

Notes on Étale Cohomology

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4. The Étale Topology and its Sheaves

■ 4.0 Short Comment on Sheaves, Bundles, and Representable Functors.

The notion of a sheaf over a topological space X generalize that of a bundle over X . We're more-or-less aware of this. I want to spend some time looking the specifics of this generalization.

I'll focus on schemes. Fix a ring A and let $\mathbb{A}^1 := \mathbb{A}_A^1$ be the affine line.

The first thing I want to point out is that the global section functor

$$\Gamma : \mathbf{Sch}_{/A}^{\text{op}} \rightarrow \mathbf{Sets} \quad \text{taking } X \mapsto \Gamma(X) := \Gamma(X, \mathcal{O}_X)$$

is representable. Specifically, $\Gamma(-) \cong \text{Hom}_{A\text{-Alg}}(-, \mathbb{A}^1)$, so that $\Gamma(X)$ really is the ring of " A -valued, regular functions on the A -scheme X ," in complete analogy with rings like the ring $\mathcal{H}(X)$ of " \mathbb{C} -valued, analytic functions on a \mathbb{C} -manifold X ." To see that \mathbb{A}^1 represents Γ , recall that for any A -scheme X and any A -algebra B , we have a natural isomorphism

$$\text{Hom}_{\mathbf{Sch}/A}(X, \text{Spec } B) \cong \text{Hom}_{A\text{-Alg}}(B, \Gamma(X))$$

(see for instance [AG, §II.2, Ex. 2.4]). In the case that $B := A[t]$, this becomes

$$\text{Hom}_{\mathbf{Sch}/A}(X, \mathbb{A}^1) \cong \text{Hom}_{A\text{-Alg}}(A[t], \Gamma(X)) \cong \Gamma(X).$$

We can generalize this fact in two directions. First, we can replace \mathbb{A}^1 by another scheme Y and consider the resulting functor $h_Y := \text{Hom}_{\mathbf{Sch}/A}^{\text{op}}(-, Y) : \mathbf{Sch}_{/A}^{\text{op}} \rightarrow \mathbf{Sets}$. Without any specific information about Y , it's hard to go anywhere interesting with this. Secondly though, we can think about "algebraic distributions" on X more general than maps $X \rightarrow \mathbb{A}^1$. A map $f : X \rightarrow \mathbb{A}^1$ of schemes is nothing but a graph in $X \times_A \mathbb{A}^1$, that is, a section of the trivial bundle $X \times_A \mathbb{A}^1 \rightarrow X$. Indeed, this is just the universal property of the fiber product:

$$\begin{array}{ccc} X & \leftarrow & X \times_A \mathbb{A}^1 & \rightarrow & \mathbb{A}^1 \\ & \nearrow & \uparrow & \nearrow & \\ & & X & & \end{array}$$

Thus if we consider the restriction of $\Gamma(-)$ to X 's Zariski topology, we have

$$\Gamma(-) \cong \text{Hom}_{\mathbf{Sch}/X}(-, X \times_A \mathbb{A}^1) \text{ restricted to } \underline{X}^{\text{op}} \subset \mathbf{Sch}_{/X}.$$

Hence the second way we can generalize the representability of Γ is by considering less trivial \mathbb{A}^1 -bundles over X .

Slightly less trivial than $X \times_A \mathbb{A}^1 \rightarrow X$ is any bundle $B \rightarrow X$ that looks only Zariski-locally like a trivial \mathbb{A}^1 -bundle, that is, any scheme map $B \rightarrow X$ for which X admits a Zariski-open covering $\bigcup_i U_i = X$ such that, over each U_i , the map $B \rightarrow X$ looks like $U_i \times_A \mathbb{A}^1 \rightarrow U_i$. In other words, such that for the pullback in

$$\begin{array}{ccc} U_i \times_X B & \rightarrow & B \\ \downarrow & & \downarrow \\ U_i & \hookrightarrow & X, \end{array}$$

we have an isomorphism $U_i \times_X B \cong U_i \times_A \mathbb{A}^1$.

If we combine the first and second directions in which we generalized Γ , we come to the quite general notion of a bundle or fibration $Y \rightarrow X$ with varying fiber Y_x over X (for "varying" to really make sense in the context of schemes, we probably need to specify that $Y \rightarrow X$ be flat, but this will not be important in our present considerations). The totality of local sections of $Y \rightarrow X$ organizes itself as the functor

$$\mathrm{Hom}_{\overline{X}}(-, Y) \text{ restricted to } \underline{X}^{\mathrm{op}} \subset \mathbf{Sch}/_X.$$

It's not difficult to check that $\mathrm{Hom}_{\mathbf{Sch}/_X}(-, Y)$'s restriction to $\underline{X}^{\mathrm{op}}$ is always a sheaf of sets over X .

Thus the most general object we might be interested in becomes any set-valued sheaf \mathcal{F} on X . We're to think of each set $\mathcal{F}(U)$ as being the set of local sections " $U \rightarrow Y$ " of some hypothetical bundle or fibration " $Y \rightarrow X$." The question of representability asks if there's actually a fibration $Y \rightarrow X$ playing this role for \mathcal{F} .

The message seems simple: a sheaf on a space X is a straightforward generalization of a fiber bundle.

Except it's not so simple. I think lots of people who first study algebraic geometry a la Hartshorne get confused about the analogy between sheaves and bundles. The two things that really confused me were the following:

(i) If a fiber bundle $Y \rightarrow X$ is the prototype of a sheaf, then why would algebro-geometric sheaves take values in **Ab** and **Rngs**? What's the prototype of a sheaf with values in a category **K** (choose your favorite category **K**)?

(ii) Sheaves must not be representable by bundles in general, or else we'd just talk about bundles themselves... but what then is a non-representable sheaf? How different are they from actual bundles?

