Notes on Étale Cohomology.

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2. First Properties of Étale Maps

We want to focus now on some of the basic properties of étale maps. In particular, we want to investigate the properties that make étale maps $U \longrightarrow X$ a "workable" replacement for Zariski open inclusions $U \hookrightarrow X$. This will include a verification of the properties that let étale maps form the basis for a new *Grothendieck pre-topology* for schemes, but it will also include some other important, related properties that will become important when we study the étale fundamental group in the next section.

The first, trivial observation is:

Observation 2.1.1 — Every (Zariski) open immersion $i : U \hookrightarrow X$ is étale. ■

In particular, every isomorphism $U \tilde{\to} X$ is étale.

We also want to point out that every étale map is open. Since étale maps are flat and locally of finite type, we can reduce this to the statement that every flat map $Y \longrightarrow X$ (of locally Noetherian schemes) that's locally of finite type is open. Some readers might recognize this as an exercise in Hartshorne [AG, §III.9, Ex. 9.1, p. 266]. It's not a simple exercise though. It's pretty involved. To prove it, we need to set up two preliminary definitions, a then a sequence of lemmas, theorems, and corollaries. I've labeled them "***ÉtOp-n***" to try to isolate this line of development.

ÉtOp-1 **Definition.** — Given a scheme X, let $\mathrm{Con}(X)$ be the smallest family of subsets of $|X|$ such that

(i) every open $U \subset |X|$ is in $\mathrm{Con}(X)$;
(ii) if $S_1, S_2 \in \mathrm{Con}(X)$, then $S_1 \bigcap S_2 \in \mathrm{Con}(X)$;
(iii) if $S \in \mathrm{Con}(X)$, then $|X| \setminus S \in \mathrm{Con}(X)$.

We call any element of $\mathrm{Con}(X)$ a *constructible subset* of X.

Think of constructible subsets of X as those subsets that a finite set-theoretical language can describe starting with the data of the Zariski topology.

The other definition we need is that of stability of specializations and generizations:

ÉtOp-2 **Definition.** — Let $x_0, x_1 \in X$, and suppose $x_1 \in \overline{\{x_0\}}$. Then we call x_1 a *specialization* of x_0, and we call x_0 a *generization* of x_1 (I don't understand why it's not called "generalization," but whatever).

Given a set $S \subset |X|$, we say that S is *stable under specialization* if every specialization of every point in S is also in S, and we say that S is *stable under generization* if every generization of every point in S is also in S.

ÉtOp-3 **Lemma.** — Let X be any scheme, and $S \subset |X|$ any subset. Then:
(i) S is Zariski closed if and only if it is constructible and stable under specialization;
(ii) S is Zariski open if and only if it is constructible and stable under generization.

Proof: **(i).** Let $S \subset |X|$ be closed. Then it's obviously constructible. If $x \in S$, then $\overline{\{x\}} \subset \overline{S} = S$, so S is stable under specialization.

Conversely, suppose S constructible and stable under specialization. We want to show that $S = \overline{S}$, so let $x \in \overline{S}$. Then every Zariski open $U \ni x$ has nonempty intersection $U \bigcap S$. In particular, every Zariski open affine $U = \operatorname{Spec} A \ni x$ does. Since $U \bigcap \overline{S}$ is the same thing as the closure of $U \bigcap S$ in U, this means that there is a prime $\mathfrak{q} \subset A$, defining a point in $U \bigcap S \neq \emptyset$, such that x is a prime $\mathfrak{p} \supset \mathfrak{q}$ in $\operatorname{Spec} A$. Indeed, the closure $U \bigcap S$ in U is the set of all primes in A containing primes in $U \bigcap S$). Since we assume S stable under specialization, this means that $\mathfrak{p} \in S$.

(ii). Using part **(i)**, all we need to show is that if S if constructible and stable under generization, then its compliment is constructible and stable under specialization. Of couse its compliment is constructible.

Let $V := |X| \backslash S$, let $v \in V$, and suppose $x \in \overline{\{v\}}$, but that $x \notin V$. Then $x \in S$. But S is stable under <u>generization</u>, so this puts $v \in V$, a contradiction. ■

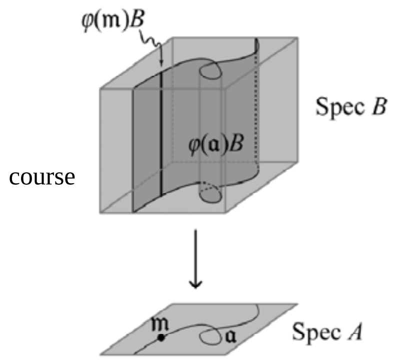
ÉtOp-4 **Theorem (Chevalley).** — If $f : Y \to X$ is a map of (locally Noetherian) schemes, locally of finite type, then f takes constructible subsets of Y to constructible subsets of X.

