

**Notes on Étale Cohomology**  
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**3. The Étale Fundamental Group  $\pi_1(X, \bar{x})$  of a Scheme  $X$ .**

■ **3.1 Introduction: One Theorem, Two Subjects.**

In both field theory and homotopy theory, a classification theorem appears establishing a bijection between, on the one hand, subgroups  $H$  of an automorphism group  $G := \text{Aut}_X(Y)$  of a particular object  $Y/X$  and, on the other, a specific class of objects "intermediate" to  $Y$  and  $X$ .

In field theory, the theorem classifies finite separable extensions of a fixed field  $k$ . In homotopy theory, it classifies finite, connected covering spaces of a fixed *non-pathological* topological space  $X$  (note: in what follows *non-pathological* := connected, locally path-connected and semi-locally simply connected). If we word these two classical theorems with some care, we really can get them to become formally identical.

For fields, we have:

**Theorem (Finite, Field-Theoretical Version).** — Let  $K/k$  be a finite separable field extension that's normal (i.e., a *Galois* extension). Then we have a one-to-one, order-reversing correspondence between the poset of intermediate fields  $K \supset E \supset k$  and the poset of subgroups  $H \subset \text{Aut}_k(K)$ , given by

$$E \mapsto \text{Aut}_E(K) \quad \text{and} \quad K^H \leftrightarrow H,$$

under which normal intermediate extensions correspond to normal subgroups  $H \triangleleft \text{Aut}_k(K)$ . ■

To state a formally identical theorem that classifies covering spaces, we have to take a bit more care, due to the fact that although sub-objects are well-defined, the dual notion of quotient objects is only well defined up to isomorphism, i.e., the first is set-theoretical, whereas the latter is category theoretical.

For this reason, given a covering space  $Y \rightarrow X$  and a subgroup  $H \subset \text{Aut}_X(Y)$ , we define  $H \backslash Y$  to be the quotient whose points are  $H$ -orbits in  $Y$ , with the quotient topology. In this way, we make a choice, once and for all, of a canonical quotient associated to each distinct subgroup  $H \subset \text{Aut}_X(Y)$ . We denote the poset of such covers  $\mathbf{Cov}(Y/X)$ , with  $H_1 \backslash Y \subseteq H_2 \backslash Y$  if and only if there's a strict surjection  $H_2 \backslash Y \twoheadrightarrow H_1 \backslash Y$ , i.e., one that's not an isomorphism. We call it the set of *covers intermediate to  $Y \rightarrow X$* .

Then for covering spaces, we have:

**Theorem (Finite, Homotopy-Theoretical Version).** — Let  $Y \rightarrow X$  be a finite, connected covering space that's normal (i.e., a "Galois" cover). Then we have a one-to-one, order-reversing correspondence between the poset  $\mathbf{Cov}(Y/X)$  of intermediate covers  $Y \rightarrow Z \rightarrow X$  and the poset of subgroups  $H \subset \text{Aut}_X(Y)$ , given by

$$Z \mapsto \text{Aut}_Z(Y) \quad \text{and} \quad H \backslash Y \leftrightarrow H,$$

under which normal intermediate covers correspond to normal subgroups  $H \triangleleft \text{Aut}_X(Y)$ . ■

Of course, the reason we call the theorem "homotopy-theoretical" is because in this case  $X$  admits a universal cover  $\tilde{X} \rightarrow X$ , for which  $\text{Aut}_X(\tilde{X}) \cong \pi_1(X, x)$ .

The appearance of formally identical theorems in these distinct settings sets up a potent analogy. For instance, we observe that the universal covering space in homotopy theory plays the same role as the separable closure  $k^s/k$ , and we can make this even more precise by reformulating the finite theorems with "infinite" versions:

**Theorem (Infinite, Field-Theoretical Version).** — Let  $k^s/k$  be  $k$ 's separable closure. Then the above correspondence sets up an isomorphism between the poset of intermediate fields  $k^s \supset E \supset k$  finite over  $k$  and the poset of subgroups  $H \subset \text{Aut}_k(k^s)$  for which  $\text{Aut}_k(k^s)/H$  is finite. Normal extensions correspond to normal subgroups under this bijection. ■

**Theorem (Infinite, Homotopy-Theoretical Version).** — Let  $\tilde{X} \rightarrow X$  be  $X$ 's universal cover. Then the above correspondence sets up an isomorphism between the sub-poset of  $\mathbf{Cov}(Y/X)$  consisting intermediate covers  $\tilde{X} \rightarrow Z \rightarrow X$  finite over  $X$  and the poset of subgroups  $H \subset \text{Aut}_X(\tilde{X})$  for which  $\text{Aut}_X(\tilde{X})/H$  is finite. Normal extensions correspond to normal subgroups under this bijection. ■

Anyone familiar with the basics of étale maps should recognize that in the algebro-geometric setting, a finite separable field extension  $K/k$  and a "finite connected covering space"  $Y \rightarrow X$  are two instances of the same thing: a surjective étale map. Since the spectrum of a field is trivially connected, they're actually both instances of connected étale covers. Hence the following "conjectural" table makes sense:

field-theoretical :	$K/k$ a...	...finite, separable field extension...	...that's normal...
homotopy-theoretical :	$Y \rightarrow X$ a...	...finite, connected covering space...	...that's normal...
scheme-theoretical :	$Y \rightarrow X$ a...	...finite connected étale cover...	...with some kind of transitive $\text{Aut}_{\text{Sch}/X}(Y)$ -action...

It suggests that the analogy between the field-theoretical and the homotopy-theoretical pictures is more than just formal, and indicates clear conjectures that would "unify" the two pictures. Letting  $X$  be a connected scheme, the finite conjecture becomes:

**Conjecture (Finite, Scheme-Theoretical Version).** — There should be notion of *étale cover* and *Galois étale cover* such that given any finite Galois étale cover  $Y \rightarrow X$  of connected schemes, we have a one-to-one, order-reversing correspondence between some poset of intermediate étale covers  $Y \rightarrow U \rightarrow X$  and the poset of subgroups  $H \subset \text{Aut}_{\text{Sch}/X}(Y)$ , given by

$$U \mapsto \text{Aut}_{\text{Sch}/U}(Y) \quad \text{and} \quad H \backslash Y \leftarrow H,$$

under which Galois étale covers correspond to normal subgroups  $H \triangleleft \text{Aut}_{\text{Sch}/X}(Y)$ . □

The infinite conjecture, for a fixed scheme  $X$ , becomes:

**Conjecture (Infinite, Scheme-Theoretical Version).** — There should be a universal étale cover  $\tilde{X} \rightarrow X$  giving a one-to-one correspondence between the set of (isomorphism classes of) finite intermediate étale covers  $\tilde{X} \rightarrow U \rightarrow X$  and the set of subgroups  $H \subset \text{Aut}_{\text{Sch}/X}(\tilde{X})$  for which  $\text{Aut}_{\text{Sch}/X}(\tilde{X})/H$  is finite. [?]

But this last conjecture breaks down at our assertion that a universal étale cover  $\tilde{X} \rightarrow X$  ought to exist. For instance, from a complex-analytic perspective the universal cover of the punctured complex-affine line  $\mathbb{A}_{\mathbb{C}}^1 \setminus \{0\}$  is  $\mathbb{A}_{\mathbb{C}}^1$ , with covering map

$$\exp : \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1 \setminus \{0\},$$

the exponential function. Yet we don't know of any scheme theoretic analog of the exponential function, and the nature of the Zariski topology on  $\mathbb{A}_{\mathbb{C}}^1$  suggests that we won't even find one in  $\mathbf{Sch}_{\mathbb{C}}$  if we try.

We can respond to this difficulty by taking two seemingly disinct perspectives:

(1) If we're familiar with pro- and ind-schemes, then we recognize in a heuristic sense that they allow for analytic considerations in  $\mathbf{Sch}_{\mathbb{Z}}$ , though in a formal analytic sense. Thinking of the universal cover of a topological space  $X$  is something like " $X$  repeated infinitely in every path direction," we suspect that the analyticity appears simply as a limit of larger and larger finite pictures. We hypothesize that given a scheme  $X$ , its "universal cover"  $\tilde{X} \rightarrow X$  will appear as the projective limit of  $X$ 's finite étale covers, that is, that the limit

$$\tilde{X} := \varprojlim \text{Hom}_{\text{Sch}/X}(U_i, -)$$

over finite étale covers  $U_i \rightarrow X$ , considered as a "generalized scheme." It clearly has the right universal property for finite étale covers at least. If this is indeed a good candidate for the scheme-theoretical universal covering space, then  $\text{Aut}_X(\tilde{X})$  should verify our infinite conjecture above.

(2) On the other hand, looking more closely at the infinite version of our conjecture, we see that the set of (isomorphism classes of) finite étale covers  $U \rightarrow X$  doesn't really depend on the presence of  $\tilde{X}$ . The only role that this hypothetical universal cover actually plays is in the definition/construction of  $\text{Aut}_{\text{Sch}/X}(\tilde{X})$  (an analog of  $\pi_1(X)$ ). Hence if we could come up with a reasonable alternative definition of  $\text{Aut}_{\text{Sch}/X}(\tilde{X})$  that makes no use of  $\tilde{X}$ , we might save the conjecture after all.

### ■ 3.2 Finite Étale Maps as "Covering Spaces."

The reader can regard this section as a first exposition on the formal similarity between finite étale maps and covering spaces.

Recall that a *separated* map  $f : Y \rightarrow X$  of schemes is one whose diagonal  $\Delta : Y \rightarrow Y \times_X Y$  is a closed immersion, and that a *proper* map  $f : Y \rightarrow X$  of schemes is one that's separated and *universally closed*. Here universally closed means that every pullback

$$\begin{array}{ccc} Z \times_X Y & \rightarrow & Y \\ \downarrow & & \downarrow f \\ Z & \rightarrow & X \end{array}$$

of  $f$  is a closed map. Since the property of a map being closed is a local property with respect to the codomain, it is enough to verify universal closedness for arbitrary maps of affine schemes  $Z = \text{Spec } A$  into  $X$ .

Now as we pointed out when we first recalled the basic finiteness properties we'd be making use of in these notes, one of the key differences between finite and locally finite type maps, apart from the heuristic idea that finite type maps have non-zero, finite dimensional fibers, is the fact that Zariski open immersions  $U \hookrightarrow X$  tend to be locally of finite type rather than finite, due roughly to the presence of an infinite descending sequence

$$\cdots \supset f^{-2}A \supset f^{-1}A \supset A \supset fA \supset f^2A \supset \cdots$$

of  $A$ -submodules of  $B \leftarrow A$  in the affine picture,  $f \in B^\times$  (think of the example  $k[t^{\pm 1}] \leftarrow k[t]$ , with  $f := t$ ).

Thus even though a Zariski open immersion  $U \hookrightarrow X$  will always have zero-dimensional fibers, every fiber being either a singleton or the empty scheme, it will in general fail to be finite when it is not proper. The precise statement, which is an exercise in Hartshorne [AG, §II.4, Ex. 4.1, p. 105], is:

**Claim.** — If  $f : Y \rightarrow X$  is finite, then it's proper.

**Proof:** If  $f$  is finite, then every affine pullback  $\text{Spec } A \times_X Y \rightarrow \text{Spec } A$  of  $f$  is a finite map. In particular,  $\text{Spec } A \times_X Y$  is the spectrum of an algebra  $B \leftarrow A$  that's finitely generated as an  $A$ -module. In particular,  $B$  is integrally dependent on  $A$  **[find the fact in A&M]**. But then by Cohen-Sidenberg Going-Up **[find the fact in A&M]**, we see that every prime  $\mathfrak{p}$  in a  $\text{Spec } A$  contained in the image of the closed set associated to an ideal  $\mathfrak{b} \subset B$  lies under a prime  $\mathfrak{q} \subset B$ . ■ **[Check this again]**

This observation becomes important in our current context because it implies that if  $f : Y \rightarrow X$  is an open, finite map of connected schemes, then it's surjective. Indeed,  $f$  is proper and thus a closed map (being universally closed), so  $f(Y)$  is both closed and open in  $X$ . Since  $X$  is connected, this implies  $f(Y) = X$ .

