Hidden Hodge symmetries and Hodge correlators

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To Don Zagier for his 60-th birthday

1 Introduction

There is a well known parallel between Hodge and étale theories, still incomplete and rather mysterious:

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<th>l-adic Étale Theory</th>
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<td>Category of l-adic</td>
<td>Abelian category $\mathcal{MH}_\mathbb{R}$</td>
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<td>Galois group $\text{Gal}(\mathbb{Q}/\mathbb{Q})$</td>
<td>Hodge Galois group $G_{\text{Hod}} := \text{Gal}(\mathbb{Q}/\mathbb{Q})$</td>
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<td>$\text{Gal}(\mathbb{Q}/\mathbb{Q})$ acts on $H^*_\text{et}(X, \mathbb{Q}_l)$,</td>
<td>$H^*(X(\mathbb{C}), \mathbb{R})$ has a functorial</td>
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<td>where $X$ is a variety over $\mathbb{Q}$</td>
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<td>étale site</td>
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<td>$\text{Gal}(\mathbb{Q}/\mathbb{Q})$ acts on the étale site, and thus</td>
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<td>on categories of étale sheaves on $X$, e.g.</td>
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<td>on the category of l-adic perverse sheaves</td>
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The current absence of the “Hodge site” was emphasized by A.A. Beilinson [B].

The Hodge Galois group. A weight $n$ pure real Hodge structure is a real vector space $H$ together with a decreasing filtration $F^*H_\mathbb{C}$ on its complexification satisfying

$$H_\mathbb{C} = \bigoplus_{p+q=n}F^pH_\mathbb{C} \cap F^qH_\mathbb{C}. $$

A real Hodge structure is a direct sum of pure ones. The category $\mathcal{PH}_\mathbb{R}$ real Hodge structures is equivalent to the category of representations of the real algebraic group $\mathbb{C}^*_\mathbb{R}$.

The group of complex points of $\mathbb{C}^*_\mathbb{R}$ is $\mathbb{C}^* \times \mathbb{C}^*$; the complex conjugation interchanges the factors.

A real mixed Hodge structure is given by a real vector space $H$ equipped with the weight filtration $W_\bullet H$ and the Hodge filtration $F^\bullet H_\mathbb{C}$ of its complexification, such that the Hodge filtration induces on $\text{gr}_n^\bullet H$ a weight $n$ real Hodge structure. The category
\( \mathcal{MH}_R \) of real mixed Hodge structures is an abelian rigid tensor category. There is a fiber functor to the category of real vector spaces
\[
\omega_{\text{Hod}} : \mathcal{MH}_R \longrightarrow \text{Vect}_R, \quad H \longrightarrow \bigoplus_{n} \text{gr}^W_n H.
\]
The Hodge Galois group is a real algebraic group given by automorphisms of the fiber functor:
\[
G_H := \text{Aut}^\circ \omega_{\text{Hod}}.
\]
The fiber functor provides a canonical equivalence of categories
\[
\omega_{\text{Hod}} : \mathcal{MH}_R \overset{\sim}{\longrightarrow} G_{\text{Hod}} - \text{modules}.
\]
The Hodge Galois group is a semidirect product of the unipotent radical \( U_{\text{Hod}} \) and \( \mathbb{C}^*_{\mathbb{C}/\mathbb{R}} \):
\[
0 \longrightarrow U_{\text{Hod}} \longrightarrow G_{\text{Hod}} \longrightarrow \mathbb{C}^*_{\mathbb{C}/\mathbb{R}} \longrightarrow 0, \quad \mathbb{C}^*_{\mathbb{C}/\mathbb{R}} \hookrightarrow G_{\text{Hod}}. \quad (1)
\]
The projection \( G_{\text{Hod}} \rightarrow \mathbb{C}^*_{\mathbb{C}/\mathbb{R}} \) is provided by the inclusion of the category of real Hodge structures to the category of mixed real Hodge structures. The splitting \( s : \mathbb{G}_m \rightarrow G_{\text{Hod}} \) is provided by the functor \( \omega_{\text{Hod}} \).

