Thick–skinned 3–manifolds

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Abstract

We show that if the totally geodesic boundary of a compact hyperbolic 3–manifold M has a collar of depth $d \gg 0$, then the diameter of the skinning map of M is no more than $Ae^{-d/2}$ for some A depending only on the genus and injectivity radius of ∂M.

Let S be a closed oriented surface of negative Euler characteristic. By the Simultaneous Uniformization Theorem [3], the space $\mathrm{QF}(S) = \mathrm{AH}(S \times \mathbb{R})^\circ$ of convex cocompact hyperbolic structures on $S \times \mathbb{R}$ is naturally homeomorphic to the product of Teichmüller spaces $\mathscr{T}(S) \times \mathscr{T}(\overline{S})$. If $(X, \overline{Y})$ is a point of $\mathscr{T}(S) \times \mathscr{T}(\overline{S})$, we let $\mathrm{qf}(X, \overline{Y})$ denote $S \times \mathbb{R}$ with the corresponding convex cocompact hyperbolic structure.

Let M be a compact hyperbolic 3–manifold with totally geodesic boundary homeomorphic to S. A generalization of the Simultaneous Uniformization Theorem due to Ahlfors, Bers, Marden, and Sullivan (see [5] or [8]) tells us that the space $\mathrm{AH}(M)^\circ$ of convex cocompact hyperbolic metrics on M° is naturally homeomorphic to the Teichmüller space $\mathscr{T}(S)$. If X is a point in $\mathscr{T}(S)$, we let M^X denote M° equipped with the corresponding convex cocompact hyperbolic structure.

The inclusion $\partial M \to M$ induces a map $\mathrm{AH}(M)^\circ \to \mathrm{QF}(S)$. Identifying $\mathrm{AH}(M)^\circ$ with $\mathscr{T}(S)$ and $\mathrm{QF}(S)$ with $\mathscr{T}(S) \times \mathscr{T}(\overline{S})$, this map is given by $X \mapsto (X, \sigma_M(X))$. The function

$$\sigma_M \colon \mathscr{T}(S) \to \mathscr{T}(\overline{S})$$

is Thurston's *skinning map* associated to M. This map is a key ingredient in Thurston's proof of Geometrization for Haken Manifolds [14, 17, 18, 11]. Thurston's Bounded Image Theorem [19, 12] states that the image of σ_M is bounded, and we call the diameter of the image with respect to the Teichmüller metric the *diameter of* σ_M. In [12], the first author proved that if ∂M has a large collar, then σ_M carries a large ball to a set of very small diameter (Theorem 29 there). We greatly improve that theorem here.

We say that the totally geodesic boundary ∂M in a hyperbolic 3–manifold M has a *collar of depth d* if the d–neighborhood of ∂M is homeomorphic to $\partial M \times [0, d]$.

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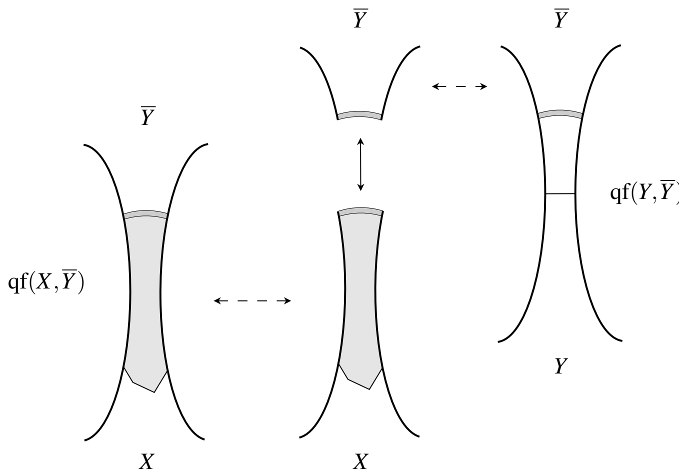

Figure 1: Building the metric η on $S \times \mathbb{R}$.

Theorem 1. *If ε and m are positive numbers, then there are positive numbers A and T such that the following holds. If M is a compact hyperbolic 3–manifold with totally geodesic boundary Y with $\chi(Y) \geq -m$ and* $\operatorname{injrad}(Y) \geq \varepsilon$*, and M contains a collar of depth $d \geq T$ about Σ, then the skinning map σ_M has diameter less than $Ae^{-d/2}$.*

We pause to sketch the proof.

Consider the hyperbolic manifolds qf$(X,\overline{Y})$ and qf$(Y,\overline{Y})$. Very far out toward their $\overline{Y}$–ends, these manifolds are very nearly isometric. In fact, the proximity of the metrics decays exponentially in the distance from the convex core. Using foliations constructed by C. Epstein, the metrics near the $\overline{Y}$–ends of qf$(X,\overline{Y})$ and qf$(Y,\overline{Y})$ may be written down explicitly in terms of the Schwarzian derivatives associated to the projective structures on $\overline{Y}$, see Section 2. This allows us to explicitly glue the "X–side" of qf$(X,\overline{Y})$ to the "$\overline{Y}$–side" of qf$(Y,\overline{Y})$ to obtain a metric η on $S \times \mathbb{R}$ which is hyperbolic away from a shallow gluing region of the form $S \times [n, n+1]$, see Figure 1. Calculations (in Sections 3.1 and 3.3) show that the resulting metric has sectional and traceless Ricci curvatures exponentially close to -1 and 0, respectively. Moreover, the L^2–norm of the traceless Ricci curvature of this metric is exponentially small (see Section 3.2).

Given a hyperbolic manifold M^Y with totally geodesic boundary Y possessing a large collar about its boundary, we may glue the "X–side" of qf$(X,\overline{Y})$ to a compact piece of M^Y in the same way to obtain a metric η on M° with the same curvature bounds, see Figure 2.

It is a theorem of Tian that a Riemannian metric on a closed 3–manifold whose sectional curvatures are very close to -1 and whose traceless Ricci curvature has very small L^2–norm is $\mathscr{C}^2$–close to a hyperbolic metric. As our manifold is noncompact, Tian's theorem is not directly applicable. A theorem of Brooks [7] allows us to circumvent this problem by performing a small quasiconformal deformation of qf$(X,\overline{Y})$ to obtain a manifold covering a closed one, and we find that ω is $\mathscr{C}^2$–close to the

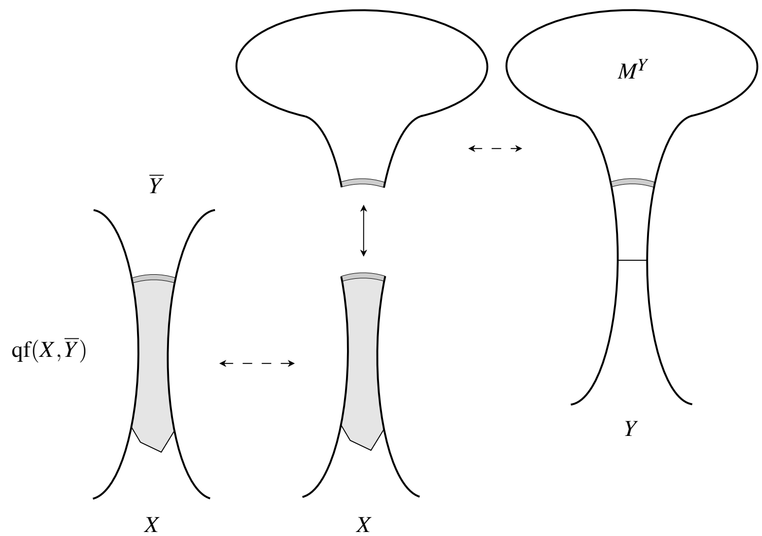

Figure 2: Building the metric η on M°.

convex cocompact manifold $M^X \cong M^\circ$ with conformal boundary X. See Section 4.2.

