots, e_n$ *such that*

$$N(e_1) \cdots N(e_n) \leq \left(\frac{4}{3}\right)^{n(n-1)/2} \det(\Lambda).$$

**Corollary**

$$\frac{N(\Lambda)^{n/2}}{\det(\Lambda)^{1/2}} \leq \left(\frac{4}{3}\right)^{n(n-1)/4}.$$

Pick $e_1$ minimal and project onto the intersection of $\Lambda$ with the orthogonal complement of $\langle e_1 \rangle$. Induct on $n$.

Hadamard’s theorem: $\det(\Lambda) \leq N(e_1) \cdots N(e_n)$. 
The Hermite Invariant

Definition

The Hermite invariant of $\Lambda$ is

$$\gamma(\Lambda) = \frac{N(\Lambda)}{\det(\Lambda)^{1/n}}.$$
The Hermite Invariant

Definition

The Hermite invariant of $\Lambda$ is

$$\gamma(\Lambda) = \frac{N(\Lambda)}{\det(\Lambda)^{1/n}}.$$  

The Hermite constant for dimension $n$ is $\gamma_n = \sup_{\Lambda} \gamma(\Lambda)$. 
The Hermite Invariant

Definition

The Hermite invariant of $\Lambda$ is

$$\gamma(\Lambda) = \frac{N(\Lambda)}{\det(\Lambda)^{1/n}}.$$ 

The Hermite constant for dimension $n$ is $\gamma_n = \sup_{\Lambda} \gamma(\Lambda)$.

A lattice for which $\gamma(\Lambda) = \gamma_n$ is called critical.
The Hermite Invariant

Definition

The Hermite invariant of $\Lambda$ is

$$\gamma(\Lambda) = \frac{N(\Lambda)}{\det(\Lambda)^{1/n}}.$$ 

The Hermite constant for dimension $n$ is $\gamma_n = \sup_{\Lambda} \gamma(\Lambda)$.

A lattice for which $\gamma(\Lambda) = \gamma_n$ is called critical.

We have already seen that $\gamma_2 = \frac{2}{\sqrt{3}}$, and $\gamma_3 = \frac{(2D)^{1/3}}{D^{1/3}} = 2^{1/3}$.
Mordell’s Theorem

In some cases, we can determine $\gamma_n$ from $\gamma_{n-1}$.

**Theorem (Mordell)**

For any $2 \leq m < n$,

$$\gamma_n \leq \frac{\gamma_{n-1}}{(m-1)}.$$
Mordell’s Theorem

In some cases, we can determine $\gamma_n$ from $\gamma_{n-1}$.

**Theorem (Mordell)**

For any $2 \leq m < n$,

$$\gamma_n \leq \gamma_{m}^{(n-1)/(m-1)}.$$

**Definition**

The dual lattice of $\Lambda$ is

$$\Lambda^* = \{ y \in \mathbb{R}^n \text{ such that } \langle x, y \rangle \in \mathbb{Z}, \ \forall \ x \in \Lambda \}.$$
Mordell’s Theorem

In some cases, we can determine $\gamma_n$ from $\gamma_{n-1}$.

**Theorem (Mordell)**

*For any $2 \leq m < n$,*

$$
\gamma_n \leq \frac{\gamma_{m-1}}{m-1}.
$$

**Definition**

The dual lattice of $\Lambda$ is

$$
\Lambda^* = \{ y \in \mathbb{R}^n \text{ such that } \langle x, y \rangle \in \mathbb{Z}, \ \forall \ x \in \Lambda \}.
$$

Pick $x$ minimal in $\Lambda^*$ and let $L = \Lambda \cap \langle x \rangle^\perp$, and then express $\gamma(M)$ in terms of $\gamma(\Lambda)$, $\gamma(\Lambda^*)$, $N(M)$, $N(\Lambda)$. We then do the same with the roles of $\Lambda$ and $\Lambda^*$ switched.
We recall that $\gamma_3 = 2^{1/3}$, so Mordell’s Theorem gives $\gamma_4 \leq \gamma_3^{3/2} = \sqrt{2}$. 
We recall that $\gamma_3 = 2^{1/3}$, so Mordell’s Theorem gives 
$\gamma_4 \leq \gamma_3^{3/2} = \sqrt{2}$.

Equality holds for the lattice $D_4$, so $\gamma_4 = \sqrt{2}$. 
We recall that $\gamma_3 = 2^{1/3}$, so Mordell’s Theorem gives $\gamma_4 \leq \gamma_3^{3/2} = \sqrt{2}$.

Equality holds for the lattice $D_4$, so $\gamma_4 = \sqrt{2}$.