The complexified Lie algebra of \( U_{\text{Hod}} \) has \textit{canonical} generators \( G_{p,q}, p, q \geq 1 \), satisfying the only relation \( \overline{C}_{p,q} = -G_{q,p} \), defined in [G1]. For the subcategory of Hodge-Tate structures they were defined in [L]. Unlike similar but different Deligne’s generators [D], they behave nicely in families. So to define an action of the group \( G_{\text{Hod}} \) one needs to have an action of the subgroup \( \mathbb{C}^*_{\mathbb{C}/\mathbb{R}} \) and, in addition to this, an action of a single operator
\[
G := \sum_{p,q \geq 1} G_{p,q}.
\]

Denote by \( \mathbb{C}^* \) the real algebraic group with the group of complex points \( \mathbb{C}^* \). The extension induced from (1) by the diagonal embedding \( \mathbb{C}^* \subset \mathbb{C}^*_{\mathbb{C}/\mathbb{R}} \) is the \textit{twistor Galois group}. It is a semidirect product of the groups \( U_{\text{Hod}} \) and \( \mathbb{C}^* \).
\[
0 \longrightarrow U_{\text{Hod}} \longrightarrow G_T \longrightarrow \mathbb{C}^* \longrightarrow 0. \quad (2)
\]

It is not difficult to prove

**Lemma 1.1** The category of representations of \( G_T \) is equivalent to the category of mixed twistor structures defined by Simpson [Si2].

We suggest the following fills the ??-marks in the dictionary related the Hodge and Galois. Below \( X \) is a smooth projective complex algebraic variety.

**Conjecture 1.2** There exists a functorial homotopy action of the twistor Galois group \( G_T \) by \( A_\infty \)-equivalences of an \( A_\infty \)-enhancement of the derived category of perverse sheaves on \( X \) such that the category of equivariant objects is equivalent to Saito’s category real mixed Hodge sheaves.\(^1\)

\(^1\) We want to have a natural construction of the action first, and get Saito’s category real mixed Hodge sheaves as a consequence, not the other way around.
Denote by $D_{\text{sm}}^b(X)$ the category of smooth complexes of sheaves on $X$, i.e. complexes of sheaves on $X$ whose cohomology are local systems.

**Theorem 1.3** There exists a functorial for pull-backs homotopy action of the twistor Galois group $G_T$ by $A_\infty$-equivalences of an $A_\infty$-enhancement of the category $D_{\text{sm}}^b(X)$.

The action of the subgroup $C^*$ is not algebraic. It arises from Simpson’s action of $C^*$ on semisimple local systems [Si1]. The action of the Lie algebra of the unipotent radical $U_{\text{Hod}}$ is determined by a collection of numbers, which we call the Hodge correlators for semisimple local systems. Our construction uses the theory of harmonic bundles [Si1]. The Hodge correlators can be interpreted as correlators for a certain Feynman integral. This Feynman integral is probably responsible for the “Hodge site”.

For the trivial local system the construction was carried out in [G2]. A more general construction for curves, involving the constant sheaves and delta-functions, was carried out in [G1].

In the case when $X$ is the universal modular curve, the Hodge correlators contain the special values $L(f,n)$ of weight $k \geq 2$ modular forms for $GL_2(\mathbb{Q})$ outside of the critical strip – it turns out that the simplest Hodge correlators in this case coincide with the Rankin-Selberg integrals for $L(f,n)$. There is an analog of Hodge correlators for the modular hyperbolic threefolds with the same properties. Its algebraic-geometric meaning is obscure.

**Acknowledgments.** I am grateful to Alexander Beilinson and Maxim Kontsevich for their interest to this project and useful discussions. This work was supported by the NSF grant DMS-1059129.