Now, the copy of $S \times \{n+1\}$ in (M°, ω) is conformally equivalent to $\overline{Y}$, and, for large n, the proximity of the metrics implies that the corresponding surface in M^X is very close to both $\sigma_M(X)$ and $\overline{Y}$ in Teichmüller space. We conclude that the image of σ_M lies in a small neighborhood of $\overline{Y}$.

Skinning maps are defined for any orientable hyperbolic manifold with incompressible boundary, and, with very little modification, our argument establishes the following generalization of Theorem 1.

Theorem 2. *Let $M^W \cong M$ be a convex cocompact hyperbolic manifold with conformal boundary W, and let $p\colon \mathrm{qf}(W,\overline{Y}) \to M^W$ be the covering map corresponding to W. There are constants A and T depending only on $\chi(S)$ and $\mathrm{injrad}(\overline{Y})$ such that if $d > T$ and p **embeds** the d–neighborhood of the convex core of $\mathrm{qf}(W,\overline{Y})$ isometrically into M^W, then σ_M has diameter no more than $Ae^{-d/2}$. In particular, the manifold M is acylindrical, the totally geodesic boundary structure on M has boundary close to Y, and that boundary has a large collar there.* $\square$

Miscellaneous notation. If f and F are functions of t, we use the Landau notation $f = \mathscr{O}(F)$ to mean that there is a constant L such that $|f(t)| \leq LF(t)$ for all t. If a, b, c, ... are objects, we write $f(t) = \mathscr{O}_{a,b,c,\ldots}(F(t))$ if $|f(t)| \leq LF(t)$ for a constant L depending only on $a, b, c, \ldots$ We use the standard notation $W^{k,p}(\mathscr{X})$ for Sobolev spaces and follow the Einstein summation convention.

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1 Tian's theorem

Theorem 3 (Tian [20])**.** *There are numbers $C_3 \geq 1$ and $\varepsilon_2 > 0$ such that the following holds. If $\varepsilon < \varepsilon_2$ and (M,ω) is a closed Riemannian 3–manifold with sectional curvatures pinched between $-1-\varepsilon$ and $-1+\varepsilon$, traceless Ricci curvature $\mathrm{Ric}^\omega + 2\omega = 0$ on the 1–thin part, and*

$$\sqrt{\int_M \|\mathrm{Ric}^\omega + 2\omega\|_\omega^2 \, \mathrm{d}V_\omega} \leq \varepsilon, \tag{1.1}$$

then M admits a hyperbolic metric ζ and $\|\omega - \zeta\|_{\mathscr{C}^2(M,\omega)} \leq C_3\varepsilon$, where $\|\cdot\|_{\mathscr{C}^2(M,\omega)}$ is the $\mathscr{C}^2$–norm with respect to ω. □

The background metric ω defines the pointwise norm of any tensor on M, and the pointwise $\mathscr{C}^2$–norm of a smoothly varying bilinear form b is defined by taking the supremum of the norm of b and its first two covariant derivatives with respect to ω. The norm $\|b\|_{\mathscr{C}^2(M,\omega)}$ is then the supremum of the pointwise norms.

2 Hyperbolic metrics and Epstein surfaces

Most of this section is a review of Section 6.1 of [6] and Sections 3.2–3.4 of [2].

Let Δ be the open unit disk in $\mathbb{C}$ parameterized by the variable $z = x+iy$. We model hyperbolic space $\mathbb{H}^3$ as $\Delta \times \mathbb{R}$ with the metric g given by

$$\mathrm{d}s^2 = \frac{4\cosh^2 t}{(1-|z|^2)^2}\,\mathrm{d}x^2 + \frac{4\cosh^2 t}{(1-|z|^2)^2}\,\mathrm{d}y^2 + \mathrm{d}t^2. \tag{2.1}$$

Note that $\Delta \times \{0\}$ is a totally geodesic hyperbolic plane. We encode g in the matrix

$$g = (g_{ij}) = \begin{pmatrix} \lambda^2\cosh^2 t & 0 & 0 \\ 0 & \lambda^2\cosh^2 t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\lambda = 2/(1-|z|^2)$.

Let Γ^Y be a fuchsian group uniformizing $\overline{Y}$ via $\Delta/\Gamma^Y = \overline{Y}$. This gives us the fuchsian hyperbolic 3–manifold $\mathrm{qf}(Y,\overline{Y}) = \mathbb{H}^3/\Gamma^Y$. Identifying Δ and $\Delta \times \{0\}$, we have a local expression for the hyperbolic metric on $\mathrm{qf}(Y,\overline{Y})$ in (2.1). We let $\mathscr{D}^Y$ be a compact fundamental domain for Γ^Y acting on Δ whose interior contains zero.

We also want to consider the Poincaré ball model $\mathbb{B}^3$ of $\mathbb{H}^3$, with boundary the Riemann sphere $\widehat{\mathbb{C}}$. There is a unique isometry $\iota : \Delta \times \mathbb{R} \to \mathbb{B}^3$ which extends continuously to $\Delta \times \{\pm\infty\}$, taking $\Delta \times \{\infty\}$ to $\Delta \subset \widehat{\mathbb{C}}$ by the identity map.

Let $\varphi\colon\ \Delta \to \widehat{\mathbb{C}}$ be a univalent function, let $\mathscr{S}\varphi(z)$ be its Schwarzian derivative, and let $\|\mathscr{S}\varphi(z)\| = |\lambda^{-2}\mathscr{S}\varphi(z)|$. Let $M_{\varphi(z)} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the osculating Möbius transformation to φ at z (the Möbius transformation with the same 2–jet as φ at z). This uniquely extends to an isometry $M_{\varphi(z)}\colon\ \mathbb{B}^3 \to \mathbb{B}^3$. There is then a map $\Phi\colon\ \Delta \times \mathbb{R} \to \mathbb{B}^3$ given by

$$\Phi(z,t) = M_{\varphi(z)}(\iota(z,t)),$$

which also admits a continuous extension to $\Delta \times (-\infty, \infty]$ with $\Phi(z, \infty) = \varphi(z)$. We henceforth identify Δ with $\Delta \times \{\infty\}$, and identify both $\Delta \times \mathbb{R}$ and $\mathbb{B}^3$ with $\mathbb{H}^3$.