I'll answer (i) in the next section, when we talk about group schemes. For now I want to discuss (ii).

When we specify any topological category like **CGHaus**, or **Diff_R**, or **Aff_A**, or **Sch_{/A}**, we're just specifying basic spaces that interact in a way we can describe, and when we then specify what "open immersions" are in it, we're specifying a rudimentary understanding of how spaces in it can glue and/or decompose along simple building blocks. Of course, every time we specify such a category, we've probably excluded a bunch of perfectly reasonable spaces that just don't quite fit our description. The fundamental idea of topos theory is that our open immersions let us define sheaves over the spaces in our category, generalizations of representable functors, and that these sheaves are in fact represented by precisely those spaces still missing from our category.

The étale topology on a scheme X , and the étale sheaves we obtain from it, are maybe the first non-trivial example of this. For a well behaved scheme X , we can actually use the language of complex manifolds to say what are the missing bundles $Y \rightarrow X$ that étale sheaves recover, or at least, we can say what the complex part $Y(\mathbb{C}) \rightarrow X(\mathbb{C})$ is: it's what's called a *Moishezon manifold*, that is, an n -dimensional, compact, connected \mathbb{C} -manifold M whose field of meromorphic functions has transcendence degree n over \mathbb{C} [**find the exact statement**]. In general, étale sheaves are not representable by schemes over X , so in some sense passing from **Sch_{/X}** to étale sheaves over X really does enlarge our category of geometric or topological spaces.

■ 4.1 Sites and Sheaves.

To keep things concrete, I'm going to work with *Grothendieck pre-topologies* rather than full-blown *Grothendieck topologies*. If we were doing topos theory, the latter would be the more beautiful to work with. But the former are more straight-forward, at least in the context of étale cohomology.

Definition 4.1.1 — Let **T** be a fixed category with pullbacks (think "category of objects that are supposed to be spaces"). A *Grothendieck pre-topology* (or just *pre-topology*) on **T** consists of a class of *covers* assigned to each object Y of **T**, where a *cover* (or *covering*) is a privileged set $\{U_i \rightarrow Y\}_{i \in I}$ of morphisms in **T** with codomain Y . To constitute a pre-topology, the system of covers has to satisfy the following three conditions:

- (i) A singleton $\{U \xrightarrow{\sim} Y\}$ consisting of an isomorphism is always a cover (**T**'s objects are "open");
- (ii) If $\{U_i \rightarrow Y\}_{i \in I}$ is a cover and $Y' \rightarrow Y$ is a morphism in **T**, then $\{Y' \times_Y U_i \rightarrow Y'\}_{i \in I}$ is a cover (**T**'s morphisms are "continuous");
- (iii) If $\{U_i \rightarrow Y\}_{i \in I}$ is a cover, and $\{U_{ij} \rightarrow U_i\}_{j \in J(i)}$ is a cover for each $i \in I$, then $\{U_{ij} \rightarrow U_i \rightarrow Y\}_{i \in I, j \in J(i)}$ is a cover (the morphisms forming a covering are "local" in nature).

We call a category \mathbf{T} equipped with a Grothendieck pre-topology a *site*.

In practice, to specify a site we start with a category \mathbf{T} that we understand well enough to isolate a class of morphisms in \mathbf{T} satisfying conditions like those we verified for étale maps in the previous §2. We then deem coverings in \mathbf{T} to be families $\{p_i : U_i \rightarrow Y\}_{i \in I}$ of morphisms from our class that are "jointly-surjective" in some suitable sense.

The two sites of primary interest in what follows will be ones where our class of covering morphisms is the class of Zariski open immersions and the class of étale maps, respectively.

Fix a scheme X once and for all, several generally important examples of sites "over X " are:

Example 1: The small Zariski site X_{zar} . — Let X_{zar} be the full subcategory of \mathbf{Sch}/X whose objects are Zariski open immersions $U \hookrightarrow X$ and whose morphisms are scheme maps $U_2 \rightarrow U_1$ over X . Put a Grothendieck pre-topology on X_{zar} by specifying that a covering of a given object $U \hookrightarrow X$ in X_{zar} is any set $\{U_i \hookrightarrow U\}_{i \in I}$ of Zariski open immersions over X for which $|U| = \bigcup_i p_i |U_i|$.

This makes X_{zar} a site, since (i) and (iii) are trivial, while (ii) is "Step 4" of [AG, §II, Theorem 3.3]. We call X_{zar} the *small Zariski site on X* . It does not differ in any practical way from the category we often denote \underline{X} , whose objects are Zariski open sub-schemes of X and whose hom-sets are either singleton inclusions $\text{Hom}_{\underline{X}}(U_2, U_1) = \{U_2 \hookrightarrow U_1\}$ or are empty, depending on whether or not $U_2 \subset U_1$ in X . Thus we think of X_{zar} as being something like a categorification of X 's underlying topology.

Example 2: The big Zariski site X_{ZAR} . — Let X_{ZAR} be either \mathbf{Sch}/X or its full subcategory \mathbf{LFT}/X of schemes locally of finite type over X . Put a Grothendieck pre-topology on X_{zar} by specifying again that a covering of any given object $Y \rightarrow X$ in X_{ZAR} is any set $\{U_i \hookrightarrow Y\}_{i \in I}$ of Zariski open immersions over X for which $|Y| = \bigcup_i p_i |U_i|$.

Then X_{ZAR} is a site for exactly the reasons X_{zar} is, and we call X_{ZAR} the *big Zariski site on X* . One should think of X_{ZAR} as being a category of topological fiber bundles over X , or rather, a category of finer bundles over X equipped with an interrelated network of topological data on each bundle.