## 2 Hodge correlators for local systems

### 2.1 An action of $G_T$ on the “minimal model” of $D_{\text{sm}}(X)$.

Tensor products of irreducible local systems are semisimple local systems. The category of harmonic bundles $\text{Har}_X$ is the graded category whose objects are semi-simple local systems on $X$ and their shifts, and morphisms are given by graded vector spaces

$$\text{Hom}_{\text{Har}_X}(V_1, V_2) := H^\bullet(X, V_1^\vee \otimes V_2).$$

Here is our main result.

**Theorem 2.1** There is a homotopy action of the twistor Galois group $G_T$ by $A_\infty$-equivalences of the graded category $\text{Har}_X$, such that the action of the subgroup $C^*$ is given by Simpson’s action of $C^*$ on semi-simple local systems.

This immediately implies Theorem 1.3. Indeed, given a small $A_\infty$-category $\mathcal{A}$, there is a functorial construction of the triangulated envelope $\text{Tr}(\mathcal{A})$ of $\mathcal{A}$, the smallest triangulated
category containing $A$. Since $D^b_{\text{sm}}(X)$ is generated as a triangulated category by semi-simple local systems, the category $\text{Tr}(\text{Har}_X)$ is equivalent to $D^b_{\text{sm}}(X)$ as a triangulated category, and thus is an $A_\infty$-enhancement of the latter. On the other hand, the action of the group $G_T$ from Theorem 2.1 extends by functoriality to the action on $\text{Tr}(\text{Har}_X)$.

Below we recall what are $A_\infty$-equivalences of DG categories and then define the corresponding data in our case.

### 2.2 $A_\infty$-equivalences of DG categories

**The Hochshild cohomology of a small dg-category $A$.** Let $A$ be a small dg category. Consider a bicomplex whose $n$-th column is

$$\prod_{[X_i]} \text{Hom}\left(A(X_0, X_1)[1] \otimes A(X_1, X_2)[1] \otimes \ldots \otimes A(X_{n-1}, X_n)[1], A(X_0, X_n)[1]\right),$$

where the product is over isomorphism classes $[X_i]$ of objects of the category $A$. The vertical differential $d_1$ in the bicomplex is given by the differential on the tensor product of complexes. The horizontal one $d_2$ is the degree 1 map provided by the composition

$$A(X_i, X_{i+1}) \otimes A(X_{i+1}, X_{i+2}) \longrightarrow A(X_i, X_{i+2}).$$

Let $\text{HC}^*(A)$ be the total complex of this bicomplex. Its cohomology are the Hochshild cohomology $\text{HH}^*(A)$ of $A$. Let $\text{Fun}_{A_\infty}(A, A)$ be the space of $A_\infty$-functors from $A$ to itself. Lemma 2.2 can serve as a definition of $A_\infty$-functors considered modulo homotopy equivalence.

**Lemma 2.2** One has

$$H^0 \text{Fun}_{A_\infty}(A, A) = \text{HH}^0(A).$$

Indeed, a cocycle in $\text{HC}^0(A)$ is the same thing as an $A_\infty$-functor. Coboundaries corresponds to the homotopic to zero functors.

**The cyclic homology of a small rigid dg-category $A$.** Let $(\alpha_0 \otimes \ldots \otimes \alpha_m)_c$ be the projection of $\alpha_0 \otimes \ldots \otimes \alpha_m$ to the coinvariants of the cyclic shift. So, if $\overline{\alpha} := \deg \alpha$,

$$(\alpha_0 \otimes \ldots \otimes \alpha_m)_c = (-1)^{\overline{\alpha}_0 + \overline{\alpha}_1 + \ldots + \overline{\alpha}_m} (\alpha_0 \otimes \alpha_1 \otimes \ldots \otimes \alpha_m \otimes \alpha_0)_c.$$

We assign to $A$ a bicomplex whose $n$-th column is

$$\prod_{[X_i]} \left( A(X_0, X_1)[1] \otimes \ldots \otimes A(X_{n-1}, X_n)[1] \otimes A(X_n, X_0)[1] \right)_c.$$

The differentials are induced by the differentials and the composition maps on Hom’s. The cyclic homology complex $\text{CC}_c(A)$ of $A$ is the total complex of this bicomplex. Its homology are the cyclic homology of $A$. 