There is an orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\frac{\partial}{\partial t}$ for the tangent space to $\mathbb{H}^3$ at (z,t) and an orthonormal basis for the tangent space to $\mathbb{H}^3$ at $\Phi(z,t)$ such that the derivative of Φ at (z,t) is given by

$$\mathrm{D}\Phi\big|_{(z,t)} = \begin{pmatrix} 1 + \frac{\|\mathscr{S}\varphi(z)\|}{4e^t \cosh t} & 0 & 0 \\ 0 & 1 - \frac{\|\mathscr{S}\varphi(z)\|}{4e^t \cosh t} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

If we normalize (by conjugation in $\mathrm{PSL}_2\mathbb{C}$) so that $z = 0$ and so that the osculating Möbius transformation at zero is the identity, we have

$$2\cosh t \cdot \mathbf{e}_k = \cos\left(\theta_0 + (k-1)\pi/2\right)\frac{\partial}{\partial x} + \sin\left(\theta_0 + (k-1)\pi/2\right)\frac{\partial}{\partial y} \tag{2.3}$$

for each k in $\{1,2\}$, where θ_0 is the argument of $\mathscr{S}\varphi(0)$, see Section 3.3 of [2].

If t is such that $4e^t\cosh(t) > \|\mathscr{S}\varphi(z)\|$, then Φ is an orientation preserving immersion. This inequality holds for $t > 0$ in our setting, by Kraus and Nehari's theorem [13, 16] that $\|\mathscr{S}\varphi(z)\| \leq 3/2$.

Now, consider the quasifuchsian manifold $\mathrm{qf}(X, \overline{Y})$. Let $\mathscr{U}^X$ be the component of the domain of discontinuity uniformizing X, and let $\mathscr{U}^{\overline{Y}}$ be the component uniformizing $\overline{Y}$. Let $\varphi\colon \Delta \to \mathscr{U}^{\overline{Y}}$ be the Riemann mapping induced by quasiconformally conjugating $\mathrm{qf}(Y, \overline{Y})$ to $\mathrm{qf}(X, \overline{Y})$. Let $t_0 > \log 2$. The restriction of Φ to $\Delta \times [t_0, \infty)$ is a $\pi_1(S)$–equivariant diffeomorphism onto its image. So, the images $\mathscr{E}_t^X$ of the $\Phi(\Delta \times \{t\})$ foliate the $\overline{Y}$–end of $\mathrm{qf}(X, \overline{Y})$ as t ranges over $[t_0, \infty)$. We call the $\mathscr{E}_t^X$ *Epstein surfaces*, as they were first studied by C. Epstein [9].

2.1 Gluing hyperbolic metrics

Let $s_0(t)$ be a smooth nonincreasing function on $\mathbb{R}$ such that $s_0(t) = 1$ when $t \leq 0$ and $s_0(t) = 0$ when $t \geq 1$. Let $s_n(t) = s_0(t - n)$. As the Sobolev norms of the s_n are independent of n, we write $s(t)$ for $s_n(t)$ and let context dictate n.

Let $h = \Phi^* g$ be the pullback of the hyperbolic metric via Φ. We interpolate between the hyperbolic metrics g and h on $\Delta \times [n, n+1]$ using the metric

$$\eta = (1 - s(t))g + s(t)h, \tag{2.4}$$

which descends to a metric on $S \times [n, n+1]$ that we also call η, see Figure 1.

By (2.2), (2.4), and the Kraus–Nehari theorem, we have the following proposition.

Proposition 4. *The identity map $\iota\colon (S \times [n, n+1], g) \to (S \times [n, n+1], h)$ is $(1 + \mathscr{O}_s(e^{-2n}))$–bilipschitz with Jacobian determinant $1 + \mathscr{O}_s(e^{-4n})$.* $\square$

Let M be a hyperbolic 3–manifold with totally geodesic boundary isometric to Y having a collar of depth at least $n+1$ about ∂M. The $S \times [n, n+1]$–portion of $\mathrm{qf}(Y, \overline{Y})$

embeds isometrically in M (in the Riemannian sense), and so we may extend the metric η on $S\times[n,n+1]$ to a complete Riemannian metric on M° which is isometric to M "above" $S\times\{n+1\}$ and isometric to $\mathrm{qf}(X,\overline{Y})$ "below" $S\times\{n\}$.

We begin by showing that the traceless Ricci curvature of our metric η is on the order of e^{-2n}. The region where η is nonhyperbolic has volume on the order of e^{2n}, and so it will follow that the L^2–norm of the traceless Ricci curvature is on the order of e^{-n}. Since we are in dimension three, having Ricci curvature on the order of e^{-2n} implies that the difference between the sectional curvatures and -1 is on the order of e^{-2n} as well, and this allows us to apply Tian's theorem.°

The intuition for the estimate of the Ricci curvature is as follows. The Ricci curvature measures the infinitesimal defect in volume of a sharp geodesic cone compared to the corresponding Euclidean cone: the volume element of a metric ω at a point p admits an asymptotic expansion in ω–geodesic normal coordinates

$$\mathrm{d}V_\omega=\left(1-\frac{1}{6}\mathrm{Ric}^\omega(\mathbf{u})\varepsilon^2+\mathscr{O}(\varepsilon^3)\right)\varepsilon^2\,\mathrm{d}\varepsilon\,\mathrm{d}A(\mathbf{u}) \tag{2.5}$$

where Ric^ω is the Ricci curvature of ω considered a quadratic form, and $\mathrm{d}A(\mathbf{u})$ is the canonical spherical measure on the unit tangent space T^1_pM, see section 3.H.4 of [10]. Our metric η is obtained by gluing two hyperbolic metrics on $S\times\mathbb{R}$ together fiberwise. The original metrics on the fibers are exponentially close, as are the original normal curvatures to the fibers, and so, after interpolating to obtain η, the volumes of cones are disturbed an exponentially small amount. One may try to make this precise using (2.5) and Proposition 4. This shows that the Ricci curvatures are close, but depends on the precise rate of decay of the $\mathscr{O}(\varepsilon^3)$ term in $\mathrm{d}V_\eta$. Fortunately, the Ricci curvatures are fairly easy to estimate directly.

3 Bounds on curvatures

If $\omega=\omega_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ is a Riemannian metric, we have Christoffel symbols

$$\Gamma^\ell_{ij}(\omega)=\frac{1}{2}\omega^{k\ell}\left(\frac{\partial}{\partial x^i}\omega_{kj}+\frac{\partial}{\partial x^j}\omega_{ik}-\frac{\partial}{\partial x^k}\omega_{ij}\right), \tag{3.1}$$

where $(\omega^{ij})=(\omega_{ij})^{-1}$.