Of course, the particular brand of open map we're interested in is the étale map. Regarding étale maps, this last observation says that if our based scheme  $X$  is connected, then all we have to do is restrict to finite étale maps  $Y \rightarrow X$  to ensure that every connected component of  $Y$  covers  $X$ .

To this end, we'll work almost exclusively in this §3 with a the category  $\mathbf{Fin\acute{E}t}_{/X}$  of finite étale maps over a connected base scheme  $X$  that we fix once and for all. By Observation 2.1.5 of the previous section, we know that every map in  $\mathbf{Fin\acute{E}t}_{/X}$  (i.e., every map giving a commutative triangle constituting a morphism in  $\mathbf{Fin\acute{E}t}_{/X}$ ) is itself étale.

We now want to argue that finite étale maps look quite like our intuitive picture of covering spaces. The algebro-geometric idea that really gets this rolling is the heuristic idea that, given a scheme  $X$ , we would think of projective  $\mathcal{O}_X$ -modules as locally free bundles of some kind over  $X$ , or said differently, as the twist of some trivial bundle over  $X$ . I think this idea runs into snags in general, but when our projective module is finitely generated, it works out quite nicely:

**Claim.** — If  $f : Y \rightarrow X$  is finite and flat, then  $\mathcal{O}_Y$  is a locally free, finitely generated  $\mathcal{O}_X$ -module.

**Proof:** First of all, a finitely generated projective module  $M$  over any local ring  $A_{\mathfrak{p}}$  is always free. Indeed, Nakayama's Lemma tells us that we have a surjective morphism  $A_{\mathfrak{p}}^{\times n} \rightarrow M$  onto a minimal set of generators for  $M$ , and thus an exact sequence

$$0 \rightarrow K \hookrightarrow A_{\mathfrak{p}}^{\times n} \rightarrow M \rightarrow 0.$$

By projectivity of  $M$ , this sequence splits:  $A_{\mathfrak{p}}^{\times n} \cong K \oplus M$ , and then a second application of Nakayama's Lemma shows us that in fact  $K = 0$ .

In general, it is not the case that flat affine modules are projective, but by a Theorem in Weibel (see [Weibel, §3, Thm. 3.2.7, p. 71]) we know that this is the case when the module is finitely presented. Thus we can cover  $X$  by open affines  $\text{Spec } A_i$  such that  $f^{-1}(\text{Spec } A_i) = \text{Spec } B_i$  with each  $B_i$  a projective, finitely generated  $A_i$ -module. Let  $B \leftarrow A$  be one such algebra. Then  $B \oplus B' \cong A^{\times n}$  for some  $n$ , and thus  $B_{\mathfrak{p}} \oplus B'_{\mathfrak{p}} \cong A_{\mathfrak{p}}^{\times n}$  at every prime  $\mathfrak{p} \subset A$ . In particular, each  $B_{\mathfrak{p}}$  is finitely generated and thus free by the observation of the previous paragraph.

Finally, by an exercise in Hartshorne [AG, §II.5, Ex. 5.7, p. 124], and using our ever-present locally Noetherian assumption, this implies that  $B$  is a locally free  $A$ -module. ■

**Corollary.** — Every finite étale map  $Y \rightarrow X$  with connected codomain comes with a well defined notion of (*locally free*) *rank* or *order*, defined to be the locally constant rank of  $\mathcal{O}_Y$  as an  $\mathcal{O}_X$ -module. ■

**Counter Example:** Don't get confused about the meaning of this last Corollary! It does not say that  $Y$  is Zariski locally free over  $X$ , but only that  $\mathcal{O}_Y$  is! For example, let  $Y := \text{Spec } k[t^{\pm 1}]$  and  $X := \text{Spec } k[t^{\pm 2}]$ . Then  $X$  is étale and  $Y \rightarrow X$  is finite étale. As we've already seen,  $Y$  cannot be Zariski locally free over  $X$ , yet

$$k[t^{\pm 1}] \cong k[t^{\pm 2}] \oplus t k[t^{\pm 2}]$$

since

$$k[t^{\pm 2}] \cong \bigoplus_{n=-\infty}^{\infty} k t^{2n} \quad \text{and} \quad t k[t^{\pm 2}] \cong \bigoplus_{n=-\infty}^{\infty} k t^{2n+1}.$$

**Corollary.** — If  $Y \rightarrow X$  is finite étale with connected codomain, then every geometric fiber  $Y_{\bar{x}}$  has the same *order*, i.e., the same cardinality, equal to the order of  $Y \rightarrow X$  itself. ■

**Corollary.** — If  $f : Y \rightarrow X$  is a finite étale map with connected codomain, and  $\bar{x} : \text{Spec } K \rightarrow X$  is a geometric point for which  $Y_{\bar{x}}$  is order-1, then  $f$  is an isomorphism.

**Proof:** By the previous Corollary [yellow], the order of  $Y_{\bar{x}}$  is the locally free rank of  $Y \rightarrow X$ . Thus if  $Y_{\bar{x}}$  is order-1, then  $\mathcal{O}_Y$  is an invertible  $\mathcal{O}_X$ -module. Since it comes from a scheme morphism  $f : Y \rightarrow X$ , this makes  $\mathcal{O}_Y$  isomorphic to  $\mathcal{O}_X$  via  $f^{\#}$ , which is to say that  $f$  is itself an isomorphism. ■

**Observation.** — Let  $f : Y \rightarrow X$  be a finite scheme map with connected codomain. Then the following are equivalent:

(i)  $f$  is étale;

(ii) there exists a finite étale map  $U \rightarrow X$  along which  $Y$ 's pullback is a trivial cover of  $U$ , specifically, such that  $U \times_X Y \rightarrow U$  is surjective with  $U \times_X Y \cong \bigsqcup_i U$ .

**Proof:** (i) $\Rightarrow$ (ii). Suppose  $f$  étale. Fix a geometric point  $\bar{y} : \text{Spec } K \rightarrow Y$  and let  $\bar{x} : \text{Spec } K \rightarrow X$  be its composition  $f \bar{y}$ , i.e., its "image in  $X$ ".

We proceed by induction. In the case that  $Y_{\bar{x}}$  is order-1, the previous Corollary [yellow] gives us the result immediately: just take  $U := Y$ . So suppose (i) $\Rightarrow$ (ii) proved in the case that  $\text{card } |Y_{\bar{x}}| = n - 1$ , and let  $Y$  now be a scheme with  $\text{card } |Y_{\bar{x}}| = n$ . Consider the pullback

$$\begin{array}{ccc} Y \times_X Y & \rightarrow & Y \\ \downarrow & & \downarrow \\ Y & \rightarrow & X. \end{array}$$

Since  $Y \rightarrow X$  is étale, Observation [ ] from the previous section tells us that each projection  $p_1, p_2 : Y \times_X Y \rightarrow Y$  is also étale. Moreover, Theorem [ ] tells us that the diagonal  $\Delta : Y \rightarrow Y \times_X Y$  is an open immersion, i.e., its image is a connected component of  $Y \times_X Y$ . In other words,

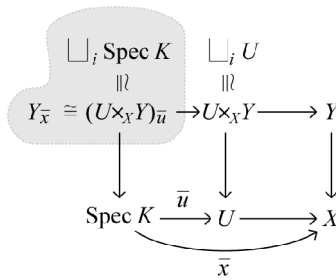
$$Y \times_X Y = \Delta(Y) \sqcup Z$$

for some scheme  $Z$ . Since  $Z \hookrightarrow Y \times_X Y$  is a Zariski open immersion, we see by Theorem [ ] that each composition

$$Z \hookrightarrow Y \times_X Y \xrightarrow{p_i} Y$$

is étale. But now it's easy to see that  $\text{card} |Z_{\bar{y}}| = n - 1$  [argue this: draw a little diagram with the geometric points in it], so by our inductive hypothesis, we can find an finite étale map  $U \rightarrow Y$  along which  $Z$  pulls back to a trivial étale cover. Since  $\Delta(Y) \rightarrow Y$  is already trivial, this means that our original cover  $Y \rightarrow X$  pulls back to a trivial cover along  $U \rightarrow Y \rightarrow X$ .

(ii)  $\Rightarrow$  (i). Let  $Y \rightarrow X$  be connected with finite codomain, and suppose  $U \rightarrow X$  another finite étale map such that  $U \times_X Y \cong \bigsqcup_{i=1}^n U$ . Let  $\bar{x} : \text{Spec } K \rightarrow X$  be an arbitrary geometric point. By our remark following Claim [ ] above, we can factor  $\bar{x}$  through  $U \rightarrow X$  along some geometric point  $\bar{u} : \text{Spec } K \rightarrow U$ .



Thus  $Y_{\bar{x}} \cong (U \times_X Y)_{\bar{u}} \cong \bigsqcup_{i=1}^n \text{Spec } K$ . In particular,  $Y \rightarrow X$  is unramified. A similar argument using stalks in place of geometric fibers shows that  $Y \rightarrow X$  is flat. ■

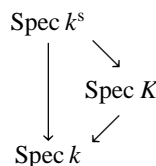
In summary, we have the following: even though the complexity of the residue fields of étale maps makes them seemingly more complicated than covering spaces, a finite étale map  $Y \rightarrow X$  with connected codomain (i.e., connected base) still come with a well defined notion of "its number  $n$  of sheets over  $X$ ," i.e., its order, and this order  $n$  coincides with the order of every geometric fiber  $Y_{\bar{x}}$  as in the homotopical case. Moreover, if we allow étale covers to play a role similar to Zariski open immersions, then finite étale maps turn out to be "locally trivial" even.

This parallel with the homotopical case, even though it's not perfect, will let us prove further theorems for finite étale covers  $Y \rightarrow X$  that mirror results on covering spaces, using techniques that come from the homotopical perspective.

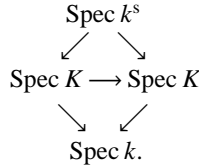
### ■ 3.3 Galois Objects in the Unified Perspective.

Recall that in field theory, a finite *Galois extension*  $K/k$  is a finite separable extension with the property that whenever we fix a particular embedding  $k^s \supset K \supset k$ , the  $k$ -automorphism of  $k^s$  induced by any other embedding  $K \hookrightarrow k^s$  restricts to a  $k$ -automorphism of  $K$ . [correct?]

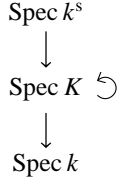
At the level of spectra, a choice of embedding  $k^s \supset K \supset k$  is a choice of commutative vertical triangle



and the  $k$ -automorphism induced by a second embedding  $K \hookrightarrow k^s$  over  $k$  produces a commutative diagram



between two such vertical triangles (the vertical map  $\text{Spec } k^s \rightarrow \text{Spec } K$  suppressed in this last diagram). Call such a diagram a *morphism from one covering of  $\text{Spec } k$  by  $\text{Spec } K$  into another*. Then a Galois extension  $K/k$  becomes a vertical triangle like the first above whose only morphisms into another covering of  $\text{Spec } k$  by  $\text{Spec } K$  are of the form



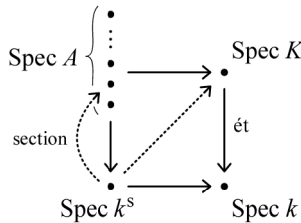
i.e., are automorphisms.

If  $\text{Spec } k^s$  plays the role of a universal étale covering space, as the analogy between the field theoretical and homotopy theoretical cases suggests, then, with respect to our conjectures connecting the two cases, it will be worthwhile to reformulate the notion of Galois extensions in a language that doesn't depend on the existence of  $k^s$ .