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Assume that there are functorial pairings
\[ \mathcal{A}(X,Y)[1] \otimes \mathcal{A}(Y,X)[1] \longrightarrow \mathcal{H}^*. \]
Then there is a morphism of complexes
\[ HC^*(\mathcal{A})^* \longrightarrow CC_*(\mathcal{A}) \otimes \mathcal{H}. \]  
For the category of harmonic bundles Har_X there is such a pairing with
\[ \mathcal{H} := H_{2n}(X)[-2]. \]
It provides a map
\[ \varphi : \text{Hom} \left( H_0(CC_*(\text{Har}_X) \otimes \mathcal{H}, \mathbb{C} \right) \longrightarrow H^0(\text{Fun}_{A_\infty}(\text{Har}_X, \text{Har}_X). \]

2.3 The Hodge correlators

Theorem 2.3 a) There is a linear map, the Hodge correlator map
\[ \text{Cor}_{\text{Har}_X} : H_0(CC_*(\text{Har}_X) \otimes \mathcal{H}) \longrightarrow \mathbb{C}. \]
Combining it with (7), we get a cohomology class
\[ \mathcal{H}_{\text{Har}_X} := \varphi(\text{Cor}_{\text{Har}_X}) \in H^0(\text{Fun}_{A_\infty}(\text{Har}_X, \text{Har}_X). \]

b) There is a homotopy action of the twistor Galois group G_T by A_\infty-autoequivalences of the category Har_X such that
- Its restriction to the subgroup C^* is the Simpson action [Si1] on the category Har_X.
- Its restriction to the Lie algebra LieU_Hod is given by a Lie algebra map
  \[ \mathbb{H}_{\text{Har}_X} : \text{LieU}_{\text{Hod}} \longrightarrow H^0(\text{Fun}_{A_\infty}(\text{Har}_X, \text{Har}_X), \]
  uniquely determined by the condition that \[ \mathbb{H}_{\text{Har}_X}(G) = \mathcal{H}_{\text{Har}_X}. \]

2.4 Construction.
To define the Hodge correlator map (8), we define a collection of degree zero maps
\[ \text{Cor}_{\text{Hod}_X} : \left( H^*(X, V_0^* \otimes V_1)[1] \otimes \ldots \otimes H^*(X, V_m^* \otimes V_0)[1] \right) \otimes \mathcal{H} \longrightarrow \mathbb{C}. \]
The definition depends on some choices, like harmonic representatives of cohomology classes. We prove that it is well defined on HC^0, i.e. its restriction to cycles is independent of the choices, and coboundaries are mapped to zero.

We picture an element in the source of the map (11) by a polygon P, see Fig 1, whose vertices are the objects V_i, and the oriented sides V_iV_{i+1} are graded vector space \[ \text{Ext}^*(V_i, V_{i+1})(1). \]
Green currents for harmonic bundles. Let $V$ be a harmonic bundle on $X$. Then there is a Doulbeaut bicomplex $(\mathcal{A}^\bullet(X,V); D', D'')$ where the differentials $D', D''$ are provided by the complex structure on $X$ and the harmonic metric on $V$. It satisfies the $D', D''$-lemma.

Choose a splitting of the corresponding de Rham complex $\mathcal{A}^\bullet(X,V)$ into an arbitrary subspace $\text{Har}^\bullet(X,V)$ isomorphically projecting onto the cohomology $H^\bullet(X,V)$ ("harmonic forms") and its orthogonal complement. If $V = \mathbb{C}_X$, we choose $a \in X$ and take the $\delta$-function $\delta_a$ at the point $a \in X$ as a representative of the fundamental class.