Example 3: The small étale site $X_{\text{ét}}$. — Let $X_{\text{ét}}$ be the full subcategory of \mathbf{Sch}/X whose objects are étale maps $Y \rightarrow X$ and whose morphisms are scheme maps $Y_2 \rightarrow Y_1$ over X . Let a covering of any given object $Y \rightarrow X$ in $X_{\text{ét}}$ be any set $\{U_i \rightarrow Y\}_{i \in I}$ of étale maps such that $|Y| = \bigcup_i p_i |U_i|$.

We've seen that $X_{\text{ét}}$ contains all fiber products, and we've already seen that it satisfies axioms (i) through (iii) in the previous §2. We call $X_{\text{ét}}$ the *small étale site on X* . This will be the fundamental site of concern for us. We will sometimes refer to the maps $U \rightarrow X$ forming the objects and components of covers in this site as *étale neighborhoods*.

Example 4: The big étale site $X_{\text{ÉT}}$. — Let $X_{\text{ÉT}}$ be \mathbf{Sch}/X or \mathbf{LFT}/X as for X_{ZAR} , but equip it now with the Grothendieck pre-topology given by coverings $\{U_i \rightarrow Y\}_{i \in I}$ consisting of étale maps such that $|Y| = \bigcup_i p_i |U_i|$. Then $X_{\text{ÉT}}$ is a site for the same reasons that $X_{\text{ét}}$ is, and we call it the *big étale site on X* . We think of $X_{\text{ÉT}}$ as being a category of fiber bundles over X , all equipped with their inter-related étale topologies.

Example 5: Flat sites. — There are a couple of sites based on coverings $\{U_i \rightarrow Y\}$ by sets of flat maps, namely, coverings by:

- fpqc: maps that are faithfully flat and quasi-compact;
- fppf: maps that are faithfully flat and locally of finite presentation.

Flat is "plat" in French, hence the first "p" in each abbreviation.

These sites are important in the theory of *descent*, which is roughly a theory about what kind of data we need in order to conclude that we can glue a fiber bundle (or sheaf) together over the glued bases of a bunch of disparate fiber bundles, or (equivalently), what kind of data one needs in order to conclude that he can take a quotient bundle over the quotient of the base of some given bundle. From another perspective, descent theory is just a theory of "higher-order sheaves," where the question isn't one of gluing sections (i.e., elements) from local ones with matching restrictions, but of gluing global objects out of local objects with transitions meeting a cocycle condition.

The arguments we'll give in the next section, giving conditions under which particular functors $X_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Ab}$ are *sheaves*, will scratch at the surface of descent theory, and the reader should notice that the arguments actually only make use of faithful flatness, which suggests perhaps why fpqc and fppf sites might be the relevant sites in descent theory.

The whole reason we introduce sites in the present context of étale cohomology is in order to obtain new categories of sheaves. In particular, we want abelian categories of \mathbf{Ab} -valued sheaves over a given scheme X that differ from the category of sheaves over X 's Zariski topology. The hope is that constant sheaves G will have new injective resolutions in these new sheaf categories, resolutions that change the values of $H^*(X, G)$.

We can define sheaves over any site. Thus fix a site \mathbf{T} .

Definition 4.1.2 — We call any functor $\mathcal{F} : \mathbf{T}^{\text{op}} \rightarrow \mathbf{Sets}$ a (*set-valued*) *presheaf* over \mathbf{T} .

A (*set-valued*) *sheaf* \mathcal{F} over \mathbf{T} is a presheaf over \mathbf{T} such that every covering $\{U_i \rightarrow Y\}_{i \in I}$ of every object Y in \mathbf{T} induces a *left exact* sequence

$$\mathcal{F}(Y) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_Y U_j), \quad (\text{SA})$$

meaning that $\mathcal{F}(Y)$ is the kernel, or equalizer, of the pair of arrows on the right. The arrows on the right are the products of \mathcal{F} 's restrictions along $U_i \hookrightarrow U_i \times_Y U_j$ and $U_j \hookrightarrow U_i \times_Y U_j$, respectively. We call left exactness of the above diagram the *sheaf axiom*, or just "(SA)" for short.

We can also formulate the sheaf axiom in a more elementary, or set-theoretical way, a presheaf over \mathbf{T} is a sheaf if, whenever we have a family $(f_i)_{i \in I}$ of sections $f_i \in \mathcal{F}(U_i)$ indexed by some covering $\{U_i \rightarrow Y\}_{i \in I}$ of an object in \mathbf{T} such that

$$f_i|_{U_i \times U_j} = f_j|_{U_i \times U_j}$$

for all $i, j \in I$, there exists a unique section $f \in \mathcal{F}(Y)$ such that $f_i = f|_{U_i}$ for all $i \in I$.

Finally, it is easy to generalize the notion of set-valued sheaves over a fixed site \mathbf{T} to that of sheaves with values in any category \mathbf{K} that admits arbitrary small products and finite equalizers (this turns out to be equivalent to requiring simply that \mathbf{K} admit all small limits). In particular, in what follows we will be primarily interested in \mathbf{Ab} -valued sheaves, \mathbf{Rng} -valued sheaves, and more specifically $A\text{-}\mathbf{Alg}$ -valued sheaves over various sites, and we will be primarily interested in sheaves over the small étale site $X_{\text{ét}}$ for certain schemes X .

■ 4.2 Étale Sheaves.

Fix a scheme X .

By *étale sheaves* on X , we mean sheaves over the small étale site $X_{\text{ét}}$. We'll refer to sheaves $X^{\text{op}} \rightarrow \mathbf{K}$, for any category \mathbf{K} admitting all small limits, as *Zariski sheaves* on X . They are nothing but sheaves $X_{\text{zar}}^{\text{op}} \rightarrow \mathbf{K}$, and we're already quite familiar with them.

Since every Zariski cover $\{U_i \rightarrow U\}$ is an étale cover, it's clear that every étale sheaf on X is de facto a Zariski sheaf on X . However the converse can fail. Still, many familiar Zariski sheaves "reappear" as étale sheaves. We'll spend some time going over the examples that will be most important to us.