To this end, we point out that another standard, equivalent definition of a finite Galois extension  $K/k$  is that it is any finite separable extension for which the fixed field  $K^G$  of  $G := \text{Aut}_k(K)$  is  $k$  itself.

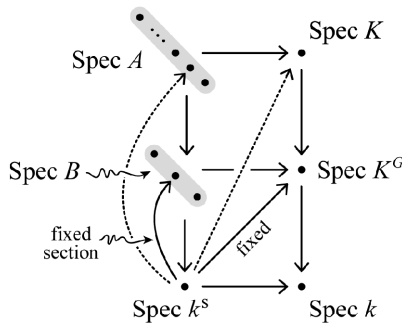
We can give this second definition a rather geometric interpretation, using the language of *geometric points*:

We first re-interpret our finite separable extension  $K/k$  as an étale cover  $\text{Spec } K \rightarrow \text{Spec } k$ , and we fix a geometric point  $\bar{x} : \text{Spec } k^s \rightarrow X := \text{Spec } k$  by simply choosing an embedding  $k \hookrightarrow k^s$ . A further choice of a particular embedding  $K \hookrightarrow k^s$  over  $k$  then becomes the dotted diagonal producing the triangle commuting at bottom-right in the picture below. Letting  $A := k^s \otimes_k K$ , we obtain the full drawing, where separability of  $K/k$  implies that  $\text{Spec } A$  looks as depicted, each point in the fiber being a copy of  $\text{Spec } k^s$ . Since the diagram is Cartesian, the choice of dotted diagonal becomes equivalent to a choice of particular section at left:



Moreover, again since the diagram is Cartesian, automorphisms of  $\text{Spec } K$  induce automorphisms of  $\text{Spec } A$ . We can keep track of each automorphism of  $\text{Spec } K$  by watching how it changes the dotted diagonal lift  $\text{Spec } k^s \rightarrow \text{Spec } K$ , which becomes the same as watching where it moves the vertical section  $\text{Spec } k^s \rightarrow \text{Spec } A$  at left. In short, we can keep track of automorphisms of  $\text{Spec } K$  by looking at their pulled-back action on a single point in the geometric fiber  $\text{Spec } A$ .

Let  $G := \text{Aut}_k(K)$ , i.e.,  $G = \text{Aut}_{\text{Sch}/\text{Spec } k}(\text{Spec } K)^{\text{op}}$ , and let  $K^G$  be the subfield of  $K$  over  $k$  fixed by  $G$ . This gives us the sequence  $\text{Spec } K \rightarrow \text{Spec } K^G \rightarrow \text{Spec } k$  of scheme maps, and an intermediate pullback  $\text{Spec } A \rightarrow \text{Spec } B \rightarrow \text{Spec } k^s$  (see drawing below). Fix again a lift  $\text{Spec } k^s \rightarrow \text{Spec } K$  (the dotted diagonal in the drawing below). This induces a lift  $\text{Spec } k^s \rightarrow \text{Spec } K^G$ , but the very definition of  $K^G$  means that automorphisms of  $\text{Spec } K$  over  $\text{Spec } k$  induce only the trivial action on  $\text{Spec } k^G$ , so this induced lift  $\text{Spec } k^s \rightarrow \text{Spec } K^G$  is not altered by automorphisms of  $\text{Spec } K$  over  $\text{Spec } k$ . In particular, although the induced section  $\text{Spec } k^s \rightarrow \text{Spec } A$  (dotted at left below) changes under automorphisms of  $\text{Spec } K$ , the section  $\text{Spec } k^s \rightarrow \text{Spec } B$  that it induces remains fixed.



In particular, if  $\text{Spec } B$  is a disjoint union of spectra isomorphic to  $\text{Spec } k^s$ , points identified under the action of  $G^{\text{op}}$  on  $\text{Spec } A$ , then the fixedness of this section  $\text{Spec } k^s \rightarrow \text{Spec } B$  means that the section  $\text{Spec } k^s \rightarrow \text{Spec } A$  isn't transitive under the  $G^{\text{op}}$ -action on the fiber  $\text{Spec } A$ .

So Galois extensions  $K/k$  of fields correspond to geometric fibers with transitive  $\text{Aut}_k(K)$ -action. This puts them in clear geometric analogy with normal covering spaces. Recall that a *normal covering space*  $Y \rightarrow X$  is a connected covering space on the fibers  $Y_x$  of which  $\text{Aut}_X(Y)$  acts transitively. This it suggests a clear formulation of the general definition of a Galois cover of an arbitrary scheme  $X$ :

**Definition.** — By an *étale cover*  $Y \rightarrow X$ , we mean an étale map of schemes that's surjective at the level of underlying spaces.

We say that a connected étale cover  $Y \rightarrow X$  is *Galois* if the  $\text{Aut}_{\text{Sch}/X}(Y)$ -action induced on its every geometric fiber is transitive.

Using Observation □ above, it is not difficult to prove the following claim:

**Claim.** — To check that  $Y \rightarrow X$  is Galois, it suffices to check transitivity at single, arbitrary fiber  $Y_{\bar{x}}$ .

**Proof:** Exercise. □

**Remark 1:** There's one funny issue here. Namely, in the field theoretical picture, the map  $\text{Spec } k^s \rightarrow \text{Spec } k$  plays the simultaneous role of universal covering  $\tilde{X}$  of the base  $X = \text{Spec } k$  and of geometric point  $\bar{x}$  in the base  $X$ .

This means that another possible way we might try to pin-down the analogy between the field-theoretical and geometric pictures would be to look not, in the homotopy-theoretical picture, at the deck transformation action on fibers  $Y_x$ , but at the pullback  $\tilde{X} \times_X Y$  of  $Y$  over the the universal cover  $\tilde{X} \rightarrow X$ .

But from a certain category-theoretical perspective, the two situations amount to the same thing. More specifically, let  $X$  be a non-pathological topological space with base point  $x \in X$ , and let  $F(-)$  be the functor taking each covering space  $Y \rightarrow X$  to the underlying set  $|Y_x|$  of  $Y$ 's fiber over  $x$ . Then it's not hard to see that the universal property of the universal covering space  $\tilde{X} \rightarrow X$  makes the pullback

$$\begin{array}{ccc} \tilde{X} \times_X Y & \rightarrow & Y \\ \downarrow & & \downarrow \\ \tilde{X} & \rightarrow & X \end{array}$$

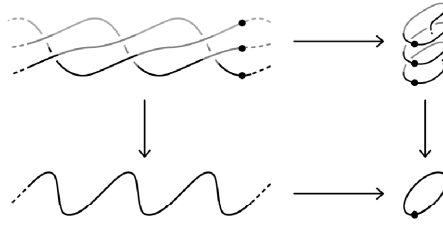
of every covering space  $Y \rightarrow X$  a trivial covering of  $\tilde{X}$  and, what's more, puts the sections  $\tilde{X} \rightarrow \tilde{X} \times_X Y$  of this pullback covering into canonical bijection with  $|Y_x|$ . In other words, we obtain a canonical isomorphism

$$\text{Hom}_{\text{Top}/\tilde{X}}(\tilde{X}, \tilde{X} \times_X (-)) \cong \text{Hom}_{\text{Top}/\{x\}}(\{x\}, \{x\} \times_X (-)) =: F(-)$$

of functors, taking arguments in the category of connected coverings spaces of  $X$ . Using the universal property of the fiber product, this becomes the simpler isomorphism

$$\text{Hom}_{\text{Cov}(X)}(\tilde{X}, -) \cong \text{Hom}_{\text{Top}/X}(\{x\}, -).$$

We'll see this "equivalence" between the two perspectives play out in what follows. Here's a cartoon example where  $X = \mathbb{S}^1$  and  $Y \rightarrow X$  is the order-3 covering  $\mathbb{S}^1 \rightarrow \mathbb{S}^1$ :



**Remark 2:** The transitivity condition in the definition of a Galois cover makes Galois étale covers the analogs of normal covering spaces, or normal field extensions. Since, for fields at least, Galois  $\approx$  simple + transitive, one might ask where the simple part is in the definition. For fields, simplicity  $\approx$  separability, and this continuous to étale covers, that is, the simplicity is already present in the fact that the cover is étale:

**Claim.** — Let

$$\begin{array}{ccc} Z & \xrightarrow{f_1, f_2} & Y \\ & \searrow & \swarrow \\ & X & \end{array}$$

be a pair of maps of finite étale covers, and let  $\bar{z} : \text{Spec } K \rightarrow Z$  a geometric point. If  $f_1 \circ \bar{z} = f_2 \circ \bar{z}$ , then  $f_1 = f_2$ .

**Proof:** Pass to the pullback

$$\begin{array}{ccc} Z \times_X Y & \rightarrow & Y \\ \downarrow & & \downarrow \\ Z & \rightarrow & X, \end{array}$$

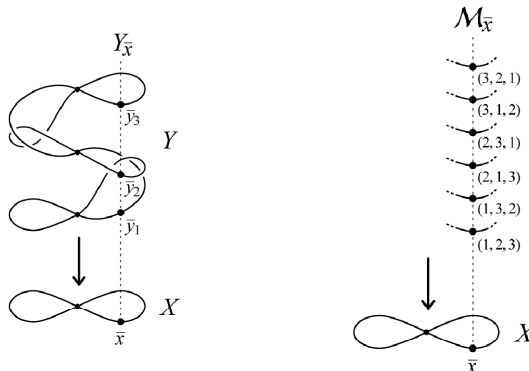
and note the pair  $f_1, f_2 : Z \rightarrow Y$  is equivalent to a pair of sections  $s_1, s_2 : Z \rightarrow Z \times_X Y$ . But sections of finite étale covers are Zariski open immersions to connected components of the cover, so the identity  $s_1 \circ \bar{z} = s_2 \circ \bar{z}$  implies that  $s_1 = s_2$ , and thus by the universal property of the pullback, that  $f_1 = f_2$ . ■

**Corollary.** — If  $Y \rightarrow X$  is a finite étale map of connected schemes,  $\bar{x} : \text{Spec } K \rightarrow X$  is any geometric point, and  $g$  is an  $X$ -automorphism of  $Y$ , then  $g$  has no fixed points in  $Y_{\bar{x}}$ , i.e., "the deck action" is simple on connected étale covers.

**Proof:** This follows immediately from the previous Lemma, with  $Z := Y$ ,  $f_1 := g$ , and  $f_2 := 1_Y$ . ■

We now want to construct, over each connected étale cover  $Y \rightarrow X$ , a canonical Galois cover  $P \rightarrow X$  factoring through  $Y \rightarrow X$ . We'll give a technical, scheme-theoretical construction of  $P$  below, but before doing so, we want to motivate the construction with its homotopy-theoretical counterpart.

For this, we'll momentarily let  $X$  be a connected, locally path-connected, semi-locally simply-connected topological space, and we'll let  $Y \rightarrow X$  be an arbitrary finite, connected covering space. In the drawing below, we take  $X = \mathbb{S}^1 \vee \mathbb{S}^1$ , the boquet of two circles. To keep with the notation we'll use below, let  $\bar{x} \in X$  be a point varying in  $X$ , and let  $Y_{\bar{x}}$  be  $Y$ 's fiber over  $\bar{x}$ . We first construct a whole new covering space  $\mathcal{M} \rightarrow X$ , which we think of as the "moduli space of marked fibers" over  $X$ . If  $Y$  is order- $n$  over  $X$ , then we let the integers  $\{1, 2, \dots, n\}$  label the set of points in each fiber  $Y_{\bar{x}}$ . Over each  $\bar{x} \in X$ , we let  $\mathcal{M}_{\bar{x}}$  be the order- $n!$  set of labelings of  $Y_{\bar{x}}$ . This at least determines  $\mathcal{M}$ 's underlying set over  $X$ . To see that  $\mathcal{M}$  comes naturally with the topology of a covering space, note that any labeling of a single fiber  $Y_{\bar{x}}$  in  $Y$  determines labelings of all other fibers of  $Y$  over a small open neighborhood of  $\bar{x}$ .