3.1 Bounding the Ricci curvature

The Ricci curvature tensor $\mathrm{Ric}^\omega=R^\omega_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ of a metric ω in coordinates x^i is given by

$$R^\omega_{ij}=\left(\frac{\partial\Gamma^\ell_{ij}}{\partial x^\ell}-\frac{\partial\Gamma^\ell_{i\ell}}{\partial x^j}+\Gamma^\ell_{ij}\Gamma^m_{\ell m}-\Gamma^m_{i\ell}\Gamma^\ell_{jm}\right)(\omega).$$

°It may seem rather lucky that the exponentially large volume and the exponentially small traceless Ricci curvature are off balance enough to give an exponentially small L^2–norm, but the situation is less delicate than it seems: Tian's theorem allows any finite p–norm [20], and since we find an exponentially small L^∞–bound on the traceless Ricci curvature, a choice of very large p yields an L^p–norm on the order of e^{-kn} for some $k>0$.

Theorem 5. *If* $\mathbf{u}$ *is an* η*–unit vector at* (z,t) *in* $S\times[n,n+1]$*, then*

$$\mathrm{Ric}^{\eta}(\mathbf{u})-\mathrm{Ric}^{g}(\mathbf{u})=\mathscr{O}(e^{-2n}). \tag{3.2}$$

Theorem 5 follows immediately from the following theorem.

Theorem 6. *We have* $\|\eta-g\|_{\mathscr{C}^2(S\times[n,n+1],g)}=\mathscr{O}(e^{-2n})$.

Proof. Since the norms $\|\mathscr{S}\varphi(z)\|$ and arguments $\arg(\mathscr{S}\varphi(z))$ are smooth functions away from the zeroes of $\mathscr{S}\varphi$, and the set of points (z,t) such that $\mathscr{S}\varphi(z)=0$ is a finite set of lines in $S\times\mathbb{R}$, we restrict attention to points (z,t) such that $\mathscr{S}\varphi(z)\neq 0$.

Let $\mathscr{Q}(\overline{Y})$ be the vector space of holomorphic quadratic differentials on $\overline{Y}$. By the Kraus–Nehari theorem, the subset of $\mathscr{Q}(\overline{Y})$ consisting of Schwarzian derivatives of developing maps of Kleinian projective structures on Y is compact, see [4]. So there is a number $\mathscr{B}_0=\mathscr{B}_0(Y)$ bounding the norms $\|\mathscr{S}\varphi(z)\|$, arguments $\arg(\mathscr{S}\varphi(z))$, and their first few partial derivatives on the fundamental domain $\mathscr{D}^Y$. In other words, the functions $\|\mathscr{S}\varphi(z)\|$ and $\arg(\mathscr{S}\varphi(z))$ have Sobolev norms $\|\cdot\|_{W^{2,\infty}(\mathscr{D}^Y)}$ at most $\mathscr{B}_0$. In fact, if we fix a compact subset $\mathscr{X}$ of $\mathscr{T}(\overline{Y})$ containing $\overline{Y}$, we obtain a uniform bound $\mathscr{B}_1=\mathscr{B}_1(\mathscr{X})$ on these Sobolev norms over all of $\mathscr{X}$. As the thick part of the moduli space $\mathscr{M}(\overline{Y})$ is compact [15], the action of the mapping class group $\mathrm{Mod}(S)$ on $\mathscr{T}(\overline{Y})$ provides a bound $\mathscr{B}=\mathscr{B}(\chi(S),\mathrm{injrad}(Y))$ on these Sobolev norms over the entire thick part of $\mathscr{T}(\overline{Y})$.

The notation $\mathscr{O}(\,\cdot\,)$ will now always mean $\mathscr{O}_{\chi(S),\mathrm{injrad}(Y)}(\,\cdot\,)$.

Given a point (w,t) with $\mathscr{S}\varphi(w)\neq 0$, we conjugate by an element of $\mathrm{PSL}_2\mathbb{C}$ fixing Δ to carry w to 0 and to ensure that the osculating Möbius transformation $M_{\varphi(0)}=\mathrm{Id}$.

We work with a small ball $B\subset\mathscr{D}^Y\subset\Delta$ centered at 0 containing no zeroes of $\mathscr{S}\varphi$.

Let z be a point of B, and let ψ_z be the hyperbolic element of $\mathrm{PSL}_2\mathbb{C}$ stabilizing Δ that carries 0 to z and whose axis in Δ contains 0. By the invariance of the Schwarzian, we have

$$\mathscr{S}(\varphi\circ\psi_z)(0)=\mathscr{S}\varphi(\psi_z(0))\,\psi_z'(0)^2=\mathscr{S}\varphi(z)\,\psi_z'(0)^2. \tag{3.3}$$

We may postcompose $\varphi\circ\psi_z$ with a Möbius transformation to ensure that the osculating Möbius transformation to $\varphi\circ\psi_z$ at 0 is the identity, and this has no effect on the Schwarzian. Let

$$\theta_z=\arg(\mathscr{S}\varphi(z))-\arg\left(\psi_z'(0)^2\right).$$

A change of variables allows us to assume that $\theta_0=0$. Let

$$A_z=\begin{pmatrix}\cos(\theta_z) & -\sin(\theta_z) & 0\\ \sin(\theta_z) & \cos(\theta_z) & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{3.4}$$

Consider coordinates $u^1 = \cosh(t)\,x$, $u^2 = \cosh(t)\,y$, and $u^3 = t$. In these coordinates, the metric g is given by

$$g = (g_{ij}) = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\lambda = 2/(1-|z|^2)$.

By (2.3), we have

$$\mathrm{D}\Phi\big|_{(z,t)} = A_z \begin{pmatrix} 1 + \frac{\|\mathscr{S}\varphi(z)\|}{4e^t\cosh t} & 0 & 0 \\ 0 & 1 - \frac{\|\mathscr{S}\varphi(z)\|}{4e^t\cosh t} & 0 \\ 0 & 0 & 1 \end{pmatrix} A_z^{-1} \tag{3.5}$$

in $B \times \mathbb{R}$ with respect to the orthonormal basis $\frac{1}{\lambda}\frac{\partial}{\partial u^1}$, $\frac{1}{\lambda}\frac{\partial}{\partial u^2}$, $\frac{\partial}{\partial u^3}$ to $\mathbb{H}^3$ at (z,t) and an orthonormal basis at $\Phi(z,t)$. Note that (g_{ij}) commutes with the matrix $\mathrm{D}\Phi^T = \mathrm{D}\Phi$.