First through, we need to establish a descent-type criterion that simplifies the general task of checking if a given presheaf over $X_{\text{ét}}$ is a full-blown sheaf.

Recall that since étale maps are open, every étale $Y \rightarrow X$ factors as an étale surjection followed by a Zariski open immersion:

$$\begin{array}{ccc} Y & & \\ \text{ét} \downarrow & \searrow & \\ U & \xrightarrow{\text{zar}} & X \end{array}$$

Taking this as a cue, our criterion breaks the check up into a "horizontal" and "vertical" part, the horizontal part being a check that \mathcal{F} is already a Zariski sheaf, and the vertical part being a check that \mathcal{F} satisfies the sheaf condition for singleton, affine étale covers $B \rightarrow \text{Spec } A$.

Theorem 4.2.1: Étale Sheaf Criterion. — A presheaf $\mathcal{F} : X_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Sets}$ is a sheaf if and only it satisfies the following two conditions:

- (i) \mathcal{F} satisfies the sheaf axiom with respect to every Zariski cover $\{U_i \rightarrow Y\}$ of every étale $Y \rightarrow X$;
- (ii) \mathcal{F} satisfies the sheaf axiom with respect to every surjective étale cover $\text{Spec } B \rightarrow \text{Spec } A$ of an étale $\text{Spec } A \rightarrow X$,

Proof: First of all, it's trivial that every étale sheaf satisfies the two conditions, since they are just particular instances of étale covers in $X_{\text{ét}}$.

In the other direction, let $\mathcal{F} : X_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Sets}$ be a presheaf satisfying conditions (i) and (ii). We point out first that the equalizer of any trivial diagram

$$S \begin{array}{c} \xrightarrow{\varphi} \\ \rightrightarrows \\ \xleftarrow{\varphi} \end{array} S$$

is isomorphic to S . In particular, if $V = \bigsqcup_j V_j$ is a disjoint union of objects in $X_{\text{ét}}$, then

$$V_j \times_V V_k = \begin{cases} V_j & \text{if } j = k \\ \emptyset & \text{if } j \neq k, \end{cases}$$

so by condition (i), the set $\mathcal{F}(V)$ equalizes a diagram isomorphic to a trivial diagram like that above, putting

$$\mathcal{F}(\bigsqcup_j V_j) \cong \prod_j \mathcal{F}(V_j).$$

This will become important in what follows.

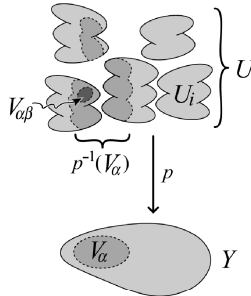
Let $Y \rightarrow X$ be an arbitrary object of $X_{\text{ét}}$, and let $\{p_i : U_i \rightarrow Y\}_{i \in I}$ be an arbitrary étale covering. Define $U := \bigsqcup_{i \in I} U_i$ and let $p := \bigsqcup_{i \in I} p_i : U \rightarrow Y$. Clearly p is étale. Let $\{V_\alpha \rightarrow Y\}_{\alpha \in \mathcal{A}}$ be a covering of Y by Zariski affine opens, and let

$$\{V_{\alpha\beta} \rightarrow p^{-1}(V_\alpha)\}_{\beta \in \mathcal{B}(\alpha)}$$

be an open cover by Zariski affine opens for each $\alpha \in \mathcal{A}$. It induces an étale cover

$$\{V_{\alpha\beta} \rightarrow V_\alpha\}_{\beta \in \mathcal{B}(\alpha)}.$$

The situation might appear as follows:



By introducing redundancies in the covering $\{V_\alpha \rightarrow Y\}_{\alpha \in \mathcal{A}}$, we can actually require the each $\mathcal{B}(\alpha)$ be finite while maintaining that each $\{V_{\alpha\beta} \rightarrow V_\alpha\}_{\beta \in \mathcal{B}(\alpha)}$ is an étale covering. Indeed, since p is étale it's open, and each V_α is affine and so quasi-compact. It is import that each $\mathcal{B}(\alpha)$ be finite because finiteness guarantees that each disjoint union

$$\bigsqcup_{\beta \in \mathcal{B}(\alpha)} V_{\alpha\beta}$$

remains affine.

All this data now fits into a commutative diagram

$$\begin{array}{ccc}
& \text{(c1)} & \text{(c2)} \\
\text{(r1)} & \mathcal{F}(Y) \xrightarrow{(\dagger)} & \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_Y U) \\
& \downarrow & \downarrow \quad \downarrow \\
\text{(r2)} & \prod_a \mathcal{F}(V_a) \longrightarrow & \prod_{\alpha,\beta} \mathcal{F}(V_{\alpha\beta}) \rightrightarrows \prod_{\alpha,\beta,\gamma} \mathcal{F}(V_{\alpha\beta} \times V_{\alpha\gamma}) \\
& \Downarrow & \Downarrow \\
& \prod_{\alpha,\beta} \mathcal{F}(V_\alpha \cap V_\beta) \xrightarrow{(\ddagger)} & \prod_{\alpha,\beta,\gamma,\delta} \mathcal{F}(V_{\alpha\gamma} \cap V_{\beta\delta})
\end{array}$$

(*)

It's commutativity follows from commutativity of related squares in $X_{\acute{e}t}$. The symbols **(r1)**, **(c1)**, **(*)**, **(†)** etc. appearing in it are for the sake of reference. We're going to argue that the diagram's top row is left-exact. Since

$$\mathcal{F}(U) \cong \prod_i \mathcal{F}(U_i) \quad \text{and} \quad \mathcal{F}(U \times_Y U) \cong \prod_{i,j} \mathcal{F}(U_i \times_Y U_j)$$

by the argument of the previous paragraph, this left-exactness is equivalent to the sheaf condition for the étale covering $\{U_i \rightarrow Y\}$. We argue as follows:

Condition **(i)** implies that the columns **(c1)** and **(c2)** are left-exact. Condition **(i)** implies that the row **(r2)** is isomorphic to the α -indexed product of the sequences

$$\mathcal{F}(V_\alpha) \rightarrow \mathcal{F}(\bigsqcup_\beta V_{\alpha\beta}) \rightrightarrows \mathcal{F}(\bigsqcup_{\beta,\gamma} V_{\alpha\beta} \times V_{\alpha\gamma}),$$

each of which is left-exact since $\bigsqcup_\beta V_{\alpha\beta} \rightarrow V_\alpha$ is an étale cover of the form specified by condition **(ii)** (here we use the fact that $\bigsqcup_\beta V_{\alpha\beta}$ is itself affine). Thus again by the argument we gave above, **(r2)** is exact.

In particular, exactness of **(c1)** and **(r2)** implies that the left-vertical and bottom-horizontal arrows of the square **(*)** that the bold arrows in the top left square are injective. Thus **(†)** must be injective as well.

This argument that **(†)** is injective is general enough that it applies to any étale covering. In particular, it applies also to the étale covering $\{V_{\alpha\gamma} \cap V_{\beta\delta} \rightarrow \bigsqcup_{\alpha,\beta} V_\alpha \cap V_\beta\}$. Thus since

$$\bigsqcup_{\alpha,\beta} V_\alpha \cap V_\beta = \bigsqcup_{\alpha,\beta,\gamma,\delta} V_{\alpha\gamma} \cap V_{\beta\delta}$$

the arrow **(‡)** is injective for the same reason that **(†)** is.

Finally, a simple diagram chase using exactness of **(r2)**, **(c1)**, and **(c2)** along with injectivity of **(‡)** shows the top row **(r1)**, to be left exact. ■

Notice that in the above proof, the only way we used the assumption that our covers are étale was in making sure that each $\{V_{\alpha\beta} \rightarrow V_\alpha\}$ is finite. It's clear that the same thing works if $p: U \rightarrow Y$ is merely quasi-compact, and thus the above criterion extends to X_{fpqc} , that is, to the small site over X constructed from faithfully flat, quasi-compact maps.

With the above criterion for $X_{\acute{e}t}$, we can now give the examples of étale sheaves that will be most important to us.

Example 1: The Structure Sheaf \mathcal{O} . — Every k -scheme X comes equipped with its structure sheaf $\mathcal{O}_X: \underline{X}^{\text{op}} \rightarrow \mathbf{Rngs}$, which serves as the algebro-geometric analog of the sheaf of holomorphic functions on a \mathbb{C} -manifold. In the previous §4.0, we pointed out that

$$\Gamma(-, \mathcal{O}_X) \cong \text{Hom}_{\mathbf{Sch}/X}(-, X \times_A A^1)$$

restricted to $\underline{X}^{\text{op}} \subset \mathbf{Sch}_X^{\text{op}}$. Forgetting the restriction, this gives us a perfectly well defined presheaf $\mathcal{O}: X_{\acute{e}t}^{\text{op}} \rightarrow \mathbf{Rngs}$ which is simply the restriction of $\text{Hom}_{\mathbf{Sch}/X}(-, X \times_A A^1)$ to $X_{\acute{e}t}^{\text{op}}$ rather than $\underline{X}^{\text{op}}$. We call \mathcal{O} the *étale structure sheaf*, though we need to verify that it's actually an étale sheaf.

We check that \mathcal{O} meets conditions **(i)** and **(ii)** of Theorem 4.2.1 above. Condition **(i)** is just the statement that each structure sheaf \mathcal{O}_Y really is a sheaf with respect to the Zariski topology. Condition **(ii)** becomes the statement that

$$\begin{array}{c}
0 \rightarrow A \rightarrow B \rightarrow B \otimes_A B \\
b \mapsto b \otimes 1 - 1 \otimes b
\end{array}$$

is exact whenever $\text{Spec } B \rightarrow \text{Spec } A$ is surjective and étale. We claim that if $\text{Spec } B \rightarrow \text{Spec } A$ is surjective and étale, then $A \rightarrow B$ is a faithfully flat, and that this implies that the above sequence is exact. This is the content of Corollary 4.2.3 below.

Recall that a flat algebra $A \rightarrow B$ is *faithfully flat* when exactness of

$$0 \rightarrow B \otimes_A M' \rightarrow B \otimes_A M \rightarrow B \otimes_A M'' \rightarrow 0$$

implies exactness of $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, or equivalently, when injectivity $B \otimes_A M' \hookrightarrow B \otimes_A M$ implies injectivity $M' \hookrightarrow M$.

Lemma 4.2.2 — If $\text{Spec } B \rightarrow \text{Spec } A$ is flat and surjective, then $A \rightarrow B$ is faithfully flat and injective.

Proof: We already proved injectivity in Lemma **ÉtOp-5** of the previous §2. Faithful flatness of $A \rightarrow B$ comes now from the fact that, given a sequence

$$M' \xrightarrow{\psi'} M \xrightarrow{\psi''} M''$$

of A -modules such that

$$B \otimes_A M' \xrightarrow{1 \otimes_A \psi'} B \otimes_A M \xrightarrow{1 \otimes_A \psi''} B \otimes_A M''$$

is exact, flatness of $A \rightarrow B$ gives us

$$B \otimes_A \text{im } \psi'' \psi' = \text{im } (1 \otimes_A \psi'') (1 \otimes_A \psi') = 0$$

and

$$B \otimes_A (\ker \psi'' / \text{im } \psi') = \ker (1 \otimes_A \psi'') / \text{im } (1 \otimes_A \psi') = 0.$$

By observations we made in the proof of **ÉtOp-5**, this implies that $\psi'' \psi' = 0$ and that $\ker \psi'' = \text{im } \psi'$. ■

Corollary 4.2.3 — If $\text{Spec } B \rightarrow \text{Spec } A$ is surjective and étale, then the sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & A & \rightarrow & B & \rightarrow & B \otimes_A B \\ & & & & & & b \mapsto b \otimes 1 - 1 \otimes b \end{array}$$

is exact.