The monodromy action of every loop in  $X$  based at  $\bar{x}$  permutes the labelings, and so, in simple cases at least, we can use it to exhibit  $\mathcal{M}$  explicitly. For the example above, we find that  $\mathcal{M}$  looks something like:



(the credit goes to Slava for figuring out what  $\mathcal{M}$  looks like here).

In general,  $\mathcal{M}$  will not be connected, so cannot possibly be Galois. Thus we have to choose a connected component of  $\mathcal{M}$  arbitrarily. We denote this component  $P$ , and it will become our canonical Galois cover of  $Y$ .

It's at least clear that the induced map  $P \rightarrow X$  is a covering space. To see that it is Galois, notice first that by definition of  $\mathcal{M}$ , we have an embedding  $\mathcal{M} \subset Y \times_X \cdots \times_X Y$  into the  $n$ -fold fiber product over  $X$ . If  $(\bar{y}_1, \dots, \bar{y}_n)$  and  $(\bar{y}_{\sigma(1)}, \dots, \bar{y}_{\sigma(n)})$  are two points in a  $P_{\bar{x}}$  in  $P$ , for some permutation  $\sigma \in \Sigma_n$ , then we know at least that the standard  $\Sigma_n$ -action on  $Y \times_X \cdots \times_X Y$  that just permutes the factors takes

$$(\bar{y}_1, \dots, \bar{y}_n) \longmapsto (\bar{y}_{\sigma(1)}, \dots, \bar{y}_{\sigma(n)}) \quad \text{in } Y \times_X \cdots \times_X Y.$$

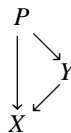
In fact, since  $\mathcal{M}$  consists of all  $n$ -tuples of distinct points in  $Y \times_X \cdots \times_X Y$ , we know that  $\sigma$ 's action on  $Y \times_X \cdots \times_X Y$  restricts to an action on  $\mathcal{M}$ . We only need to show that it restricts all the way to  $P \subset \mathcal{M}$ . But this is trivial, because  $P$  is connected, so  $\sigma$ 's action on the point  $(\bar{y}_1, \dots, \bar{y}_n)$  implies that  $\sigma(P) = P$ . Thus  $P$ 's deck action is transitive on fibers, i.e.,  $P$  is Galois over  $X$ .

It's not hard to see that  $P$  is also "canonical" or "semi-universal" over  $Y$ , but rather than argue this here, we leave it for the scheme-theoretical construction below. The argument is almost identical in both cases.

In fact, the scheme-theoretical construction closely matches the one we've just described. The primary complication comes from the fact that it is not as easy to verify, in the scheme-theoretical case, that the moduli space  $\mathcal{M}$  exists and is a covering space of  $X$ . This is not a terrible difficulty though, because a review of the previous argument reveals that the real issue is that of picking a good component  $P$  of the  $n$ -fold fiber product  $Y \times_X \cdots \times_X Y$ . Most of the work in the scheme-theoretical construction goes into making a good choice of  $P$  and verifying that it behaves like a component in the moduli space  $\mathcal{M}$ , without explicit recourse to  $\mathcal{M}$ .

The construction is as follows:

**Theorem (Existence of Finite Galois Covers).** — If  $X$  is a connected scheme and  $Y \rightarrow X$  is a connected, finite étale cover, then there exists a finite Galois cover  $P \rightarrow X$  fitting into a commutative triangle



with the semi-universal property that if  $Z \rightarrow X$  is any other finite, étale Galois cover of  $X$ , then it factors (non-uniquely) through  $P \rightarrow X$ .

**Proof (Serre):** Let  $Y \rightarrow X$  be a connected, finite étale cover. Fix a geometric point  $\bar{x} : \text{Spec } k^s \rightarrow X$ . Then by Theorem □,  $|Y_{\bar{x}}|$  is a finite, discrete set of singletons isomorphic to  $\text{Spec } k^s$ . Let  $n$  be the cardinality of this geometric fiber, and consider the fiber product

$$Y_{/X}^{\times n} := \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n \text{ factors}}$$

Let  $\bar{y} := (\bar{y}_1, \dots, \bar{y}_n)$  be an enumeration of  $|Y_{\bar{x}}|$ . Then  $\bar{y}$  defines a geometric point in  $Y_{/X}^{\times n}$ , namely the point  $\text{Spec } k^s \rightarrow Y_{/X}^{\times n}$  whose  $i^{\text{th}}$  component is  $\bar{y}_i : \text{Spec } k^s \rightarrow Y$  over  $\bar{x}$ . We define  $P$  to be the connected component of  $Y_{/X}^{\times n}$  containing the image of  $\bar{y}$ .

We clearly have a projection  $P \rightarrow Y$ , induced by the projection of the first factor of  $Y_{/X}^{\times n}$  onto  $Y$ . To see that it is étale, use permanence of étale maps under base change, along with the fact that  $Y \rightarrow X$  is étale by assumption, on

$$\begin{array}{ccc}
 Y \times_X Y \rightarrow Y & & Y_{/X}^{\times(m+1)} \rightarrow Y_{/X}^{\times m} \\
 \downarrow & \text{and then inductively} & \downarrow \\
 Y \rightarrow X & & Y \rightarrow X
 \end{array}$$

to see at least that the first projection  $Y_{/X}^{\times n} \rightarrow Y$  is étale. Our definition of  $P$  as a connected component of  $Y_{/X}^{\times n}$  makes  $P \hookrightarrow Y_{/X}^{\times n}$  a Zariski open immersion, hence étale, and thus by permanence of étale maps under composition,  $P \hookrightarrow Y_{/X}^{\times n} \rightarrow Y$  becomes étale. Composing with  $Y \rightarrow X$ , we obtain an étale cover  $P \rightarrow X$  (note: for a while I was confused about why we're assured that  $P \rightarrow X$  is actually a cover, i.e., is surjective, but this comes from the fact that  $P \rightarrow X$  is finite, thus proper, thus surjective as discussed in § [ ] above).

We can now consider the geometric fiber  $P_{\bar{x}}$  over our original geometric point  $\bar{x} : \text{Spec } k^s \rightarrow X$ . Since  $P \rightarrow X$  is finite étale, we know that  $P_{\bar{x}} \cong \bigsqcup_i \text{Spec } k^s$ . In general, points  $\text{Spec } k^s \rightarrow Y \times_X \dots \times_X Y$  can have repeating entries, that is, can have components  $\bar{y}_i \cong \bar{y}_j$ . In the fiber  $P_{\bar{x}}$  however, all points/sections  $\text{Spec } k^s \rightarrow P_{\bar{x}}$  have non-repeating indices, and are thus permutations  $(\bar{y}_{\sigma(1)}, \dots, \bar{y}_{\sigma(n)})$  of our original point  $\bar{y} := (\bar{y}_1, \dots, \bar{y}_n)$ . Since  $Y \rightarrow X$  is étale to begin with, Theorem [ ] says that the diagonal  $\Delta : Y \rightarrow Y \times_X Y$  is an open immersion. Letting

$$p_{ij} : Y_{/X}^{\times n} \rightarrow Y \times_X Y$$

be the projection onto the  $(i, j)^{\text{th}}$  component of  $Y_{/X}^{\times n}$ , this implies that  $p_{ij}^{-1}(\Delta(Y))$  is open in  $Y_{/X}^{\times n}$ . If it were the case that

$$p_{ij}^{-1}(\Delta(Y)) \cap P \neq \emptyset \quad \text{in } Y_{/X}^{\times n},$$

then the fact that  $P$  is connected by construction would imply that

$$p_{ij}^{-1}(\Delta(Y)) \supset P,$$

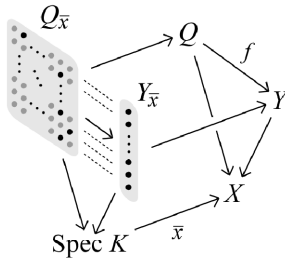
contradicting the fact, again by construction, that  $(\bar{y}_1, \dots, \bar{y}_n)$  is a geometric point in  $P$ . Thus the  $\text{Spec } k^s$ -valued points in the fiber  $P_{\bar{x}}$  all have non-repeating indices.

Now what's really beautiful is that the canonical permutation action of  $\Sigma_n$  on  $Y_{/X}^{\times n} := Y \times_X \dots \times_X Y$  must actually permute the components of  $Y_{/X}^{\times n}$ , since every  $\sigma$  acts as a homeomorphism on  $Y_{/X}^{\times n}$ . Thus if  $(\bar{y}_{\sigma(1)}, \dots, \bar{y}_{\sigma(n)})$  is some point in  $P_{\bar{x}}$ , then  $\sigma(P) = P$ , which is to say that  $\sigma$ 's restriction to  $P$  is an element of  $\text{Aut}_{\text{Sch}/X}(P)$ , and  $\text{Aut}_{\text{Sch}/X}(P)$  acts transitively on  $P_{\bar{x}}$ . Since it's enough to check the Galois condition on a single fiber in  $P$  (Corollary [ ] above), this proves that  $P$  is Galois.

To establish the "semi-universal" property, let  $Q \rightarrow X$  be Galois and suppose given a triangle

$$\begin{array}{ccc}
 Q & \xrightarrow{f} & Y \\
 \searrow & & \swarrow \\
 & X &
 \end{array}$$

Fix a geometric point  $\bar{x} : \text{Spec } K \rightarrow X$ , and consider the fibers  $Y_{\bar{x}}$  and  $Q_{\bar{x}}$ . Choose a geometric point  $\bar{q}_i$  "in"  $Q_{\bar{x}}$  over each geometric point  $\bar{y}_i$  "in"  $Y_{\bar{x}}$ .



Since  $Q$  is Galois, each  $\bar{q}_i$  admits an automorphism  $g_i \in \text{Aut}_{\text{Sch}/X}(Q)$  with induced action on  $Q_{\bar{x}}$  giving

$$g_i \circ \bar{q}_1 = \bar{q}_i.$$

Combining these automorphisms with  $f$  on the right, we obtain the map

$$Q \xrightarrow{\prod_i f \circ g_i} \underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n \text{ factors}}$$

and by our very choice of the  $g_i$  and the  $\bar{q}_i$ , we find that

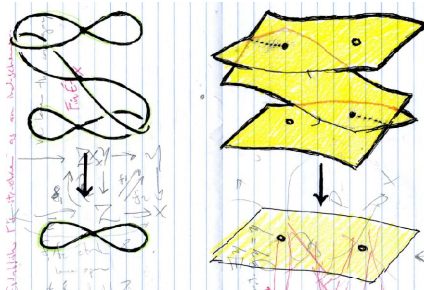
$$(\prod_i f \circ g_i) \circ \bar{q}_1 = (f \circ \bar{q}_1, f \circ \bar{q}_2, \dots, f \circ \bar{q}_n) = (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n).$$

But by construction of  $P$ , this point  $(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n)$  lies in  $P \subset Y_{/X}^{\times n}$ , and thus since  $Q$  is connected, this implies that  $Q \rightarrow Y_{/X}^{\times n}$  factors through  $P \hookrightarrow Y_{/X}^{\times n}$ . ■

Galois covers will become important because they will let us construct the pro-finite scheme  $\tilde{X} \rightarrow X$ , that will serve as  $X$ 's universal cover, in a way that lets us control/keep track of  $\text{Aut}_{\text{Sch}/X}(\tilde{X})$ , which will be our algebro-geometric manifestation of the "fundamental group  $\pi_1(X)$ ."