We note that the distance from $(1/2, \ldots, 1/2)$ to a lattice point of $D_8$ is $\sqrt{2}$, exactly the radius of one sphere. We can add in a translated copy of our lattice, giving $E_8$. Luckily, this is a lattice as well!
γ₄ and γ₈

We recall that γ₃ = 2¹/³, so Mordell’s Theorem gives
γ₄ ≤ γ₃² = \sqrt{2}.

Equality holds for the lattice D₄, so γ₄ = \sqrt{2}.

We note that the distance from (1/2, . . . , 1/2) to a lattice point of D₈ is \sqrt{2}, exactly the radius of one sphere. We can add in a translated copy of our lattice, giving E₈. Luckily, this is a lattice as well!

Let ν be a minimal vector of E₈. Let

\[ E₇ = \{ x \in E₈ \text{ such that } \langle x, ν \rangle = 0 \}. \]

We get γ(E₇) = 2⁶/₇.
\(\gamma_4\) and \(\gamma_8\)

We recall that \(\gamma_3 = 2^{1/3}\), so Mordell’s Theorem gives
\[
\gamma_4 \leq \gamma_3^{3/2} = \sqrt{2}.
\]

Equality holds for the lattice \(D_4\), so \(\gamma_4 = \sqrt{2}\).

We note that the distance from \((1/2, \ldots, 1/2)\) to a lattice point of \(D_8\) is \(\sqrt{2}\), exactly the radius of one sphere. We can add in a translated copy of our lattice, giving \(E_8\). Luckily, this is a lattice as well!

Let \(v\) be a minimal vector of \(E_8\). Let
\[
E_7 = \{x \in E_8 \text{ such that } \langle x, v \rangle = 0\}.
\]

We get \(\gamma(E_7) = 2^{6/7}\).

If we show that in fact \(\gamma_7 = 2^{6/7}\) then Mordell implies \(\gamma_8 \leq 2\), but equality holds for \(E_8\)!
Minkowski’s Convex Body Theorem

Theorem (Minkowski)

A bounded convex region in $\mathbb{R}^n$ symmetric about a lattice point with volume greater than $2^n$ contains at least 3 lattice points.
Theorem (Minkowski)

A bounded convex region in $\mathbb{R}^n$ symmetric about a lattice point with volume greater than $2^n$ contains at least 3 lattice points.

Corollary

We have $\gamma_n \leq 4\omega_n^{-2/n}$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. 
Minkowski’s Convex Body Theorem

Theorem (Minkowski)

A bounded convex region in $\mathbb{R}^n$ symmetric about a lattice point with volume greater than $2^n$ contains at least 3 lattice points.

Corollary

We have $\gamma_n \leq 4\omega_n^{-2/n}$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

A careful consideration of $\omega_n$ gives $\gamma_n \leq 1 + n/4$ for all $n$. 
Korkine and Zolotareff used a different reduction of quadratic forms to compute $\gamma_n$ for $n \leq 5$. Blichfeldt extended their methods to determine $\gamma_n$ for $n = 6, 7, 8$. 
Korkine and Zolotareff used a different reduction of quadratic forms to compute $\gamma_n$ for $n \leq 5$. Blichfeldt extended their methods to determine $\gamma_n$ for $n = 6, 7, 8$.

Since we have a metric on unimodular lattices, we can ask when $\gamma(\Lambda)$ has a local maximum. Such a lattice is called \textit{extreme}.
Local Density of Lattices

Korkine and Zolotareff used a different reduction of quadratic forms to compute $\gamma_n$ for $n \leq 5$. Blichfeldt extended their methods to determine $\gamma_n$ for $n = 6, 7, 8$.

Since we have a metric on unimodular lattices, we can ask when $\gamma(\Lambda)$ has a local maximum. Such a lattice is called extreme.

**Definition**

Let $\min(\Lambda) = \{x \in \Lambda$ such that $\langle x, x \rangle = N(\Lambda)\}$. 

$\Lambda$ is perfect if $\{xx^T =: \pi_x, \text{such that } x \in \min(\Lambda)\}$ spans $\mathbb{R}^{n\times n}_{\text{sym}}$. 

$\Lambda$ is eutactic if $\exists \{\lambda_x\}$, such that $\lambda_x \geq 0$ for all $x \in \min(\Lambda)$ and $I_n = \sum_{x \in \min(\Lambda)} \lambda_x \cdot \pi_x$. 
Voronoi’s Theorem

Theorem (Voronoï)

Λ is extreme if and only if it is perfect and eutactic.
Theorem (Voronoi)

\( \Lambda \) is extreme if and only if it is perfect and eutactic.

Proposition (K-Z)

A perfect lattice is proportional to an integral lattice.