Proof: Suppose $\text{Spec } B \rightarrow \text{Spec } A$ étale and surjective. Then in particular $B \leftarrow A$ is flat, and so by the last Lemma 4.2.2, $B \leftarrow A$ is faithfully flat and injective. It remains to show that $A \supset \ker(B \rightarrow B \otimes_A B)$ (is obvious that $A \subset \ker(B \rightarrow B \otimes_A B)$).

Since $A \rightarrow B$ is faithfully flat, exactness of $A \rightarrow B \rightarrow B \otimes_A B$ is equivalent to exactness of

$$\begin{array}{ccccccc} f & \mapsto & f \otimes 1 \\ B & \xrightarrow{\iota} & B \otimes_A B & \rightarrow & B \otimes_A B \otimes_A B \\ f \otimes b & \mapsto & f \otimes b \otimes 1 - f \otimes 1 \otimes b \end{array}$$

Consider the morphism $B \leftarrow B \otimes_A B : \mu$ taking $fb \leftarrow f \otimes b$. It's clearly a section of $\iota : B \rightarrow B \otimes_A B$. Define

$$\begin{array}{ccc} B \otimes_A B & \xleftarrow{\psi} & B \otimes_A B \otimes_A B \\ \mu(f \otimes b_1) \cdot b_2 & \leftarrow & f \otimes b_1 \otimes b_2 \end{array}$$

Then $\psi(f \otimes b \otimes 1 - f \otimes 1 \otimes b) = fb \otimes 1 - f \otimes b$. Thus if $f \otimes b \otimes 1 - f \otimes 1 \otimes b = 0$, we have

$$f \otimes b = fb \otimes 1,$$

and therefore $f \otimes b \in \text{im}(B \rightarrow B \otimes_A B)$. This completes the proof of exactness. ■

Let us add at this point that on the face of it, the étale structure sheaf contains nothing new, since it remains the sheaf represented by the trivial bundle $X \times \mathbb{A}^1 \rightarrow X$, just as it is over X 's Zariski topology. But we will discover below that \mathcal{O} 's stalks do not coincide with \mathcal{O}_X 's stalks. That is, étale stalks and Zariski stalks are different rings in general. This will ultimately dictate that certain inexact sequences of Zariski sheaves become exact as sequences of étale sheaves, and we will use just such a sequence to calculate $H^1(X_{\text{ét}}, \mathbb{Z}/n)$ for curves X .

The next important examples are **Ab**-valued sheaves represented by abelian group schemes. I promised above that I'd explain what the prototype of a "sheaf with values in the category **C**" is, relative to the idea that a bundle $Y \rightarrow X$ is a prototype for a set-valued sheaf.

The definition of a group, as a set-with-added-structure, is actually quite diagrammatic in nature, making a "group" something that's well defined in many categories beside **Sets**. In any category **T** containing finite products (including a terminal object $\mathbb{1}$), we can define a *group object* to be an object G in **T** equipped with morphisms

$$\mathbb{1} \xrightarrow{\eta} G, \quad G \xrightarrow{\iota} G, \quad \text{and} \quad G \times G \xrightarrow{\mu} G$$

i.e., morphisms $G^{\times 0}, G^{\times 1}, G^{\times 2} \xrightarrow{\eta, \iota, \mu} G$, respectively, such that the following diagrams commute:

$$\begin{array}{ccc} G \cong \mathbb{1} \times G & \xrightarrow{\eta \times 1} & G \times G & \xleftarrow{1 \times \eta} & G \times \mathbb{1} \cong G \\ & \searrow & \downarrow \mu & \swarrow & \\ & & G & & \end{array}$$

(unit axiom)

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{1 \times \mu} & G \times G \\ \mu \times 1 \downarrow & & \downarrow \mu \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

(associativity axiom)

$$\begin{array}{ccccc} G & \rightarrow & \mathbb{1} & \xrightarrow{\eta} & G & \xleftarrow{\eta} & \mathbb{1} & \leftarrow & G \\ \Delta \downarrow & & & & \uparrow \mu & & & & \downarrow \Delta \\ G \times G & \xrightarrow{\iota \times 1} & G \times G & \xleftarrow{1 \times \iota} & G \times G & & & & \end{array}$$

(inversion axiom)

We can also define abelian group objects by letting $G \times G \xrightarrow{\tau} G \times G$ be the canonical "transposition" of the product's factors (it's induced using mismatched projections and $G \times G$'s universal property), and introducing a fourth axiom given by commutativity of the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\tau} & G \times G \\ \mu \searrow & & \swarrow \mu \\ & G & \end{array}$$

(commutativity axiom).

The framework is very general here: we can imagine all kinds of objects in **Sets** defined by commutative diagrams of morphisms between finite limits, and the possibility of translating those definitions diagrammatically to any other category **T** containing the appropriate limits. The universal property of the limit takes the form of a natural isomorphism

$$\text{Hom}_{\mathbf{T}}(-, \varinjlim^{\mathbf{T}} G_i) \cong \varinjlim^{\mathbf{Sets}} \text{Hom}_{\mathbf{T}}(-, G_i).$$

Thus employing the Yoneda embedding, we see that a representable **Ab**-valued sheaf \mathcal{F} is nothing but a set valued sheaf \mathcal{F} represented not merely by an object B in **T**, but by an abelian group object G in **T**. In this way, the prototypical **Ab**-valued sheaf over a space X is a bundle $G \rightarrow X$ for which G is a topological abelian group over X , meaning a topological abelian group G whose group structure is compatible with its fibration over X .