■ **Aside on a Non-Galois Cover.**

The drawing at left below is the non-normal covering space of the boquet  $\mathbb{S}^1 \vee \mathbb{S}^1$  that we drew in class when we were trying to think about (non-) Galois covers. Using the fact that  $\mathbb{C} \setminus \{\pm 1\} \simeq \mathbb{S}^1 \vee \mathbb{S}^1$ , the drawing at right is my intuitive attempt to come up with a Riemann surface  $Y \rightarrow \mathbb{C} \setminus \{\pm 1\}$  such that  $Y \simeq$  the non-normal covering space at left:



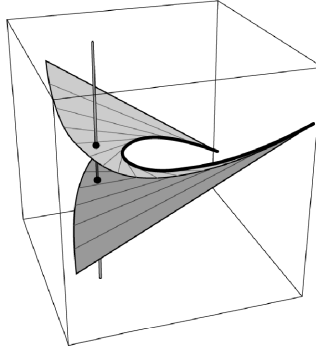
It's clear that  $Y$  is a non-normal covering space of  $\mathbb{C} \setminus \{\pm 1\}$ . However, the drawing only describes, at best, a topological construction of this covering. The goal of the present aside is to realize  $Y$  explicitly as a non-Galois étale cover of a

doubly-punctured affine line, that is, to give an equation defining  $Y \subset \mathbb{A}_k^2$  over  $\mathbb{A}_k^1$  (say for  $k$  algebraically closed) with ramification matching that in the drawing at right.

Let  $\mathbb{A}^n := \mathbb{A}_{\mathbb{C}}^n$ . We consider the space  $\mathbb{A}^3$  as being the  $\mathbb{C}$ -vector space of monic, cubic polynomials in a single variable  $t$ . In other words, we identify the coordinates in  $\mathbb{A}^3$  with the coefficients  $a_i$  of the general polynomial  $p(t) = a_0 + a_1 t + a_2 t^2 + t^3$ .

Similarly, we interpret  $\mathbb{A}^4$  as the space of polynomials  $y - p(t)$ , where  $p(t)$  is monic and cubic, by keep in mind a particular projection  $\mathbb{A}^4 \rightarrow \mathbb{A}^3$ .

Our interpretation of  $\mathbb{A}^3$  lets us consider the locus  $Z$  in  $\mathbb{A}^3$  consisting of all monic, cubic polynomials with a root of multiplicity at least 2. Expanding  $(t - a)^2(t - b)$ , we get an  $\mathbb{A}^2$ -parametrization of this locus in the variables  $a$  and  $b$ . If we restrict attention to " $\mathbb{R}^3 \subset \mathbb{A}^3$ ," this locus  $Z$  becomes the ruled surface depicted below (the vertical line is not a component of it):



Consider an arbitrary (complex) line in  $\mathbb{A}^3$ , say parametrized by a coordinate  $x$ . It parametrizes a linear family of monic, cubic polynomials

$$p_x(t) = a_0(x) + a_1(x)t + a_2(x)t^2 + t^3.$$

Wherever this line intersects  $Z$ , two or more roots of  $p_x(t)$  coincide.

Consider now the particular case of a line in  $\mathbb{A}^3$  that intersects  $Z$  exactly twice, at distinct points  $z_1, z_2 \in Z$ , such that the polynomials described by  $z_1$  and  $z_2$  each have a root of multiplicity exactly two. Since this is the "general" situation for  $p_x(t)$  when it has multiple roots, we guess that the complex line  $L$  depicted by the vertical line in the previous image is such an example (modulo the possibility that it intersects  $Z$  at complex points that we can't see).

The particular line  $L$  we've chosen corresponds to the family of polynomials

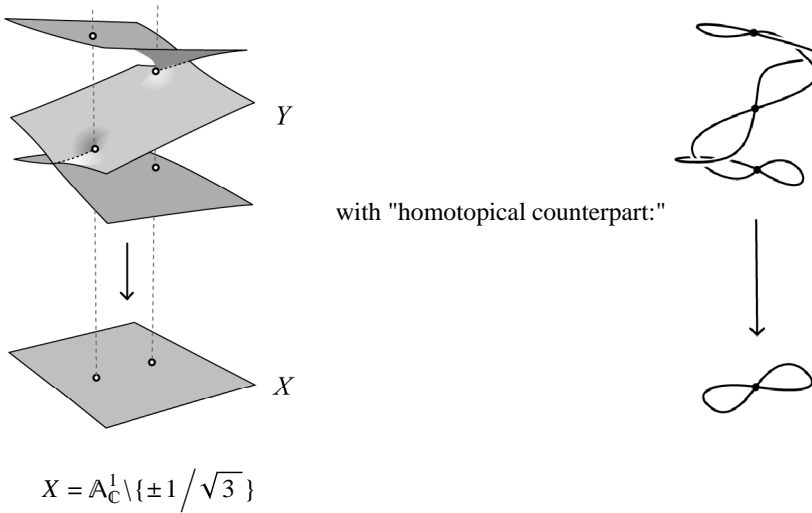
$$p_x(t) = x - t + t^3.$$

Recalling our projection  $\mathbb{A}^4 \rightarrow \mathbb{A}^3$ , and writing our line  $L$  as a map  $p_{(-)} : \mathbb{A}^1 \rightarrow \mathbb{A}^3$ , we can consider the vertical map at left in the pullback

$$\begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & \mathbb{A}^4 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{A}^3 \\ & & p_{(-)} \end{array}$$

We obtain the polynomial  $p_x(y) = x - y + y^3$  in  $\mathbb{A}^2$ , and consider its locus  $Y := V(x - y + y^3)$ . The two intersections of  $L$  with  $Z$  correspond now to two points  $x$  in  $\mathbb{A}^1$  at which the generally order-3 fiber  $Y_x$  collapses to an order-2 fiber. In other words, the intersection  $L \cap Z \subset L$  coincides precisely with the locus in  $X$  of points over which  $Y \rightarrow X$  is ramified. Playing with the equation  $x - y + y^3 = 0$  directly, we see that our guess was correct:  $x = \pm 1 / \sqrt{3}$  are the only points in  $\mathbb{A}^1$  over which  $Y$  is ramified, of order-2 at each. Thus defining  $X := \mathbb{A}^1 \setminus \{ \pm 1 / \sqrt{3} \}$ , we obtain the following Riemann surface, the clear algebraic counterpart to our cover of the bouquet hat we were originally after:

$$Y = V(x - y + y^3) \subset \mathbb{A}_{\mathbb{C}}^2$$

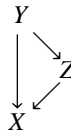


**[How do we verify, abstractly, that it's non-Galois?]**

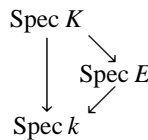
Probably look at the generic fiber  $k(x, y)/(x - y + y^3) \leftarrow k(x)$ .

■ **3.4 The Finite Fundamental Theorem.**

Some of this section may seem belaboured on a first read. However, the care we take comes from the meaningless of the phrase "the set of covers intermediate to a cover  $Y \rightarrow X$ ." The phrase is supposed to refer to a set of commutative triangles



But this set is not a set. Rather, it's the class underlying the category of such triangles. When we work with fields, we don't run into this problem because our triangles of the above form, namely triangles



come from specific intermediate fields  $K \supset E \supset k$ . Thus in the larger algebro-geometric setting, we need to make canonical choices of intermediate covers in some way analogous to our ability to consider the set of fields intermediate to any extension  $K/k$ .

In one line: the way we do it is by re-interpreting covers intermediate to  $Y \rightarrow X$  as certain  $\mathcal{O}_X$ -subalgebras of  $\mathcal{O}_Y$ .

Fix a connected base scheme  $X$ . Galois covers of  $X$  exist, so let  $f : Y \rightarrow X$  be such a cover. As part of the definition,  $Y$  is connected and its structure map  $f$  is finite. In particular,  $f$  is affine and also, by Theorem [ ] above,  $f$  is proper and thus an étale covering.

Let  $G := \text{Aut}_{\text{Sch}/X}(Y)^{\text{op}}$ . Interpret the structure sheaf  $\mathcal{O}_Y$  as an  $\mathcal{O}_X$ -algebra, defined over each open  $U \subset X$  according to

$$\Gamma(U, \mathcal{O}_Y) := \Gamma(f^{-1}(U), \mathcal{O}_Y).$$

In other words, interpret  $\mathcal{O}_Y$  as a sheaf over  $\mathcal{O}_X$  by conflating it with  $f_* \mathcal{O}_Y$ . Then given any subgroup  $H \subset G$ , we can consider the  $\mathcal{O}_X$ -algebra  $\mathcal{O}_Y^H \subset \mathcal{O}_Y$  defined at each open  $U \subset X$  according to

$$\Gamma(U, \mathcal{O}_Y^H) := \Gamma(U, \mathcal{O}_Y)^H = \Gamma(f^{-1}(U), \mathcal{O}_Y)^H.$$

Since  $f : Y \rightarrow X$  is affine, this  $\mathcal{O}_X$ -algebra  $\mathcal{O}_Y^H$  determines a well defined scheme

$$H \backslash Y := \text{Spec } \mathcal{O}_Y^H$$

over  $X$ , which we call  $Y$ 's *quotient by  $H$* . Note: this is our definition of  $H \backslash Y$ . Although  $\text{Spec } \mathcal{O}_Y^H$  has the universal property of a certain left-quotient, when we write  $H \backslash Y$  we mean the specific realization of this left-quotient as the spectrum of a specific subsheaf of  $\mathcal{O}_Y$ . This is not category-theoretical. It's set-theoretical.

Since  $\mathcal{O}_Y^H$  is an unambiguously defined subsheaf of  $\mathcal{O}_Y$ , we see that every subgroup  $H \subset G$  determines a scheme  $H \backslash Y \rightarrow X$ , giving us a well defined set

$$\mathbf{IntCov}(Y/X) := \{H \backslash Y \rightarrow X : H \subset G\}.$$

We call this the *set of covers intermediate to  $f : Y \rightarrow X$* . In fact, since a pair of subgroups  $H_2 \subset H_1 \subset G$  determines an inclusion

$$\mathcal{O}_Y^{H_2} \supset \mathcal{O}_Y^{H_1},$$

we realize that  $\mathbf{IntCov}(Y/X)$  actually has the structure of a poset, induced by the poset structure opposite the one on  $G$ 's poset of subgroups. Apriori, we have no indication that this poset structure induced on  $\mathbf{IntCov}(Y/X)$  is a faithful image of that on  $G$ 's poset of subgroups.

Now we want to say that every such  $H \backslash Y \rightarrow X$  is a finite étale cover, and moreover that every finite étale cover  $Z \rightarrow X$  fitting into a commutative triangle

$$\begin{array}{ccc} & Y & \\ & \downarrow f & \searrow \\ & X & \\ & & \nearrow \\ & & Z \end{array}$$

is isomorphic to one of these particular covers  $H \backslash Y \rightarrow X$  for some subgroup  $H \subset G$ .

To prove this, we first need a lemma which we can view as a formulation, in a general situation, of the observation we made in §0, namely that finite étale maps look like finite-order bundles as long as we're willing to use a notion of "local triviality" more general than that given by Zariski open immersions.

**Lemma (Finite Étale Quotients).** — Let  $Y \rightarrow X$  be a finite étale map of connected schemes,  $H \subset \text{Aut}_{\text{Sch}/X}(Y)$  an arbitrary subgroup. Then  $H \backslash Y$  is connected, and the scheme map  $H \backslash Y \rightarrow X$  defined above is a finite étale cover.

**Proof:** S **[Finish]**

**[... The following theorem]** includes in the Fundamental Theorem of Galois Theory as a special case, while simultaneously formulating the familiar classification of covering spaces in the new context of schemes.