Writing the metric $\eta = (1-s(t))\,g + s(t)\,h$ in these coordinates, we have

$$(\eta_{ij}) = (1-s(t))(g_{ij}) + s(t)\,\mathrm{D}\Phi^T \cdot (g_{ij}) \cdot \mathrm{D}\Phi \tag{3.6}$$

$$= (1-s(t))(g_{ij}) + s(t) \cdot (g_{ij}) \cdot (\mathrm{D}\Phi)^2. \tag{3.7}$$

Expanding this, we have

$$(\eta_{ij}) = (g_{ij}) + \frac{1}{e^t\cosh t} \cdot \frac{s(t)\lambda^2\|\mathscr{S}\varphi(z)\|}{2} \cdot E, \tag{3.8}$$

where

$$E = \begin{pmatrix} \cos(2\theta_z) + \frac{\|\mathscr{S}\varphi(z)\|}{8e^t\cosh t} & \sin(2\theta_z) & 0 \\ \sin(2\theta_z) & -\cos(2\theta_z) + \frac{\|\mathscr{S}\varphi(z)\|}{8e^t\cosh t} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.9}$$

Proposition 7. *For all i, j, k, and ℓ, we have*

$$\left|\eta_{ij} - g_{ij}\right| = \mathscr{O}(e^{-2n}),$$

$$\left|\frac{\partial}{\partial x^k}\eta_{ij} - \frac{\partial}{\partial x^k}g_{ij}\right| = \mathscr{O}(e^{-2n}),$$

and

$$\left|\frac{\partial}{\partial x^\ell}\frac{\partial}{\partial x^k}\eta_{ij} - \frac{\partial}{\partial x^\ell}\frac{\partial}{\partial x^k}g_{ij}\right| = \mathscr{O}(e^{-2n}),$$

in coordinates $u^1 = \cosh(t)\,x$, $u^2 = \cosh(t)\,y$, and $u^3 = t$ on $S \times [n, n+1]$.

Proof. The estimates hold on B by inspection of (3.8) and (3.9), and, since the zeroes of $\mathscr{S}\varphi$ are isolated and our metrics are smooth, they hold on all of $S\times[n,n+1]$ by continuity. $\square$

Now, in the coordinates u^i, the first few derivatives of the g_{ij} and g^{ij} are $\mathscr{O}(1)$, and so Proposition 7 implies that $\|\eta-g\|_{\mathscr{C}^2(S\times[n,n+1],g)}=\mathscr{O}(e^{-2n})$, by definition of the $\mathscr{C}^2$–norm. $\square$

3.2 L^2–norm of the traceless Ricci curvature

To apply Tian's theorem, we need to estimate the L^2–norm of the traceless Ricci curvature of η. We begin by estimating the volume of the nonhyperbolic part.

Lemma 8. *We have*

$$\int_{S\times[n,n+1]} 1\ \mathrm{d}V_\eta \le -18\pi\chi(S)e^{2n}. \tag{3.10}$$

Proof. Let $\iota\colon (S\times[n,n+1],g)\to(S\times[n,n+1],\eta)$ be the identity map. By (2.2), the Jacobian determinant of ι at (z,t) is

$$|\operatorname{Jac}\iota| = 1-\left(\frac{\|\mathscr{S}\varphi(z)\|}{4e^t\cosh t}\right)^2. \tag{3.11}$$

So

$$\begin{aligned}
\int_{S\times[n,n+1]} 1\ \mathrm{d}V_\eta &= \int_{S\times[n,n+1]} |\operatorname{Jac}\iota|\ \mathrm{d}V_g \\
&\le \int_{S\times[n,n+1]} 1\ \mathrm{d}V_g \\
&= \int_{\mathscr{D}^Y\times[n,n+1]} \sqrt{\det g}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= \int_{\mathscr{D}^Y\times[n,n+1]} \lambda^2\cosh^2 t\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= \int_S\left(\int_{[n,n+1]}\cosh^2 t\ \mathrm{d}t\right)\mathrm{d}A_Y \\
&\le -2\pi\chi(S)\,e^{2n+2} \\
&\le -18\pi\chi(S)\,e^{2n}.
\end{aligned}$$

$\square$

For an η–unit vector $\mathbf{u}$ at (z,t), we have

$$\operatorname{Ric}^\eta(\mathbf{u})+2\eta(\mathbf{u}) = \operatorname{Ric}^g(\mathbf{u})+2g(\mathbf{u})+\mathscr{O}(e^{-2t}) = \mathscr{O}(e^{-2t}), \tag{3.12}$$

by Theorem 5. So there is a constant $A_1=A_1(\chi(S),\operatorname{injrad}(Y))$ such that

$$\|\operatorname{Ric}^\eta+2\eta\|_\eta \le A_1e^{-2n}. \tag{3.13}$$

Lemma 9. *We have*

$$\sqrt{\int_M \|\mathrm{Ric}^\eta + 2\eta\|_\eta^2 \, \mathrm{d}V_\eta} \le -18\pi A_1 \chi(S)\, e^{-n}. \tag{3.14}$$

Proof. Since η is hyperbolic away from $S \times [n, n+1]$, Lemma 8 and (3.13) give us

$$\begin{aligned}
\sqrt{\int_M \|\mathrm{Ric}^\eta + 2\eta\|_\eta^2 \, \mathrm{d}V_\eta} &\le \sqrt{\int_{S\times[n,n+1]} A_1^2 e^{-4n} \, \mathrm{d}V_\eta} \\
&= A_1 e^{-2n} \sqrt{\int_{S\times[n,n+1]} 1 \, \mathrm{d}V_\eta} \\
&\le A_1 e^{-n} \sqrt{-18\pi\chi(S)} \\
&\le -18\pi A_1 \chi(S)\, e^{-n}.
\end{aligned}$$

□

3.3 Sectional curvatures

In dimension three, the sectional curvatures are determined by the Ricci curvatures. More specifically, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are orthonormal tangent vectors at a point in a 3–manifold, we have

$$2K(\mathbf{u},\mathbf{v}) = \mathrm{Ric}(\mathbf{u}) - \mathrm{Ric}(\mathbf{w}) + \mathrm{Ric}(\mathbf{v}). \tag{3.15}$$

Theorem 5 and (3.15) give us

$$2K^\eta(\mathbf{u},\mathbf{v})\big|_{(z,t)} = -2 + 2 - 2 + \mathscr{O}(e^{-2t}) \tag{3.16}$$

for any η–orthonormal vectors $\mathbf{u}$ and $\mathbf{v}$ at any (z,t) in $\Delta \times [n, n+1]$. So

$$K^\eta(\mathbf{u},\mathbf{v})\big|_{(z,t)} = -1 + \mathscr{O}(e^{-2n}) \tag{3.17}$$

for all (z,t), since $K^\eta = -1$ on $M - (\Delta \times [n, n+1])$. So there is a constant $A_2 = A_2(\chi(S), \mathrm{injrad}(Y)) \ge A_1$ such that

$$-1 - A_2 e^{-2n} \le K^\eta \le -1 + A_2 e^{-2n}. \tag{3.18}$$

Let $A_3 = \max\{-18\pi A_1 \chi(S), A_2, 1\}$.

4 Curvatures of surfaces and the proof of Theorem 1

4.1 Normal projections

Letting $w = x_1 + ix_2$, we compactify hyperbolic space by attaching the Riemann sphere $\widehat{\mathbb{C}}$ via the upper half-space model

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}.$$

Let $\mathscr{F}$ be a smooth surface in $\mathbb{H}^3$ equipped with a smooth unit normal field, and let q be in $\mathscr{F}$. Applying an element of $\mathrm{PSL}_2(\mathbb{C})$, we assume that $q = (0,0,1)$, that the unit normal to $\mathscr{F}$ at q is $-\mathbf{k}$, and that the principal directions at q are $\mathbf{i}$ and $\mathbf{j}$. Let ν be the normal projection of $\mathscr{F}$ to $\mathbb{C}$. Picking an orthonormal basis for $\mathrm{T}_q\mathscr{F}$ along its principal directions and the usual basis for $\mathrm{T}_0\mathbb{C}$, the derivative of ν at q is given by

$$\mathrm{D}\nu_q = \begin{pmatrix} \frac{1+\kappa_1}{2} & 0 \\ 0 & \frac{1+\kappa_2}{2} \end{pmatrix} \tag{4.1}$$

where the κ_i are the principal curvatures of $\mathscr{F}$ at q. Our convention is that normal curvatures are *positive* when the surface is curving *away* from the normal vector.