Let $\delta_\Delta$ be the Schwarz kernel of the identity map $V \to V$ given by the $\delta$-function of the diagonal, and $P_{\text{Har}}$ the Schwarz kernel of the projector onto the space $\text{Har}^\bullet(X,V)$, realized by an $(n,n)$-form on $X \times X$. Choose a basis $\{\alpha_i\}$ in $\text{Har}^\bullet(X,V)$. Denote by $\{\alpha^\vee_i\}$ the dual basis. Then we have

$$P_{\text{Har}} = \sum \alpha^\vee_i \otimes \alpha_i, \quad \int_X \alpha_i \wedge \alpha^\vee_j = \delta_{ij}.$$ 

Let $p_i : X \times X \to X$ be the projections onto the factors.

**Definition 2.4** A Green current $G(V; x, y)$ is a $p_1^*V^* \otimes p_2^*V$-valued current on $X \times X$,

$$G(V; x, y) \in \mathcal{D}^{2n-2}(X \times X, p_1^*V^* \otimes p_2^*V), \quad n = \dim_{\mathbb{C}}X,$$

which satisfies the differential equation

$$(2\pi i)^{-1}D''D'G(V; x, y) = \delta_\Delta - P_{\text{Har}}. \quad (12)$$

The two currents on the right hand side of (12) represent the same cohomology class, so the equation has a solution by the $D'D''$-lemma.

**Remark.** The Green current depends on the choice of the "harmonic forms". So if $V = \mathbb{C}$, it depends on the choice of the base point $a$. Solutions of equation (12) are well defined modulo $\text{Im}D'' + \text{Im}D' + \text{Har}^\bullet(X,V)$. 

6
Construction of the Hodge correlators. Trees. Take a plane trivalent tree $T$ dual to a triangulation of the polygon $P$, see Fig 1. The complement to $T$ in the polygon $P$ is a union of connected domains parametrized by the vertices of $P$, and thus decorated by the harmonic bundles $V_i$. Each edge $E$ of the tree $T$ is shared by two domains. The corresponding harmonic bundles are denoted $V_{E-}$ and $V_{E+}$. If $E$ is an external edge, we assume that $V_{E-}$ is before $V_{E+}$ for the clockwise orientation.

Given an internal vertex $v$ of the tree $T$, there are three domains sharing the vertex. We denote the corresponding harmonic bundles by $V_i$, $V_j$, $V_k$, where the cyclic order of the bundles agrees with the clockwise orientation. There is a natural trace map

$$\text{Tr}_v : V_i^* \otimes V_j \otimes V_j^* \otimes V_k \otimes V_k^* \otimes V_i = \rightarrow \mathbb{C}. \quad (13)$$

It is invariant under the cyclic shift.

Decorations. For every edge $E$ of $T$, choose a graded splitting of the de Rham complex

$$\mathcal{A}^*(X, V_{E-}^* \otimes V_{E+}) = \mathcal{H}ar^*(X, V_{E-}^* \otimes V_{E+}) \bigoplus \mathcal{H}ar^*(X, V_{E-}^* \otimes V_{E+})^\perp.$$

Then a decomposable class in \( \bigotimes_{i=0}^m H^*(X, V_i^* \otimes V_{i+1})[1] \otimes C \) has a harmonic representative

$$W = (\alpha_{0,1} \otimes \alpha_{1,2} \otimes \ldots \otimes \alpha_{m,0})_C.$$

We are going to assign to $W$ a top degree current $\kappa(W)$ on $X$ (internal vertices of $T$).

