Problem 1. For a squared matrix $A \in M_{n \times n}$, let us define the trace of $A$ to be $tr(A) := \sum_{i=1}^{n} a_{ii}$. That is, the trace of $A$ is the sum of all the elements on the diagonal. Prove that:

(i) $tr(A) = tr(A^{t})$, and

(ii) $tr(cA) = c \cdot tr(A)$ (where $c$ is a scalar), and

(iii) $tr(A + B) = tr(A) + tr(B)$, and

(iv) $tr(AB) = tr(BA)$.

A bit harder task: prove that there are no two matrices $A$ and $B$ for which $AB - BA = I$.

Problem 2. Given a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ and a squared matrix $A$, we define $p(A) = a_n A^n + a_{n-1} A^{n-1} + \ldots + a_0 \cdot I$. Show that for a $2 \times 2$ matrix $A$ we have $p(A) = 0$, where $p(x) = x^2 - tr(A)x + det(A)$.

Problem 3. Try to prove (by providing an explanation) or disprove (by giving an example) each of the following statements. Suppose that $A, B \in M_{n \times n}$ and satisfy $B = A^2 - A$.

(a) $Ax = 0$ has a unique solution if and only if $det(A) \neq 0$.

(b) If $Bx = 0$ has a unique solution, then so does $Ax = 0$.

(c) If $Ax = 0$ has a unique solution, then so does $Bx = 0$.

(d) If $A^2x = 0$ has a non-trivial solution, then so does $B^2x = 0$.

Problem 4. A squared matrix $A \in M_{n \times n}$ is said to be antisymmetric if $A^{t} = -A$ (and symmetric if $A^{t} = A$). Show that if $A$ is antisymmetric and $n$ is odd, then for sure $A$ is not invertible.

Problem 5. Let $A, B \in M_{n \times n}$ and consider the two systems of homogenous linear equations $ABx = 0$ and $BAx = 0$. Show that if one of them has a non-trivial solution, then so does the other.

Problem 6. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}.$$ 

Find two matrices $P$ and $Q$ for which $B = PAQ$.

Problem 8. Let $A \in M_{n \times n}$ be a matrix such that for each $i$, $\sum_j a_{ij} = 0$. That is, the sum of the elements in each row equal zero. Show that $A$ in not invertible.

Problem 9. Show that if $A$ and $B$ are two matrices for which there exists an invertible matrix $P$ such that $A = P^{-1}BP$, then they have the same eigenvalues.