Lemma 10. *Let $S \times [0,\infty)$ be a closed smoothly concave neighborhood of a convex cocompact end E of a hyperbolic manifold and let Z be the conformal boundary at E. If the principal curvatures of $\mathscr{G} = S \times \{0\}$ are within ε of 1 for some $0 < \varepsilon < 1$, then $\mathscr{G}$ and Z are $(1+\varepsilon)^2$–quasiconformal.*

Proof. Lift $\mathscr{G}$ to a surface $\widetilde{\mathscr{G}}$ in $\mathbb{H}^3$ and normalize as above so that the derivative of the normal projection at a point q in $\widetilde{\mathscr{G}}$ is

$$\mathrm{D}\nu_q = \begin{pmatrix} \frac{1+\kappa_1(q)}{2} & 0 \\ 0 & \frac{1+\kappa_2(q)}{2} \end{pmatrix}$$

where the $\kappa_i(q)$ are the principal curvatures of $\widetilde{\mathscr{G}}$ at q. The usual Euclidean metrics on the tangent spaces $\mathrm{T}_q\widetilde{\mathscr{G}}$ and $\mathrm{T}_0\mathbb{C}$ are conformally compatible with the Riemannian metrics on $\widetilde{\mathscr{G}}$ and $\widetilde{Z}$, respectively. Since $\mathrm{D}\nu_q$ is a $(1+\frac{\varepsilon}{2})$–bilipschitz linear map, the dilatation of the quasiconformal map ν is $(1+\frac{\varepsilon}{2})^2$ at q, see Chapter 1 of [1]. ☐

We need estimates on the principal curvatures of Epstein surfaces.

Let $\mathrm{qf}(W,Z)$ be a quasifuchsian manifold and let $\mathscr{S}\varphi$ be the Schwarzian derivative of the developing map $\psi\colon \Delta \to \mathscr{U}^Z$. Let $\mathscr{E}_t$ be the Epstein surface at time t in the Z–end of $\mathrm{qf}(W,Z)$. Whenever $t > 0$, the surface $\mathscr{E}_t$ is smoothly immersed and has well–defined continuously–varying principal curvatures $\kappa_i(z,t)$.

Lemma 11. *Let $\mathrm{qf}(W,Z)$ and $\mathscr{E}_t$ be as above. If $t \geq \log 9$, then*

$$|\kappa_i(z,t) - 1| \leq e^{-t}. \tag{4.2}$$

Proof. Let

$$\mathfrak{f}_1(z) = -\frac{\|\mathscr{S}\psi(z)\|}{\|\mathscr{S}\psi(z)\| - 1} \quad \text{and} \quad \mathfrak{f}_2(z) = -\frac{\|\mathscr{S}\psi(z)\|}{\|\mathscr{S}\psi(z)\| + 1}.$$

By the Kraus–Nehari theorem, we have $\|\mathscr{S}\psi(z)\| \leq 3/2$, and so

$$\mathfrak{f}_i(z) \notin \left(-3, -\frac{3}{5}\right). \tag{4.3}$$

The principal curvatures of $\mathscr{E}_t$ at (z,t) are given by

$$\kappa_i(z,t) = \frac{\sinh t + \mathfrak{f}_i(z)\cosh t}{\cosh t + \mathfrak{f}_i(z)\sinh t}, \tag{4.4}$$

when this is defined, see Proposition 6.3 of [6].°° If $\mathscr{S}\psi(z) = 1$, then $\kappa_1(z,t) = 1$ by continuity of the principal curvatures. These curvatures are positive provided $t > \log 2$, and so $\mathscr{E}_t$ is convex for such t.•

When $\mathfrak{f}_i(z) = 1$, we have $\kappa_i(z,t) = 1$, and there is nothing to prove.

Suppose that $\mathfrak{f}_i(z) \neq 1$ and $t > \log 2$. Then

$$\begin{aligned}
\left|\frac{\sinh t + \mathfrak{f}_i(z)\cosh t}{\cosh t + \mathfrak{f}_i(z)\sinh t} - 1\right| &= \left|\frac{\sinh t + \mathfrak{f}_i(z)\cosh t - \cosh t - \mathfrak{f}_i(z)\sinh t}{\cosh t + \mathfrak{f}_i(z)\sinh t}\right| && (4.5)\\
&= \left|\frac{-e^{-t} + \mathfrak{f}_i(z)\,e^{-t}}{\cosh t + \mathfrak{f}_i(z)\sinh t}\right| && (4.6)\\
&= 2\left|\frac{(\mathfrak{f}_i(z)-1)e^{-t}}{(\mathfrak{f}_i(z)+1)e^{t} + (1-\mathfrak{f}_i(z))e^{-t}}\right| && (4.7)\\
&= 2\left|\frac{1}{\left(\frac{\mathfrak{f}_i(z)+1}{\mathfrak{f}_i(z)-1}\right)e^{2t} - 1}\right| && (4.8)\\
&\le 2\left|\frac{1}{\frac{1}{4}e^{2t} - 1}\right| && (4.9)
\end{aligned}$$

by (4.3) and a little calculus. We conclude that $|\kappa_i(z,t) - 1| \le e^{-t}$ when $t \ge \log 9$. □

4.2 Proof of Theorem 1

Proof. Let $M = M^Y$ be a compact hyperbolic 3–manifold with totally geodesic boundary Y possessing a collar of depth d. Let $n = \lfloor d \rfloor - 5$.••

First assume that X is such that the $2n$–neighborhood of the convex core of $\mathrm{qf}(X,\overline{Y})$ together with the $\overline{Y}$–end of $\mathrm{qf}(X,\overline{Y})$ isometrically embeds into a convex cocompact hyperbolic 3–manifold R^X with conformal boundary $\overline{Y}$, see Figure 3. Running the entire argument above with R^X in place of $\mathrm{qf}(X,\overline{Y})$ produces a closed manifold N^X with a metric η satisfying the estimates (3.14) and (3.18). It follows from Tian's theorem that there is a $T = T(\chi(S), \mathrm{injrad}(Y)) \ge \log 9$ such that, when $n \ge T$, the metric η on N^X is within $C_3A_3e^{-n} < 1/2$ of the unique hyperbolic metric ρ on N^X in the $\mathscr{C}^2$–norm.△ In particular, an η–unit vector has ρ–length within $\sqrt{C_3A_3e^{-n}}$ of 1. The metrics are then $(1+\sqrt{C_3A_3e^{-n}})$–bilipschitz, and hence $(1+A_4e^{-n/2})$–bilipschitz for $A_4 = \sqrt{C_3A_3}$. The metrics η and ρ satisfy

$$\|\eta - \rho\|_{\mathscr{C}^2(N^X,\eta)} \le A_4 e^{-n}. \tag{4.10}$$

°°One may also compute this formula from our expression for the metric h.