Each external edge $E$ of the tree $T$ is decorated by an element

$$\alpha_E \in \mathcal{H}ar^*(X, V_{E-}^* \otimes V_{E+}).$$

Put the current $\alpha_E$ to the copy of $X$ assigned to the internal vertex of the edge $E$, and pull it back to (14) using the projection $p_{\alpha_E}$ of the latter to the $X$. Abusing notation, we denote the pull back by $\alpha_E$. It is a form on (14) with values in the bundle $p_{\alpha_E}^*(V_{E-}^* \otimes V_{E+})$

Green currents. We assign to each internal edge $E$ of the tree $T$ a Green current

$$G(V_{E-}^* \otimes V_{E+}; x_-, x_+). \quad (15)$$

The order of $(x_-, x_+)$ agrees with the one of $(V_{E-}^*, V_{E+})$ as on Fig 2: the cyclic order of $(V_{E-}^*, x_-, V_{E+}^*, x_+)$ agrees with the clockwise orientation. The Green current (15) is symmetric:

$$G(V_{E-}^* \otimes V_{E+}; x_-, x_+) = G(V_{E+}^* \otimes V_{E-}; x_+, x_-). \quad (16)$$

So it does not depend on the choice of orientation of the edge $E$.

The map $\xi$. There is a degree zero map

$$\xi : \mathcal{A}^*(X, V_0)[-1] \otimes \ldots \otimes \mathcal{A}^*(X, V_m)[-1] \rightarrow \mathcal{A}^*(X, V_0 \otimes \ldots \otimes V_m)[-1]; \quad (17)$$
The graded symmetrization in (18) is defined via isomorphisms $V_{\sigma(0)} \otimes \ldots \otimes V_{\sigma(m)} \rightarrow V_0 \otimes \ldots \otimes V_m$, where $\sigma$ is a permutation of $\{0, \ldots, m\}$. It is essential that $\deg D^C \varphi = \deg \varphi + 1$.

An outline of the construction. We apply the operator $\xi$ to the product of the Green currents assigned to the internal edges of $T$. Then we multiply on (14) the obtained local system valued current with the one provided by the decoration $W$, with an appropriate sign. Applying the product of the trace maps (13) over the internal vertices of $T$, we get a top degree scalar current on (14). Integrating it we get a number assigned to $T$. Taking the sum over all plane trivalent trees $T$ decorated by $W$, we get a complex number $\text{Cor}_{\text{Har}}(W \otimes \mathcal{H})$. Altogether, we get the map (8). One checks that its degree is zero. The signs in this definition are defined the same way as in [G2].

**Theorem 2.5** The maps (11) give rise to a well defined Hodge correlator map (8).

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3 Hodge correlators and Rankin-Selberg integrals

Let $X$ be a smooth projective curve. An irreducible holonomic $\mathcal{D}$-module with regular singularities on $X$ is either the $\delta$-functions $\delta_a$ at a point $a$, or the intermediate extension of an irreducible local system on $X - S$, where $S$ is a finite (perhaps empty) collection of
points. Given a cyclic collection of irreducible holonomic $\mathcal{D}$-modules on $X$ one can define Hodge correlators for cyclic tensor products of the consecutive $\text{Ext}^1$’s. The simplest Hodge correlator of this kind is given by the Green function assigned to the diagram on Fig 3. It tells the Feynman rule needed to incorporate the $\mathcal{D}$-modules $\delta_a$.

Figure 4: The Hodge correlator for the special value $L(f, k+m+2)$ of a weight $k$ modular form.

Now let $Y(N)$ be the level $N$ modular curve, and $X(N)$ its smooth compactification. Let $p : \mathcal{E} \longrightarrow Y(N)$ be the universal elliptic curve. Consider the local system $\mathcal{H} := R^1 p_* \mathbb{C}(1)$ on $Y(N)$. Then for each cusp $a \in X(N) - Y(N)$ there are extension classes

$$E(a, S^m\mathcal{H}) \in \text{Ext}^1(\delta_a, S^m\mathcal{H}), \quad E(S^m\mathcal{H}, a) \in \text{Ext}^1(S^m\mathcal{H}, \delta_a)$$

Let $f$ be a weight $k+2$ modular form. It provides a class $f \in \text{Ext}^1(\mathcal{C}, S^k\mathcal{H})$ Now the Hodge correlator

References


[Si1] Simpson C.: 