Proof: See either Exposé 7 of [*Géometrie Algébrique*, Séminaire Cartan-Chevalley. Secrétariat Math., Paris (1955/56)], or [Matsumura, *Commutative Algebra*. Chp. 2, §6.E, Thm. 6, p. 42] (the former is hard to find… Here's something interesting, from Wikipedia: "*the Séminaire Cartan-Chevalley of the academic year 1955/6* […] *dealt with topics on algebraic groups and the foundations of algebraic geometry, as well as pure abstract algebra. The Cartan-Chevalley seminar was the genesis of scheme theory, but its subsequent development in the hands of Alexander Grothendieck was so rapid, thorough and inclusive that its historical tracks can appear well covered. Grothendieck's work subsumed the more specialised contribution of Serre, Chevalley, Goro Shimura, and others such as Kähler and Masayoshi Nagata*.")

ÉtOp-5 **Lemma.** — If $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is flat and its image contains $max\operatorname{Spec} A$, then $A \longrightarrow B$ is injective.

Proof: Suppose $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ flat and surjective, and let it be dual to $\varphi : A \to B$.

We're first going to prove something more general, namely that if $N \subset M$ be a nonzero submodule of any given A-module M, then $B \otimes_A N \neq 0$. To see that this is so, fix $s \in N$ and consider $As \cong A/\mathfrak{a} \subset M$. Since $B \otimes_A A/\mathfrak{a} \cong B/\varphi(\mathfrak{a})B$, to have $B \otimes_A N = 0$ we would have to have $\varphi(\mathfrak{a})B = B$. But since $max\operatorname{Spec} A$ is in the image of $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$, choosing any maximal ideal $\mathfrak{m} \supset \mathfrak{a}$ in A we're aware that the fiber $\operatorname{Spec} B/\varphi(\mathfrak{m})\, B$ of $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ over $\mathfrak{m} \in \operatorname{Spec} A$ is nonempty.



(To realize that $B/\varphi(\mathfrak{m})\, B$ is the coordinate ring of the fiber, note that $\varphi^{-1}(\mathfrak{p}) = \mathfrak{m}$ implies $\mathfrak{p} \supset \varphi(\mathfrak{m})\, B$). In other words, $\varphi(\mathfrak{m})\, B \neq B$. Since $\varphi(\mathfrak{a}) \subset \varphi(\mathfrak{m})$, this implies that $B/\varphi(\mathfrak{a})B \neq 0$.

This proves the general claim of the previous paragraph. To apply it to the claim that $A \longrightarrow B$ is injective, note that a nontrivial kernel $\mathfrak{a} \hookrightarrow A$ is just such a nonzero submodule with $B \otimes_A \mathfrak{a} = 0$. ■

ÉtOp-6 **Corollary.** — Going-down holds for every flat algebra $B \longleftarrow A$.

Proof: By *going-down*, we mean that whenever we have the picture

$$\begin{array}{ccccccccccccc} B: & & & & & & \mathfrak{q}_n \subseteq \cdots & \subseteq & \mathfrak{q}_1 & \subseteq & \mathfrak{q}_0 \\ \uparrow & & & & & & | & \cdots & | & & | \\ A: & \mathfrak{p}_m \subseteq \cdots & \subseteq & \mathfrak{p}_{n+1} & \subseteq & & \mathfrak{p}_n \subseteq \cdots & \subseteq & \mathfrak{p}_1 & \subseteq & \mathfrak{p}_0, \end{array}$$

of primes in A and B, each $\mathfrak{p}_i$ being the preimage of $\mathfrak{q}_i$ in A (we say $\mathfrak{q}_i$ *lies over* $\mathfrak{p}_i$), we can find $\mathfrak{q}_m \subseteq \cdots \subseteq \mathfrak{q}_{n+1}$ in B lying over $\mathfrak{p}_m \subseteq \cdots \subseteq \mathfrak{p}_{n+1}$ to complete the picture.