•The statement in [6] that the $\mathscr{E}_t$ are convex when $t > 0$ is false when $-\coth t \le \mathfrak{f}_i \le -\tanh t$.

••We make the crude choice of 5 to be sure that $S \times [n, n+1]$ avoids the thin part of M^Y.

△Note that T does depend on $\mathrm{injrad}(Y)$, for we must be at a certain depth in the collar to ensure that the traceless Ricci curvature of our metric vanishes on the thin parts.

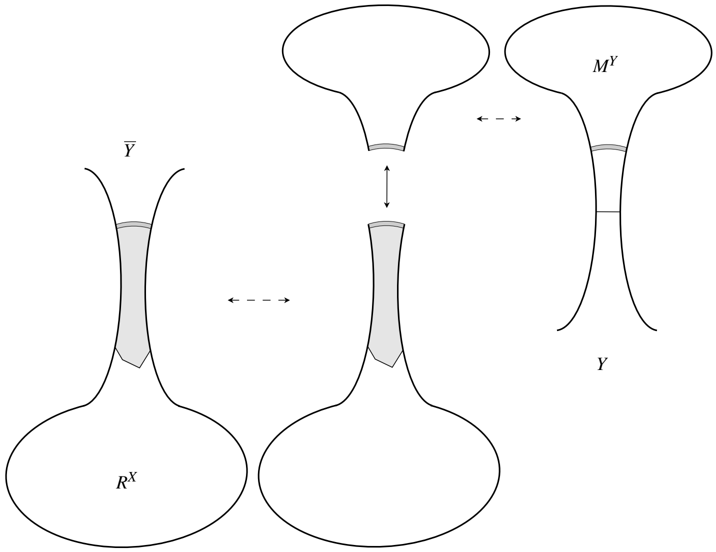

Figure 3: Building N^X with the metric η.

Again this yields a nice bilipschitz constant.

Proposition 12. *The metrics η and ρ are $(1+A_4e^{-n/2})$–bilipschitz.* $\square$

Let $N^X_\eta = (N^X, \eta)$ and $N^X_\rho = (N^X, \rho)$, and henceforth identify their tangent spaces.

The cover of N^X_ρ corresponding to ∂M is a quasifuchsian manifold $\mathrm{qf}(Z, \sigma_M(Z))$, see Figure 4.

We now show that there is a constant $A_5 = A_5(\chi(S), \mathrm{injrad}(Y)) \geq A_4$ such that the Teichmüller distance between $\sigma_M(Z)$ and $\overline{Y}$ is less than $2A_5e^{-n/2}$.

Consider the surface $\mathscr{E}^Y_{n+1}$ in N^X_η. Let $\mathscr{F}_n$ be the image of $\mathscr{E}^Y_{n+1}$ in N^X_ρ, see Figure 4. A small neighborhood of $\mathscr{F}_n$ lifts isometrically to $\mathrm{qf}(Z, \sigma_M(Z))$, and we continue to use $\mathscr{F}_n$ to denote this lift. Let $\mathbf{n}$ be the field of η–unit normal vectors to $\mathscr{E}^Y_{n+1}$ in N^X_η pointing toward $\overline{Y}$. Let $\mathbf{u}$ be an η–unit vector field tangent to $\mathscr{E}^Y_{n+1}$ on some open patch of $\mathscr{E}^Y_{n+1}$. The normal curvature of $\mathscr{E}^Y_{n+1}$ along $\mathbf{u}$ is given by

$$\mathscr{N}_{\mathscr{E}^Y_{n+1}}(\mathbf{u}) = \frac{\mathrm{II}_{\mathscr{E}^Y_{n+1}}(\mathbf{u},\mathbf{u})}{\|\mathbf{u}\|^2_\eta} = -\frac{\eta\left(\mathbf{n}, \nabla^\eta_{\mathbf{u}}\mathbf{u}\right)}{\|\mathbf{u}\|^2_\eta} = -\eta\left(\mathbf{n}, \nabla^\eta_{\mathbf{u}}\mathbf{u}\right) \tag{4.11}$$

where ∇^η is the Levi–Civita connection for η. This is easily computed to be

$$\mathscr{N}_{\mathscr{E}^Y_{n+1}}(\mathbf{u}) = \tanh(n+1), \tag{4.12}$$

which is within e^{-n} of 1.

Letting $\mathbf{m}$ be the ρ–unit normal field to $\mathscr{F}_n$ pointing toward the skinning surface $\sigma_M(Z)$, the normal curvature of $\mathscr{F}_n$ is given by

$$\mathscr{N}_{\mathscr{F}_n}(\mathbf{u}) = \frac{\mathrm{II}_{\mathscr{F}_n}(\mathbf{u},\mathbf{u})}{\|\mathbf{u}\|^2_\rho} = -\frac{\rho\left(\mathbf{m}, \nabla^\rho_{\mathbf{u}}\mathbf{u}\right)}{\|\mathbf{u}\|^2_\rho} \tag{4.13}$$

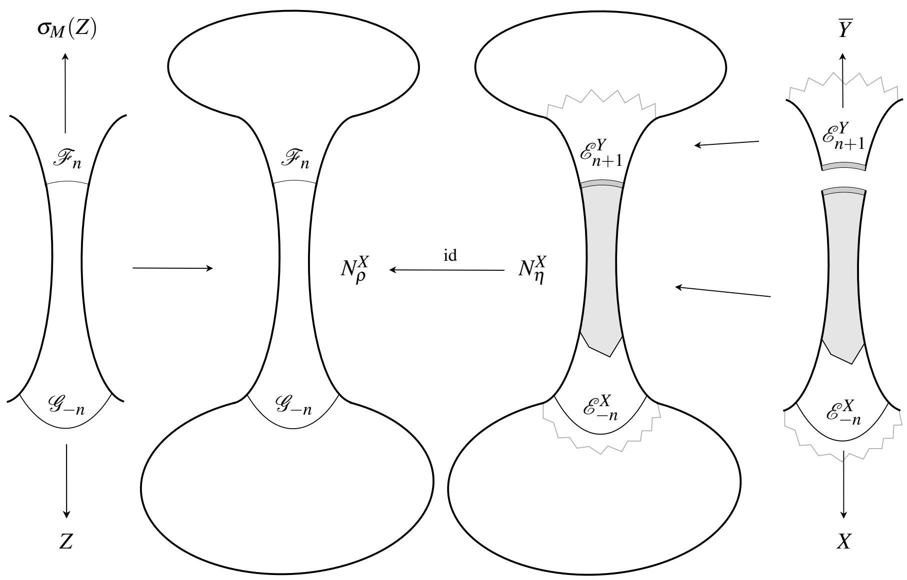

Figure 4: The manifold N^X with its two metrics ρ and η. At left is the covering map $\mathrm{qf}(Z, \sigma_M(Z)) \to N^X_\rho$, together with the projections $\mathscr{G}_{-n} \to Z$ and $\mathscr{F}_n \to \sigma_M(Z)$. At right are partial covering maps from subsets of $\mathrm{qf}(Y, \overline{Y})$ and $\mathrm{qf}(X, \overline{Y})$ to N^X_η with corresponding projections $\mathscr{E}^Y_{n+1} \to \overline{Y}$ and $\mathscr{E}^X_{-n} \to X$.

where ∇^ρ is the Levi–Civita connection for ρ. The proximity (4.10) of the metrics provides the following estimate.