Corollary

\( \gamma_n^n \) is rational
Voronoi’s Theorem

Theorem (Voronoï)

Λ is extreme if and only if it is perfect and eutactic.

Proposition (K-Z)

A perfect lattice is proportional to an integral lattice.

Corollary

γ_n^n is rational

Theorem (Voronoï)

There are only finitely many non-isometric unimodular lattices in a given dimension and there is an algorithm to compute them.

For n = 7 there are 33, but for n = 8 there are over 10,000.
Example of Voronoi’s Theorem

There are 6 minimal vectors in the hexagonal lattice. Let $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $x_1 = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$, and $x_3 = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$.
Example of Voronoi’s Theorem

There are 6 minimal vectors in the hexagonal lattice. Let

\[ x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \text{and} \quad x_3 = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}. \]

We see that

\[ \pi x_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \frac{2}{3}(\pi x_2 + \pi x_3) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{2}{\sqrt{3}}(\pi x_2 - \pi x_3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

so the hexagonal lattice is perfect and eutactic, hence extreme.
We know $\gamma_n$ only for $n \leq 8$ and $n = 24$. The last result is recent work of Cohn and Kumar, building on techniques of Cohn and Elkies.
We know $\gamma_n$ only for $n \leq 8$ and $n = 24$. The last result is recent work of Cohn and Kumar, building on techniques of Cohn and Elkies.

For $f \in L^1$, $f : \mathbb{R}^n \to \mathbb{R}$, $\hat{f}(t) = \int_{\mathbb{R}^n} f(x) e^{2\pi i \langle t, x \rangle} \, dx$.

**Proposition (Poisson Summation for Lattices)**

*Let $f$ be a Schwartz function*

$$\sum_{x \in \Lambda} f(x) = \frac{1}{\text{Vol}(\mathbb{R}^n/\Lambda)} \sum_{t \in \Lambda^*} \hat{f}(t).$$
Theorem (Cohn, Elkies)

Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a Schwartz function, \( \hat{f}(0) \neq 0 \). If \( f(x) \leq 0 \) for \( |x| \geq 1 \) and \( \hat{f}(t) \geq 0 \) for all \( t \), then the packing density is at most

\[
\frac{\pi^{n/2}}{2^n \cdot \Gamma(n/2 + 1)} \cdot \frac{f(0)}{\hat{f}(0)}.
\]
Upper Bound for Packing Density

Theorem (Cohn, Elkies)

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a Schwartz function, \( \hat{f}(0) \neq 0 \). If \( f(x) \leq 0 \) for \( |x| \geq 1 \) and \( \hat{f}(t) \geq 0 \) for all \( t \), then the packing density is at most

\[
\frac{\pi^{n/2}}{2^n \cdot \Gamma(n/2 + 1)} \cdot \frac{f(0)}{\hat{f}(0)}.
\]

It is enough to prove this for periodic packings, (unions of translates of lattices), since we can use these to approximate general packings. For lattices we sketch an argument. The density of centers per unit volume is \( \frac{1}{\text{Vol}(\mathbb{R}^n/\Lambda)} \). We have \( \hat{f}(0) \leq \sum \hat{f}(t) \) and also \( \sum f(x) \leq f(0) \). Then

\[
\frac{1}{\text{Vol}(\mathbb{R}^n/\Lambda)} \leq \frac{f(0)}{\hat{f}(0)}.
\]
It is conjectured that for \( n = 2, 8, 24 \) there exists an \( f \) giving a sharp bound.

Such a sharp \( f \) must vanish at every point of \( \Lambda \), and \( \hat{f} \) must vanish at every point of \( \Lambda^* \). If \( f \) were sharp, rotations of \( f \) would be as well, so these functions vanish identically on spheres of radius equal to the radius of any lattice point. The lengths of lattice points imply that neither \( f \) nor \( \hat{f} \) have compact support. It seems likely that \( f \) would have to be a radial function.
It is conjectured that for \( n = 2, 8, 24 \) there exists an \( f \) giving a sharp bound.

Such a sharp \( f \) must vanish at every point of \( \Lambda \), and \( \hat{f} \) must vanish at every point of \( \Lambda^* \). If \( f \) were sharp, rotations of \( f \) would be as well, so these functions vanish identically on spheres of radius equal to the radius of any lattice point. The lengths of lattice points imply that neither \( f \) nor \( \hat{f} \) have compact support. It seems likely that \( f \) would have to be a radial function.

In practice, linear combinations of orthogonal polynomials are used to find good \( f \), and double zeros are imposed in certain places to minimize its last sign change. Extensive linear programming led to an upper bound powerful enough to show that no lattice in \( \mathbb{R}^{24} \) is denser than the Leech lattice.