We give some examples that will be important in étale cohomology.

Example 3: The Additive Group Sheaf \mathbb{G}_a . — This isn't really a new example: we simply point out that \mathbb{A}_k^1 comes with a standard group scheme structure, with structure maps

$$\begin{array}{ccc} \text{Spec } k \xrightarrow{\eta} \mathbb{A}_k^1 & \mathbb{A}_k^1 \xrightarrow{\iota} \mathbb{A}_k^1 & \mathbb{A}_k^1 \times \mathbb{A}_k^1 \xrightarrow{\mu} \mathbb{A}_k^1 \\ \text{dual to} & \text{dual to} & \text{dual to} \\ k \leftarrow k[t] & k[t] \leftarrow k[t] & k[u, v] \leftarrow k[t] \\ 0 \leftarrow t & -t \leftarrow t & u + v \leftarrow t \end{array} \quad , \quad \text{and}$$

It's easy to check that the abelian group scheme diagrams commute for these maps. We call it the *additive group structure* on \mathbb{A}_k^1 , denoting \mathbb{A}_k^1 as \mathbb{G}_a when we consider it with this group structure.

Since we've already seen that \mathbb{A}_k^1 represents the étale structure sheaf \mathcal{O} , this means that \mathcal{O} has an **Ab**-valued structure induced by this group structure. Actually, it's not hard to see that this **Ab**-valued structure is nothing but the **Ab**-valued structure induced by the forgetful functor $\mathbf{Rngs} \rightarrow \mathbf{Ab}$. One can go even further and define ring-objects in \mathbf{Sch}/X to see that the abelian group structure on \mathbb{A}_k^1 extends to a ring structure. It is this structure that makes \mathcal{O} a full blown sheaf of rings on $X_{\text{ét}}$.

Most often we won't really distinguish between the étale structure sheaf \mathcal{O} the sheaf represented by \mathbb{G}_a . \square

For further examples, we're going to need another étale sheaf criterion, namely that functors represented by schemes are always étale sheaves.

Theorem 4.2.4 — Let $Y \rightarrow X$ be an arbitrary map of schemes. The presheaf $Y(-) := \text{Hom}_{\mathbf{Sch}/X}(-, Y)$ restricted to $X_{\text{ét}}^{\text{op}} \subset \mathbf{Sch}/X^{\text{op}}$ is always a sheaf.

Proof: We'll make use of the criterion of Theorem 4.2.1 above, but the check is not as straight-forward as one might expect. We have to divided into two cases, the first specific and the second more general:

Case 1: Y Affine. Suppose $Y = \text{Spec } C$. In this special case, we can simply check the two conditions of Theorem 4.2.1:

1.i The Zariski condition is classical.

1.ii Looking carefully at the proof of Theorem 4.2.1 above, one realizes that all we really need to establish in condition (ii) is that (SA) holds for all $\text{Spec } B_i \rightarrow \text{Spec } A_i$ in some affine decomposition of each étale covering in $X_{\text{ét}}$. By starting with an affine covering of X , we can guarantee that each such $\text{Spec } B_i \rightarrow \text{Spec } A_i$ exists over an open affine S in X , and thus that $\text{Hom}_{\mathbf{Sch}/X}(\text{Spec } A_i, Y) \cong \text{Hom}_{\mathbf{S-Alg}}(C, A_i)$, and similarly for B_i and $B_i \otimes_A B_i$

Let $\text{Spec } B \rightarrow \text{Spec } A$ be any such étale cover. Then Corollary 4.2.3 tells us that the sequence $A \rightarrow B \rightarrow B \otimes_A B$ is exact. Hence by left-exactness of $\text{Hom}_{\mathbf{S-Alg}}(C, -)$, we see that the sequence

$$0 \rightarrow \text{Hom}_{\mathbf{S-Alg}}(C, A) \rightarrow \text{Hom}_{\mathbf{S-Alg}}(C, B) \rightarrow \text{Hom}_{\mathbf{S-Alg}}(C, B \otimes_A B)$$

is exact, and thus so is $\text{Hom}_{\mathbf{Sch}/X}(\text{Spec } A, Y) \rightarrow \text{Hom}_{\mathbf{Sch}/X}(\text{Spec } B, Y) \rightrightarrows \text{Hom}_{\mathbf{Sch}/X}(\text{Spec } B \otimes_A B, Y)$.

Case 2: Arbitrary Y . This case is a bit trickier than the last. Again we break it up into two sub-cases, but here they do not coincide with the conditions of Theorem 4.2.1. Rather, letting $\{U_i \rightarrow V\}$ be an arbitrary étale cover in $X_{\text{ét}}$, and defining $U := \bigsqcup_i U_i$ with covering map $f : U \rightarrow V$, the sub-cases are the two checks required to verify that the sequence

$$\text{Hom}_{\mathbf{Sch}/X}(V, Y) \xrightarrow{f^*} \text{Hom}_{\mathbf{Sch}/X}(U, Y) \xrightleftharpoons[p_2^*]{p_1^*} \text{Hom}_{\mathbf{Sch}/X}(U \times_V U, Y) \quad (\text{SA})_Y$$

is left-exact.