Claim. *The normal curvatures satisfy*

$$\left| \mathcal{N}_{\mathscr{E}^Y_{n+1}}(\mathbf{u}) - \mathcal{N}_{\mathscr{F}_n}(\mathbf{u}) \right| = \mathscr{O}(e^{-n/2}). \tag{4.14}$$

Proof of claim. Formula (4.12) tells us that $\left\| \nabla^\eta_{\mathbf{u}} \mathbf{u} \right\|_\eta = \mathscr{O}(1)$, and so (4.10) gives us

$$\left\| \nabla^\rho_{\mathbf{u}} \mathbf{u} \right\|_\eta + \left\| \nabla^\rho_{\mathbf{u}} \mathbf{u} \right\|_\rho = \mathscr{O}(1). \tag{4.15}$$

Together with (4.10), this yields

$$\left| \eta\left(\mathbf{n}, \nabla^\eta_{\mathbf{u}} \mathbf{u}\right) - \rho\left(\mathbf{n}, \nabla^\rho_{\mathbf{u}} \mathbf{u}\right) \right| = \mathscr{O}(e^{-n}). \tag{4.16}$$

Proposition 12 gives us

$$\|\mathbf{n} - \mathbf{m}\|_\eta + \|\mathbf{n} - \mathbf{m}\|_\rho = \mathscr{O}(e^{-n/2}), \tag{4.17}$$

and then (4.15) and the Cauchy–Schwarz inequality give us

$$\left| \rho\left(\mathbf{n}, \nabla^\rho_{\mathbf{u}} \mathbf{u}\right) - \rho\left(\mathbf{m}, \nabla^\rho_{\mathbf{u}} \mathbf{u}\right) \right| \leq \|\mathbf{n} - \mathbf{m}\|_\rho \left\| \nabla^\rho_{\mathbf{u}} \mathbf{u} \right\|_\rho = \mathscr{O}(e^{-n/2}). \tag{4.18}$$

Now,

$$\left| \|\mathbf{u}\|^2_\rho - 1 \right| \leq A_4 e^{-n} \leq \frac{1}{2},$$

by (4.10), and we conclude that

$$\left| \mathcal{N}_{\mathscr{E}^Y_{n+1}}(\mathbf{u}) - \mathcal{N}_{\mathscr{F}_n}(\mathbf{u}) \right| = \left| \frac{\eta\left(\mathbf{n}, \nabla^\eta_{\mathbf{u}} \mathbf{u}\right)}{\|\mathbf{u}\|^2_\eta} - \frac{\rho\left(\mathbf{m}, \nabla^\rho_{\mathbf{u}} \mathbf{u}\right)}{\|\mathbf{u}\|^2_\rho} \right| = \mathscr{O}(e^{-n/2}) \tag{4.19}$$

by the triangle inequality. $\boxed{\text{Claim}}$

So there is an $A_5 = A_5(\chi(S), \mathrm{injrad}(Y)) \geq A_4$ such that the normal curvatures of $\mathscr{F}_n$ are within $A_5 e^{-n/2}$ of 1. It follows that there is an $m = m(\chi(S), \mathrm{injrad}(Y))$ such that, for $n \geq m$, the normal projection $\nu_n \colon \mathscr{F}_n \to \sigma_M(Z)$ is defined and nonsingular. Lemma 10 tells us that ν_n is $(1 + A_5 e^{-n/2})^2$–quasiconformal. But $\mathscr{E}^Y_{n+1}$ is conformally equivalent to $\overline{Y}$, and the map $\mathscr{E}^Y_{n+1} \to \mathscr{F}_n$ is $(1 + A_4 e^{-n/2})^2$–quasiconformal. So the Teichmüller distance between $\sigma_M(Z)$ and $\overline{Y}$ is no more than $\log(1 + A_4 e^{-n/2}) + \log(1 + A_5 e^{-n/2}) \leq 2A_5 e^{-n/2}$.

We now claim that there is an $A_6 = A_6(\chi(S), \mathrm{injrad}(Y)) \geq A_5$ such that Z is within $3A_6 e^{-n/2}$ of X. To see this, let $\mathscr{E}^X_{-n}$ be the Epstein surface in the X–end of $\mathrm{qf}(X, \overline{Y})$ at distance n. By our assumption on X, a small neighborhood of this surface embeds isometrically in R^X, and hence N^X_η. We let $\mathscr{G}_{-n}$ denote this surface considered in N^X_ρ. By (4.2), the principal curvatures of $\mathscr{E}^X_{-n}$ are within e^{-n} of 1. So the principal curvatures of $\mathscr{G}_{-n}$ are $1 + \mathscr{O}(e^{-n/2})$, as in the above argument. By Lemma 10, the projection

$\nu_{-n}\colon \mathscr{G}_{-n} \to Z$ is $(1+A_6e^{-n/2})^2$–quasiconformal for some A_6. Since proximity of the metrics tells us that $\mathscr{G}_{-n}$ and $\mathscr{E}^X_{-n}$ are $(1+A_4e^{-n/2})^2$–quasiconformal, and $\mathscr{E}^X_{-n}$ is $(1+e^{-n})^2$–quasiconformally equivalent to X, we conclude that the Teichmüller distance between Z and X is less than $3A_6e^{-n/2}$.

Since the skinning map is 1–lipschitz, the distance between $\sigma_M(X)$ and $\sigma_M(Z)$ is at most $3A_6e^{-n/2}$, and so the distance between $\sigma_M(X)$ and $\overline{Y}$ is at most $5A_6e^{-n/2}$.

Using circle packings with very small circles in the proof of Brooks's theorem [7] (as done in Theorems 31 and 33 of [12]) demonstrates that any W is within $A_6e^{-n/2}$ of an X such that the $2n$–neighborhood of the convex core of $Q(X,\overline{Y})$ together with the $\overline{Y}$–end of $Q(X,\overline{Y})$ embeds into a convex cocompact hyperbolic 3–manifold R^X with conformal boundary $\overline{Y}$. Since skinning maps are 1–lipschitz, we conclude that the diameter of σ_M is no more than $6A_6e^{-n/2} = 6A_6e^{-(\lfloor d\rfloor -5)/2} \leq 6A_6e^{-d/2+3}$. Letting $A = 162A_6$ completes the proof of Theorem 1. $\square$

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