**Theorem (Finite, Scheme-Theoretical Version).** — Let  $f : Y \rightarrow X$  be a finite Galois cover of one connected scheme by another. Then we have a one-to-one, order-reversing correspondence between the poset  $\mathbf{IntCov}(Y/X)$  and the poset of subgroups  $H \subset \text{Aut}_{\text{Sch}/X}(Y)$ , given by

$$\left( \begin{array}{ccc} & Y & \\ & \downarrow f & \searrow \\ & X & \\ & & \nearrow \\ & & Z \end{array} \right) \longmapsto \text{Aut}_{\text{Sch}/Z}(Y) \quad \text{and} \quad H \backslash Y \longleftarrow H,$$

under which intermediate Galois covers correspond to normal subgroups  $H \triangleleft \text{Aut}_{\text{Sch}/X}(Y)$ .

**Proof:** Let  $G := \text{Aut}_{\text{Sch}/X}(Y)$ , and let  $\mathbf{SubGrp}(G)$  be the poset of  $G$ 's subgroups.

What's already clear? ...It's of course immediate from the definition of  $\mathbf{IntCov}(Y/X)$  that the assignment  $H \mapsto H \backslash Y$  gives a surjection  $\mathbf{SubGrp}(G) \twoheadrightarrow \mathbf{IntCov}(Y/X)$ . Since

$$H_2 \subset H_1 \quad \text{implies} \quad \mathcal{O}_Y^{H_2} \supset \mathcal{O}_Y^{H_1},$$

it's also clear that this surjection is a contravariant morphism of posets.

Thus the first order of business to show is that  $H = \text{Aut}_{\text{Sch}/(H \setminus Y)}(Y)$ . By our construction of  $H \setminus Y$ , it's immediate that  $H \subset \text{Aut}_{\text{Sch}/(H \setminus Y)}(Y)$ . Since  $H \setminus Y \rightarrow X$  is a finite étale by Lemma [ ] above, and since  $Y \rightarrow X$  is by assumption, we know by Observation [ ] that  $Y \rightarrow H \setminus Y$  is itself a finite étale cover of connected spaces. In fact, it's a Galois cover. Indeed, fix a geometric point  $\bar{z} : \text{Spec } K \rightarrow H \setminus Y$ , along with geometric points  $\bar{y}_1, \bar{y}_2 : \text{Spec } K \rightarrow Y_{\bar{x}}$  over it:

$$\begin{array}{ccc} \text{Spec } K \xrightarrow{\bar{y}_1, \bar{y}_2} Y_{\bar{x}} & \longrightarrow & Y \\ & \searrow \downarrow & \downarrow \\ \bar{z} : \text{Spec } K & \longrightarrow & H \setminus Y \\ & & \downarrow \\ & & X \end{array}$$

Since  $Y \rightarrow X$  is Galois, we can find an  $X$ -automorphism  $g$  such that  $g \circ \bar{y}_1 = \bar{y}_2$ . We need only show that  $g$  is then in fact an  $H \setminus Y$ -automorphism of  $Y$ . In other words, letting  $p : Y \rightarrow H \setminus Y$  be our structure map, we need to verify that  $p \circ g = p$ . But by our choice of  $\bar{y}_1$  and  $\bar{y}_2$ , we have

$$p \circ g \circ \bar{y}_1 = p \circ \bar{y}_2 = \bar{z} = p \circ \bar{y}_1,$$

thus by Corollary [ ] above,  $p \circ g = p$  indeed. Hence  $Y \rightarrow H \setminus Y$  is Galois.

Knowing that  $Y \rightarrow H \setminus Y$  is Galois, we can now show that  $H = \text{Aut}_{\text{Sch}/(H \setminus Y)}(Y)$ . Let  $g \in \text{Aut}_{\text{Sch}/(H \setminus Y)}(Y)$ , fix a geometric point  $\bar{z} : \text{Spec } K \rightarrow H \setminus Y$ , and fix another geometric point  $\bar{y} : \text{Spec } K \rightarrow Y_{\bar{z}}$  in its fiber. Combining the Galois theory of fields with the fact that  $|H \setminus Y| \cong H \backslash |Y|$ , one realizes that even the geometric fiber  $Y_{\bar{z}}$  is an  $H$ -orbit of  $\bar{y}$  **[if you find enough time, explain this in detail]**. Since  $Y \rightarrow H \setminus Y$  is Galois, this implies that there exists some  $h \in H$  such that  $g \circ \bar{y} = h \circ \bar{y}$ , and thus by Corollary [ ] above,  $g = h$ . In short,  $H = \text{Aut}_{\text{Sch}/(H \setminus Y)}(Y)$ , establishing that the functions

$$\mathbf{IntCov}(Y/X) \rightleftarrows \mathbf{SubGrp}(G)$$

are inverse to one another.

It only remains to establish that under this correspondence, Galois coverings  $H \setminus Y \rightarrow X$  correspond to normal subgroups  $H \triangleleft G$ . **[finish the normality part of the proof]**.

### ■ 3.5 The Étale Fundamental Group.

Fix a connected scheme  $X$ , and fix once and for all a geometric point  $\bar{x} : \text{Spec } K \rightarrow X$ . Let  $\mathbf{FinÉt}_{/X}$  be the category of finite étale covers  $Y \rightarrow X$ . More specifically,  $\mathbf{FinÉt}_{/X}$  is the full subcategory of  $\mathbf{Sch}_{/X}$  whose objects are maps  $Y \rightarrow X$  that are finite, étale, and surjective. Notice that the domain  $Y$  of each étale cover in  $\mathbf{FinÉt}_{/X}$  need not be connected, and can decompose as  $Y \cong \bigsqcup_i Y_i$  in such a way that no restriction  $Y_i \rightarrow X$  need be a cover, even though it's étale.

Since each cover  $Y \rightarrow X$  is finite, the fiber  $Y_{\bar{x}}$  is a finite disjoint union of singleton spectra  $\{\bar{y}_i\} \cong \text{Spec } K$ , and we define  $F(Y \rightarrow X) := \{\bar{y}_1, \dots, \bar{y}_n\}$  (we denote  $F(Y)$  for short) to be the set of points in  $Y_{\bar{x}}$ . A map

$$\begin{array}{ccc} Y_1 & \longrightarrow & Y_2 \\ & \searrow & \swarrow \\ & X & \end{array}$$

of covers in  $\mathbf{FinÉt}_{/X}$  induces a map  $Y_{1, \bar{x}} \rightarrow Y_{2, \bar{x}}$  functorially, so that  $F$  becomes a functor

$$F : \mathbf{FinÉt}_{/X} \longrightarrow \mathbf{Sets}$$

that we call  $X$ 's *fiber functor* (at  $\bar{x}$ ).

We define  $X$ 's *étale fundamental group*  $\pi_1(X, \bar{x})$ , based at the geometric point  $\bar{x}$ , to be simply the automorphism group of the fiber functor. That is, letting  $\mathbf{PSh}$  be the category of functors  $\mathbf{FinÉt}_{/X}^{\text{op}} \rightarrow \mathbf{Sets}$ , we define

$$\pi_1(X, \bar{x}) := \text{Aut}_{\mathbf{PSh}}(F).$$

By the very definition of an automorphism of a functor as a natural transformation, we see that every set  $F(Y)$  in  $F$ 's image comes equipped with a canonical  $\pi_1(X, \bar{x})$ -action and that every function  $F(Y_1) \rightarrow F(Y_2)$  induced by a map of covers is a  $\pi_1(X, \bar{x})$ -set morphism with respect to these canonical actions. Thus  $F$  actually factors through a canonical functor

$$F : \mathbf{Fin\acute{E}t}_{/X}^{\text{op}} \rightarrow \pi_1(X, \bar{x})\text{-Sets}.$$

**Theorem.** — (i)  $\pi_1(X, \bar{x})$  is a profinite group;

(ii)  $F$  induces an equivalence  $\mathbf{Fin\acute{E}t}_{/X} \simeq$  category of finite sets with continuous  $\pi_1(X, \bar{x})$ -action.

• Example 1:  $X := \text{Spec } k$ , with  $k$  a field;

Then  $\pi_1(X, \bar{x}) = \text{Gal}(k^s/k)$ ;

The main theorem becomes the Fundamental Theorem of Galois Theory;

• Example 2:  $X :=$  connected, locally path connected, semi-locally simply connected;

If the main theorem held, then we would have  $\pi_1(X, \bar{x}) = \widehat{\pi_1(X, x)}$ ;

The main theorem becomes the classification of covering spaces;

• Proof of Main Theorem in 4 Parts:

Outline: **Part 1:** Realization of  $F$  as an ind-scheme;

**Part 2:** Proof of (i)  $\pi_1(X, \bar{x})$  is a profinite group;

**Part 3:**  $F$  factors through  $F' : \mathbf{Fin\acute{E}t}_{/X} \rightarrow \{\text{finite sets with continuous } \pi_1(X, \bar{x})\text{-action}\}$ ;

**Part 4:**  $F'$  is an equivalence of categories;

• **Part 1:** Realization of  $F$  as an ind-scheme:

• Define: *Galois covers*

- Explain:
  - Galois covers  $\sim$  covering spaces with simple + transitive deck action;
  - They will provide us with a cofiltered system realizing  $F$  as an ind-scheme;
  - We need to construct Galois covers with a certain "semi-universal" property;
  - The construction depends on some facts about  $\mathbf{Fin\acute{E}t}_{/X}$ :

• Preliminaries on  $\mathbf{Fin\acute{E}t}_{/X}$

- Claim:  $Y \rightarrow X$  finite  $\Rightarrow$  proper;
- Corollary:  $Y \rightarrow X$  finite étale,  $X$  connected  $\Rightarrow Y \rightarrow X$  surjective;
  - Thus  $\mathbf{Fin\acute{E}t}_{/X}$  is a category of étale covers
- Claim:  $Y \rightarrow X$  finite, flat  $\Rightarrow \mathcal{O}_Y$  a locally free, finite rank  $\mathcal{O}_X$ -module;
- Corollary:  $Y \rightarrow X$  finite étale,  $X$  connected  $\Rightarrow Y \rightarrow X$  has a well defined *rank*;
  - We interpret this rank as  $Y$ 's "number of sheets" over  $X$
  - Counter Example:  $\mathbb{C}[t^{\pm 1}] \leftarrow \mathbb{C}[t^{\pm 2}] \Rightarrow$  not Zariski-local triviality of  $Y$  over  $X$ ;  
only Zariski-local triviality of  $\mathcal{O}_Y$  over  $X$ ;
- Corollary:  $Y \rightarrow X$  finite étale,  $X$  connected  $\Rightarrow \text{card } |Y_{\bar{x}}|$  is constant = rank;
- Claim:  $Y \rightarrow X$  finite étale  $\Rightarrow U \times_X Y \cong \sqcup_i U$  for some finite étale  $U \rightarrow X$ ;
  - Note that this is exactly Serre's observation that we cited on the first day;
- Corollary: To check that  $Y \rightarrow X$  is Galois,  $\text{Aut}_X(Y)$ -transitivity on one  $Y_{\bar{x}}$  suffices;

• Theorem: existence of Galois covers with "semi-universal" property;

- Heuristic construction for a covering space  $Y \rightarrow X$  of a connected, locally path...
  - $\mathcal{M} :=$  "moduli space" of marked fibers, covers  $X$ ;
  - $P :=$  any connected component of  $\mathcal{M}$ , covers  $X$ ;
  - Intuitive argument about why  $P \rightarrow X$  is Galois using  $\mathcal{M} \subset Y \times_X \cdots \times_X Y$ ;

- (Serre's) technical construction: difficulty comes from absence of  $\mathcal{M}$ ;
  - Fix fiber  $Y_{\bar{x}} \Rightarrow$  geometric point  $\bar{y} := (\bar{y}_1, \dots, \bar{y}_n)$  in  $Y \times_X \cdots \times_X Y$
  - $P :=$  connected component containing  $\bar{y}$ ;
  - Every point in  $P_{\bar{x}}$  is a permutation of  $(\bar{y}_1, \dots, \bar{y}_n)$ ;
    - $\Rightarrow P \rightarrow X$  Galois;
  - "semi-universal" property:  $Q \rightarrow X$  is Galois, factoring through  $Y \rightarrow X$ ;
    - Suffices to construct  $Q \rightarrow Y \times_X \cdots \times_X Y$  with non-empty intersection with  $P$ ;
    - Choose points  $\bar{q}_i \in Q_{\bar{x}}$  over  $\bar{y}_i \in Y_{\bar{x}}$  plus appropriate  $g_i \in \text{Aut}_X(Q)$
    - to construct  $Q \rightarrow Y \times_X \cdots \times_X Y$ . ■

### ■ Quotients by Group Actions in $\mathbf{Fin\acute{E}t}/X$ .

We cannot be as cavalier with the notion of group quotients of schemes as we can be with group quotients of topological spaces. Colimits are not well behaved in  $\mathbf{Sch}/\mathbb{Z}$ : they do not exist in general, and can be tricky to work with when they do. Yet with suitable restrictions on the nature of the group action, we can be assured that they exist.