2.a Injectivity of f^* . Suppose given a diagram of the form

$$\begin{array}{ccc} V & \xleftarrow{f} & U \\ & \searrow^{g_1} & \downarrow^{g_2} \\ & & Y \end{array} \quad \text{with } g_1 f = g_2$$

Then in particular $|g_1 f| = |g_2 f|$ on the underlying topologies. Since $|f|$ is surjective, this implies that $|g_1| = |g_2|$ as continuous maps. Thus letting $\{Y_j\}$ be an affine covering of Y , we obtain corresponding coverings $\{V_j\}$ and $\{U_j\}$ of V and U , respectively, via

$$V_j := g_1^{-1}(Y_j) = g_2^{-1}(Y_j) \quad \text{and} \quad U_j := f^{-1}(V_j).$$

Since Y_j is affine, this reduces us to **Case 1** over each Y_j , and we see at least that $g_1|_{V_j} = g_2|_{V_j}$ for each j . But then by the classical case, that is, the case of the Zariski topology, this implies that $g_1 = g_2$.

2.b Exactness at $\text{Hom}_{\mathbf{Sch}/X}(U, Y)$. The fact that our functions f^* , p_1^* , and p_2^* come from the commutative diagram

Example 5: The Sheaf μ_n of n^{th} -roots of Unity. — Let $\mu_n := \text{Spec } k[\zeta] := \text{Spec } k[t]/(t^n - 1)$, and give it group scheme structure maps

$$\begin{array}{ccc} \text{Spec } k \xrightarrow{\eta} \mu_n & \mu_n \xrightarrow{\iota} \mu_n & \mu_n \times \mu_n \xrightarrow{\mu} \mu_n \\ \text{dual to} & \text{dual to} & \text{dual to} \\ k \leftarrow k[\zeta] & k[\zeta] \leftarrow k[\zeta] & k[\xi, \eta] \leftarrow k[\zeta] \\ 1 \leftarrow \zeta & \zeta^{-1} \leftarrow \zeta & \xi\eta \leftarrow \zeta \end{array}, \quad \text{and}$$

It's easy to see that these are well defined and to check that they make μ_n an abelian group scheme, which we call the *group scheme of n^{th} -roots of unity*.

We call the sheaf $\mu_n : X_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Ab}$ that it represents the *sheaf of n^{th} -roots of unity*. Observe that given any object $U \rightarrow X$ in $X_{\text{ét}}$, we have

$$\begin{aligned} \mu_n(U) &:= \text{Hom}_{\text{Sch}/X}(U, X \times \mu_n) \cong \\ &\cong \text{Hom}_{\text{Sch}}(U, \mu_n) \cong \\ &\cong \text{Hom}_{\mathbf{Rng}}(k[\zeta], \Gamma(U, \mathcal{O}_U)) \cong \\ &\cong \{s \in \mathcal{O}(U) : s^n = 1\} \end{aligned}$$

which justifies the name.

An important example of this sheaf will be $\mu_n : X_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Ab}$ on any nonsingular, integral curve X over an algebraically closed field k . In this case, each irreducible component $U_i \rightarrow X$ of any étale neighborhood is itself a nonsingular, integral k -curve, and thus $\Gamma(U_i, \mathcal{O}_{U_i})$ is a unique factorization, Dedekind domain over k . In particular, the set of n^{th} -roots of unity in $\Gamma(U_i, \mathcal{O}_{U_i})$ is none other than the set $\mu_n(k) = \{1, \zeta_1, \dots, \zeta_{n-1}\}$ of n^{th} roots of unity in k itself, that is,

$$\mu_n(U) = \mu_n(k),$$

with all restriction maps along irreducible étale neighborhoods simply the identity $1 : \mu_n(k) \rightarrow \mu_n(k)$. One can check that, in the language of the following example, this implies that over a nonsingular, integral k -curve X , where k is closed, μ_n is nothing but the *constant sheaf \mathbb{Z}/n* . This observation will let us calculate the étale cohomology of a curve with constant \mathbb{Z}/n -coefficients as

$$H^*(X, \mathbb{Z}/n) \cong H^*(X, \mu_n). \quad \square$$

Example 6: Constant Sheaves. — Let F be a finite set. The constant F -valued sheaf $F : X_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Sets}$ is the sheaf represented by the bundle $X \times F \rightarrow X$, where

$$X \times F := \bigsqcup_{f \in F} X$$

and $X \times F \rightarrow X$ takes $(x, f) \mapsto x$. Since $X \times F \rightarrow X$ is a well defined X -scheme, it does indeed represent an étale sheaf. In fact, it's clear that $X \times F \rightarrow X$ is étale.

We call $F : X_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Sets}$ the *constant sheaf F* , and we call the set F its *constant fiber*.

An important special case of this occurs when our constant fiber is a group G . In this case, G 's group structure induces a group structure on the X -scheme $X \times G$ via G 's structure maps applied to the indices in the direct sum

$$X \times G = \bigsqcup_{g \in G} X.$$

It's clear that this does indeed make $X \times G$ into a group scheme over X , and thus $X \times G \rightarrow X$ represents a group-valued étale sheaf $G : X_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Grp}$. When G is abelian, its constant sheaf becomes the abelian-valued $G : X_{\text{ét}}^{\text{op}} \rightarrow \mathbf{Ab}$. \square

The reason the sheaves \mathbb{G}_m and μ_n are so important is because we're eventually going to calculate the étale cohomology $H^*(X, \mathbb{Z}/n)$ of a nonsingular, complete curve X (assuming $\text{char } k \nmid n$) using an exact sequence

$$0 \rightarrow \mu_n \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0$$

called the *Kummer sequence*.

Finally, let me set down some notation for what follows. We will let $\mathbf{PSh}(E_{\acute{e}t})$ denote the category of abelian-valued presheaves over $X_{\acute{e}t}$, and we will let $\mathbf{Sh}(X_{\acute{e}t})$ denote the category of abelian-valued étale sheaves on X . Set-valued étale sheaves will also be important to us, but \mathbf{Ab} -valued étale presheaves and sheaves will be more important to us as a category, so we reserve the succinct notations " $\mathbf{PSh}(X_{\acute{e}t})$ " and " $\mathbf{Sh}(X_{\acute{e}t})$ " for them, respectively.