Consider first the affine case. The anti equivalence  $\mathbf{Rng}^{\text{op}} \simeq \mathbf{Aff}$  reduces the affine case to that of group actions on commutative rings. Let  $A$  be any ring with a left  $G^{\text{op}}$ -action. It's easy to see that the ring

$$A^G := \{a \in A : g a = a \text{ for all } g \in G\}$$

of invariants satisfies the universal property opposite that of  $G \backslash \text{Spec } A$ . Specifically, every ring morphism  $\varphi : B \rightarrow A$  for which

$$g \varphi(-) = \varphi(-) \quad \text{for all } g \in G$$

factors uniquely through the inclusion  $A^G \hookrightarrow A$ . Thus

$$\text{Spec } A^G \cong G \backslash \text{Spec } A$$

within the category of affine schemes.

Note though that this does not state that  $\text{Spec } A^G$  is the quotient of  $\text{Spec } A$  in  $\mathbf{Sch}/\mathbb{Z}$ .

We want to argue that when  $Y \rightarrow X$  is a finite étale map and  $H$  is a subgroup of  $\text{Aut}_{\mathbf{Sch}/X}(Y)$ , there's an induced  $H$ -action on  $\mathcal{O}_Y$  for which have a similar isomorphism

$$\text{Spec}_X \mathcal{O}_Y^H \cong H \backslash Y$$

in the category of schemes, where  $\mathcal{O}_Y^H$  is the subsheaf of  $\mathcal{O}_Y$  consisting of all  $H$ -invariant sections (the specifics of the  $H$ -action on  $\mathcal{O}_Y$  come from the fact that  $\mathcal{O}_Y$  is nothing but the restriction to open subsets of  $Y$  of the representable functor  $\text{Hom}_{\mathbf{Sch}/X}(-, \mathbb{A}^1)$ ). Here the object  $H \backslash Y$  on the left is by definition, in a fixed category  $\mathbf{C}$  of schemes, the scheme (co-) representing the functor

$$\text{Hom}_{\mathbf{C}}(Y, -)^H,$$

where the induced  $H$ -action on the functor  $\text{Hom}_{\mathbf{C}}(Y, -)$  takes  $f(-) \mapsto f(h(-))$ . This definition of  $H \backslash Y$  is nothing but the statement that every map  $f : Y \rightarrow Z$  in  $\mathbf{C}$  factors uniquely through  $Y \rightarrow H \backslash Y$ , which in turn is nothing but the statement that  $H \backslash Y$  is the colimit of the diagram with one object  $Y$ , and with morphisms given by  $H$ 's action on  $Y$ .

The basic idea underlying our proof in  $\mathbf{C} = \mathbf{Fin\acute{E}t}/X$  is that despite the apparent complexity for arbitrary  $Y$  in  $\mathbf{Fin\acute{E}t}/X$ , we're really still in the affine case. Realize first that the one-line affine proof generalizes trivially to the case of  $\text{Spec } A$ -relative  $G$ -actions on  $\text{Spec } B$ , using the anti-equivalence  $A\text{-Alg}^{\text{op}} \simeq \mathbf{Aff}/_{\text{Spec } A}$ . In fact, it is not affineness of the base scheme that is crucial here, but only affineness in the "vertical direction:" over any base scheme  $X$ , we have a similar anti-equivalence  $\mathcal{O}_X\text{-QCoh}^{\text{op}} \simeq \mathbf{Aff}/_X$  between the category  $\mathcal{O}_X\text{-QCoh}$  of quasi-coherent  $\mathcal{O}_X$ -algebras and the category  $\mathbf{Aff}/_X$  of affine maps  $Y \rightarrow X$ . We call the contravariant functor  $\mathcal{O}_X\text{-QCoh} \rightarrow \mathbf{Aff}/_X$  establishing the equivalence the *X*-relative spectrum, and denote it  $\text{Spec}_X(-)$ . Using this anti-equivalence, we can immediately observe the following theorem.

**Theorem.** — Let  $Y$  be an affine  $X$ -scheme with the  $X$ -relative action of a group  $H$ . Then  $\mathrm{Spec}_X \mathcal{O}_Y^H \cong H \backslash Y$  where the scheme at right is the quotient in the category  $\mathbf{Aff}/X$ .

**Proof:** The proof is formally identical to the observation we made in **Aff**. Namely, the subsheaf  $\mathcal{O}_Y^H \subset \mathcal{O}_Y$  is a quasi-coherent  $\mathcal{O}_X$ -algebra with the universal property in  $\mathcal{O}_X$ -**QCoh** dual to that of  $H \backslash Y$  in  $\mathbf{Aff}/X$ . ■

The Theorem is not as strong as one might hope: it does not say that  $\mathrm{Spec}_X \mathcal{O}_Y^H$  is the quotient of  $Y$  by  $H$  in  $\mathbf{Sch}/Z$ . But we will see that for our purposes it suffices. Since the category  $\mathbf{FinÉt}/X$  is a full subcategory of  $\mathbf{Aff}/X$ , we see that this last Theorem will immediately imply existence of quotients in  $\mathbf{FinÉt}/X$  as long as we can establish that the quotient  $H \backslash Y \rightarrow X$  in  $\mathbf{Aff}/X$  is finite étale whenever  $Y \rightarrow X$  is. For this we need a lemma:

**Lemma.** — If  $B' \leftarrow A$  is a flat algebra and  $B \leftarrow A$  is an algebra representing a finite group  $H$ , then

$$B' \otimes_A B^H \cong (B' \otimes_A B)^H.$$

**Proof:** Let  $B \rightarrow \bigoplus_{g \in G} B$  be the morphism of  $B$ -modules taking  $b \mapsto \bigoplus_{g \in G} (b - gb)$ . Then the sequence

$$0 \rightarrow B^H \hookrightarrow B \rightarrow \bigoplus_{g \in G} B \quad \text{is exact.}$$

Thus flatness of  $B' \leftarrow A$  implies that

$$0 \rightarrow B' \otimes_A B^H \hookrightarrow B' \otimes_A B \rightarrow \bigoplus_{g \in G} B' \otimes_A B \quad \text{is exact.}$$

Since the map at right in this last sequence takes  $b' \otimes b \mapsto \bigoplus_{g \in G} b' \otimes (b - gb) = \bigoplus_{g \in G} (b' \otimes b - b' \otimes gb)$ , its kernel is  $(B' \otimes_A B)^H$ , and thus  $B' \otimes_A B^H \cong (B' \otimes_A B)^H$ . ■

**Theorem.** — If  $Y \rightarrow X$  is finite étale and  $H$  is a subgroup of  $\mathrm{Aut}_{\mathbf{Sch}/X}(Y)$ , then  $H \backslash Y \rightarrow X$  (the quotient in  $\mathbf{Aff}/X$ ) is finite étale, hence also the quotient in  $\mathbf{FinÉt}/X$ .

**Proof:** We need to check three things. Namely, that:

- (i) if  $Y \rightarrow X$  is finite, then  $H \backslash Y \rightarrow X$  is finite;
- (ii) if  $Y \rightarrow X$  is flat, then  $H \backslash Y \rightarrow X$  is flat;
- (iii) if  $Y \rightarrow X$  is unramified, then  $H \backslash Y \rightarrow X$  is unramified.

By the previous Lemma **[last lemma]**, it is enough to check these in the affine case, that is, when  $Y = \mathrm{Spec} B$  over  $X = \mathrm{Spec} A$ .

**Part (i):** We can further assume that  $B$  is a finitely generated  $A$  module. Since  $X$  is locally Noetherian by assumption, we can assume  $A$  Noetherian. Thus  $B$  is a Noetherian  $A$ -module, so its every  $A$ -submodule is finitely generated. In particular,  $B^H \subset B$  is finitely generated.

**Part (ii):** If  $B \leftarrow A$  is flat, then  $1 \otimes_A \varphi : B \otimes_A M' \rightarrow B \otimes_A M$  is injective whenever  $\varphi : M' \hookrightarrow M$  is. If  $B^H \leftarrow A$  is not flat, then  $\sum_i b_i \otimes \varphi(m_i) = 0$  for coefficients  $b_i \in B^H$  and elements  $m_i \in M'$  such that some  $\varphi(m_i) \neq 0$ , contradicting flatness of  $B$ , since  $B^H \subset B$ .

**Part (iii):** Let  $\bar{x} : \mathrm{Spec} K \rightarrow X$  be any geometric point, factoring through a point  $x \in X$ . Since  $\mathcal{O}_{X,x} \leftarrow \mathcal{O}_X$  is flat, we can restrict attention to the coherent  $\mathcal{O}_{X,x}$ -algebra  $B := \mathcal{O}_{X,x} \otimes_X \mathcal{O}_Y$ . Suppose  $b \neq hb$  for  $b \in B$  and  $h \in H$ . Then by Nakayama's Lemma, it is not possible to have  $b = hb$  in  $k(x) \otimes B$ . Thus  $(k(x) \otimes B)^H \cong k(x) \otimes B^H$ . But now  $\bar{x} : \mathrm{Spec} K \rightarrow \mathrm{Spec} k(x)$  is trivially flat, so this implies, by the previous Lemma **[last lemma]**, that  $(K \otimes B)^H \cong K \otimes B^H$ . Yet  $K \otimes B$  is the coordinate algebra of the geometric fiber  $Y_{\bar{x}}$ . Since  $Y \rightarrow X$  is unramified, it is of the form  $K \times \cdots \times K$  over  $K$ , and so  $H$  can only act on it by permutation of the factors. Since  $(K \otimes B)^H$  is the quotient  $H \backslash Y_{\bar{x}}$  in  $\mathbf{Aff}/\mathrm{Spec} K$ , this makes it clear that  $H \backslash Y_{\bar{x}}$  is of the form  $\bigsqcup_i \mathrm{Spec} K$ , indexed by the orbits of  $H$  in  $Y_{\bar{x}}$ . Since  $K \otimes B^H \cong (H \backslash Y)_{\bar{x}}$  simultaneously, this implies that  $H \backslash Y$  is unramified. ■

**Corollary.** — The fiber functor  $F : \mathbf{FinÉt}/X \rightarrow \mathbf{Sets}$  preserves quotients. More specifically, if  $Y \rightarrow X$  is finite étale,  $H \subset \mathrm{Aut}_{\mathbf{Sch}/X}(Y)$ , then  $F(H \backslash Y) \cong H \backslash F(Y)$ , the  $H$ -action by which we're quotienting on the right being the induced action on  $Y_{\bar{x}}$ .

**Proof:** This is nothing but the observation  $(H \backslash Y)_{\bar{x}} \cong H \backslash Y_{\bar{x}}$  that we made in the course of Part (iii) of the previous proof. ■

**Corollary.** — If  $P \rightarrow Y$  is finite étale over  $X$ , with  $Y$  connected and  $P$  Galois over  $X$ , and if  $H := \text{Aut}_{\text{Sch}/Y}(P)$ , then  $H \backslash P \cong Y$ .

**Proof:** Since  $\mathbf{FinÉt}_Y$  is a full subcategory of  $\mathbf{FinÉt}_{/Y}$ , we can take the quotient in  $\mathbf{FinÉt}_{/Y}$ . Since  $P$  is Galois,  $H \backslash F(P)$  is a singleton, and by the previous Corollary [last corollary],  $H \backslash F(P) \cong F(H \backslash P)$ . Thus  $P \rightarrow Y$  is order-1, and so by [find previous result above], is an isomorphism. ■

**Corollary.** — If  $P \rightarrow Y$  is finite étale over  $X$ , with  $Y$  connected and  $P$  Galois, then the induced  $F(P) \rightarrow F(Y)$  is surjective.

**Proof:** We actually proved this in the course of the last Corollary [last corollary]. ■

### Continuation of Proof:

Recall that we're working with a fixed connected scheme  $X$  with a choice of geometric point  $\bar{x} : \text{Spec } K \rightarrow X$ .

**Definition.** — Let  $\mathbf{Gal}_{\text{pt}/X}$  be the category whose objects are Galois covers  $P \rightarrow X$  equipped with a choice of geometric point  $\bar{p} : \text{Spec } K \rightarrow X$  over  $\bar{x}$ , that is, fitting into a commutative diagram

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\bar{p}} & P \\ & \searrow \bar{x} & \downarrow \\ & & X \end{array}$$

We will also sometimes denote objects in  $\mathbf{Gal}_{\text{pt}/X}$  as pairs  $(P, \bar{p})$ . Morphisms in  $\mathbf{Gal}_{\text{pt}/X}$  are commutative diagrams

$$\begin{array}{ccc} & \text{Spec } K & \\ \bar{p}_1 \swarrow & & \searrow \bar{p}_2 \\ P_1 & \longrightarrow & P_2 \\ \swarrow & & \searrow \\ & X & \end{array}$$

Note that since a map of finite étale spaces over  $X$  takes its complete determination from its action on a single geometric point, every homset  $\mathbf{Gal}_{\text{pt}/X}$  is either empty or a singleton.

The category  $\mathbf{Gal}_{\text{pt}/X}$  comes with an obvious functor  $U : \mathbf{Gal}_{\text{pt}/X} \rightarrow \mathbf{FinÉt}_{/X}$  that forgets the points  $\bar{p}$ . Note that this functor is not an inclusion in general: if  $n := |P_{\bar{x}}|$ , then there are  $n$ -many distinct objects  $(P, \bar{p})$  in  $\mathbf{Gal}_{\text{pt}/X}$  with image  $P$  in  $\mathbf{FinÉt}_{/X}$ .

**Claim.** — The category  $\mathbf{Gal}_{\text{pt}/X}$  is cofiltered.

**Proof:** Let  $(P_1, \bar{p}_1)$  and  $(P_2, \bar{p}_2)$  be objects in  $\mathbf{Gal}_{\text{pt}/X}$ . Form  $P_1 \times_X P_2$  and let  $Y \subset P_1 \times_X P_2$  be any connected component in it. Then  $Y$  admits a canonical Galois cover  $P$ , and we get a span

$$\begin{array}{ccc} & P & \\ \swarrow & & \searrow \\ P_1 & & P_2 \end{array}$$

Pick any point  $\bar{p} \in P_{\bar{x}}$ . Since  $P_1$  and  $P_2$  are Galois, we can furthermore use automorphisms of  $P_1$  and  $P_2$  to make sure that this span takes  $\bar{p}$  to  $\bar{p}_1$  and  $\bar{p}_2$  in  $P_1$  and  $P_2$ , respectively.

The fact that nonempty homsets in  $\mathbf{Gal}_{\text{pt}/X}$  are singletons makes the " $\bullet \implies \bullet$ " part of cofinality trivial. ■

**Claim.** — The fiber functor  $F$  is isomorphic to the ind-scheme (or "co-pro-scheme")

$$\varinjlim_{\mathbf{Gal}_{\text{pt}/X}} \text{Hom}_{\text{Sch}/X}(U(P_i, \bar{p}_i), -).$$

**Proof:** Since  $\mathbf{Gal}_{\text{pt}/X}$  is cofiltered, this limit is an ind-scheme (by definition). The real point is to check that this limit is isomorphic to  $F$ . For this we need a lemma:

**Lemma.** — Let  $Y$  be connected and finite étale over  $X$ . If  $P$  is Galois over  $X$ , with  $P \rightarrow X$  factoring through  $Y \rightarrow X$ , then  $\text{Hom}_{\text{Sch}/X}(P, Y) \cong F(Y)$  (canonically with choice of point  $\bar{p}$  in  $P_{\bar{x}}$ ).

**Proof:** Fix  $\bar{p}$  in  $P_{\bar{x}}$ , and define a function  $\text{Hom}_{\text{Sch}/X}(P, Y) \rightarrow F(Y)$  taking  $f \mapsto f \bar{p}$ . Fix a particular map  $f_0 : P \rightarrow Y$ . By Corollary [ ] above, the induced  $f_0 : P_{\bar{x}} \rightarrow Y_{\bar{x}}$  is surjective. Thus since  $P$  is Galois, we can find an automorphism  $g \in \text{Aut}_{\text{Sch}/X}(P)$  such that  $f_0 g \bar{p} = \bar{y}$  for any given  $\bar{y} \in Y_{\bar{x}}$ . ■



**Claim.** —  $F$  is essentially surjective.

**Proof:** Let  $S$  be a finite set with continuous  $\pi_1(X, \bar{x})$ -action. Decomposing  $S$  into its  $\pi_1(X, \bar{x})$  orbits, we can assume that  $\pi_1(X, \bar{x})$  acts transitively on  $S$ . Fixing  $s \in S$ , the continuity of the  $\pi_1(X, \bar{x})$ -action becomes the statement that  $s$ 's stabilizer is an open subgroup  $U \subset \pi_1(X, \bar{x})$ , and we have  $S \cong \pi_1(X, \bar{x})/U$  as a set.

Letting  $\varphi_i : \pi_1(X, \bar{x}) \rightarrow \text{Aut}_{\text{Sch}/X}(P_i)$  be the arbitrary structural projection for an arbitrary Galois cover  $P_i \rightarrow X$ , we know that the open subgroups of the form  $U_i := \varphi_i^{-1}(1)$  form a basis of open neighborhoods of the identity in  $\pi_1(X, \bar{x})$ . Furthermore, each  $U_i$  is normal in  $\pi_1(X, \bar{x})$  since for each  $u \in U_i$  and each  $g \in \pi_1(X, \bar{x})$ , we have

$$\varphi_i(g^{-1}ug) = \varphi_i(g)^{-1} \varphi_i(u) \varphi_i(g) = \varphi_i(g)^{-1} \varphi_i(g) = 1.$$

In this way we obtain a normal open subgroup  $U_i \triangleleft U \subset \pi_1(X, \bar{x})$ , and hence a subgroup

$$H := U/U_i \subset \pi_1(X, \bar{x})/U_i \cong \text{Aut}_{\text{Sch}/X}(P_i).$$

From this, in combination with the fact that  $\text{Aut}_{\text{Sch}/X}(P_i) \cong F(P_i)$  (Corollary □ and Corollary □), we obtain

$$S \cong \frac{\pi_1(X, \bar{x})}{U} \cong \frac{\pi_1(X, \bar{x})/U_i}{U/U_i} \cong \frac{\text{Aut}_{\text{Sch}/X}(P_i)}{H} \cong \frac{F(P_i)}{H} \cong F(H \backslash P_i),$$

where the last identity comes from Corollary □. ■

**Claim.** —  $F$  is fully faithful.

**Proof:** Without loss of generality, we can restrict attention to homsets  $\text{Hom}_{\text{Sch}/X}(Y, Z)$  for connected  $Y$ , since  $F(Y_1 \amalg Y_2) \cong F(Y_1) \amalg F(Y_2)$ .

Faithfulness of  $F$  is then an immediate consequence of the fact that every map  $Y \rightarrow Z$  over  $X$  in  $\mathbf{FinÉt}_X$  takes its complete determination from its action on a single geometric point  $\bar{y}$  in  $Y_{\bar{x}}$ .

For fullness of  $F$ , we want to argue first that  $F(Y)$  contains a point  $\bar{y}_1$  such that every morphism  $F(Y) \rightarrow F(Z)$  of finite, continuous  $\pi_1(X, \bar{x})$ -sets takes its unique determination from its action on  $\bar{y}_1$ . Since  $F(Y) \cong \pi_1(X, \bar{x})/U$  as a set for some (not necessarily normal) open subgroup  $U \subset \pi_1(X, \bar{x})$ , we define  $\bar{y}_1 \in F(Y)$  to be the image of 1 in  $\pi_1(X, \bar{x})/U$ .

Let  $\varphi : F(Y) \rightarrow F(Z)$  be an arbitrary  $\pi_1(X, \bar{x})$ -set morphism. Every other  $\bar{y} \in F(Y)$  has a representative  $g\bar{y}_1$  for  $g \in \pi_1(X, \bar{x})$ . Thus since  $\varphi$  is a  $\pi_1(X, \bar{x})$ -set morphism, we have

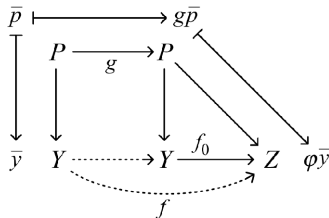
$$\varphi\bar{y} = \varphi g\bar{y}_1 = g\varphi\bar{y}_1,$$

and  $\varphi$  takes its complete determination from the image of  $\bar{y}_1$ .

This lets us reduce the demonstration that  $F$  is full to a demonstration that for every continuous  $\pi_1(X, \bar{x})$ -set morphism  $\varphi : F(Y) \rightarrow F(Z)$ , there's a scheme a map  $f : Y \rightarrow Z$  over  $X$  taking

$$\bar{y}_1 \xrightarrow{f} \varphi\bar{y}_1.$$

If we replace  $Y$  with its canonically associated Galois cover  $\psi : P \rightarrow Y$ , then this last claim follows trivially from the definition of Galois covers. For the general case, we need the following lemma and its corollary:



**Lemma (Galois-type "Fixed-Field Conjugacy" for Invariant Subschemes).** — If  $G$  is a finite automorphism

group acting on some affine  $Y \rightarrow X$ , then  $g\mathcal{O}_Y^H \cong \mathcal{O}_Y^{g^{-1}Hg}$  in  $\mathcal{O}_X$ -QCoh.

**Proof:** If  $g^{-1}hg s = s$ , then  $hg s = gs$ , and conversely. ■

**Corollary.** — If  $H$  is any subgroup in  $\text{Aut}_{\text{Sch}/Z}(P)$ , then  $H \backslash P$  is canonically isomorphic to  $g^{-1}Hg \backslash P$  in  $\text{FinÉt}/X$  along a map fitting into the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{g} & P \\ \downarrow & & \downarrow \\ H \backslash P & \rightarrow & g^{-1}Hg \backslash P. \end{array}$$

**Proof:** Immediate consequence of the last Lemma via previous observations. ■

**Completion of Proof of Claim [ ]:** Since  $F(P) \rightarrow F(Y)$  is surjective, we can find  $\bar{p}$  in  $P_{\bar{x}}$  over  $\bar{y}_1$  along with a map  $\tilde{f} : P \rightarrow Z$  taking  $\bar{p} \mapsto \varphi \bar{y}_1$ . In general through, the automorphism group of  $P$  over this new map  $P \rightarrow Z$  will differ from the automorphism group of  $P$  over the original

$$P \xrightarrow{\psi} Y \rightarrow Z$$

will differ by conjugation. Thus we can factor  $\tilde{f}$  canonically through a copy of  $H \backslash P$  "of  $Y$ ," and to obtain the map  $f : Y \rightarrow Z$  we're really after, we have have to ... **[make sure this last part is stated clearly]**