The classical Cohen-Seidenberg Going Down Theorem says that we have going-down when A is integrally closed. With the present Lemma, we're asserting going-down when $B \longleftarrow A$ is flat.

Let $\mathfrak{p}' \subseteq \mathfrak{p}$ be primes in A, and suppose $\mathfrak{q}$ a prime in B lying over $\mathfrak{p}$. Since $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ is a flat map, we know that $B_\mathfrak{q} \longleftarrow A_\mathfrak{p}$ is flat. Since $B_\mathfrak{q}$ and $A_\mathfrak{p}$ are local, $max\operatorname{Spec} A_\mathfrak{q}$ is in the image of $\operatorname{Spec} B_\mathfrak{q} \longrightarrow \operatorname{Spec} A_\mathfrak{q}$ (the preimage of $B_\mathfrak{p}$'s maximal ideal has to be maximal in $A_\mathfrak{q}$), and so the last Lemma implies that $B_\mathfrak{q} \longleftarrow A_\mathfrak{p}$ is injective. Consider the ideal $\mathfrak{p}' A_\mathfrak{p} \subset A_\mathfrak{p}$. Since $B_\mathfrak{q} \longleftarrow A_\mathfrak{p}$ is injective, it extends to the ideal $\mathfrak{p}' B_\mathfrak{p} \subset B_\mathfrak{p}$ lying over $\mathfrak{p}' A_\mathfrak{p}$. Let $\mathfrak{q}' \subset B$ be the lift of $\mathfrak{p}' B_\mathfrak{p}$ along $B \longrightarrow B_\mathfrak{p}$. Then $\mathfrak{q}'$ is a prime lying over $\mathfrak{p}'$ [A&M, §3, Prop. 3.11.iv, p. 41]. ■

Finally, we can make the original observation we were after:

Observation 2.1.2 — Every étale map is open.

Proof: As we already pointed out, this reduces to the statement that that every flat map $f : Y \to X$ (of locally Noetherian schemes) that's locally of finite type is open. By the Theorem ***ÉtOp-4*** above, we already know that f preserves constructible sets, and thus by Lemma ***ÉtOp-3***, we only need to show that if $U \subset Y$ is open, then $f(U)$ is stable under generization.

Furthermore, since f is locally of finite type, it's not hard to see that we can reduce to the case that f is of the form $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ (this involves covering U by its intersections with the open affines coming from f's status as locally of finite type, and then restricting to a basic affine open $\operatorname{Spec} B$ in the domain), and can reduce our claim to the claim that $f(\operatorname{Spec} B)$ is stable under generization.

Since we're now in the affine picture, a generization $x_1 \in \overline{\{x_0\}}$ in $f(\operatorname{Spec} B) \bigcap \operatorname{Spec} A$ gives us the going-down diagram

$$\begin{array}{lccc} B: & & & y_1 \\ \uparrow & & & | \\ A: & x_0 & \subseteq & x_1 \end{array}$$

By Corollary ***ÉtOp-6***, we know that f satisfies going-down locally, so we can find $y_1 \in \operatorname{Spec} B$ giving the picture

$$\begin{array}{lccc} B: & y_0 & \subseteq & y_1 \\ \uparrow & | & & | \\ A: & x_0 & \subseteq & x_1. \end{array}$$

Thus x_0 is in the image of f. In particular, $x_0 \in f(\operatorname{Spec} B)$, which is to say that $f(\operatorname{Spec} B)$ is stable under generization, hence open since we already know it's constructible. ■

In particular, every étale map $Y \longrightarrow X$ is a "cover" of a Zariski open subset in X.

I will now begin to denote étale maps "$U \longrightarrow X$," since they will ultimately replace Zariski open immersions.

Observation 2.1.3 (Permanence Under Base Change). — The pullback $U_Y \longrightarrow Y$ of an étale map $p : U \to X$ along <u>any</u> scheme map $f : Y \to X$ is again étale.

Proof: This consists of two separate observations, namely:

(i) the pullback of a flat map remains flat;

(ii) the pullback of an unramified map remains unramified;

Claim **(i)** is familiar, but we prove it anyway. Since we construct the pullback locally, we can reduce claim **(i)** to the affine case: we replace the map $Y \longrightarrow X$ with an algebra $B \longleftarrow A$, replace the étale map $U \longrightarrow X$ with an algebra $A' \longleftarrow A$, and consider the pullback

$$\begin{array}{ccc} B \otimes_A A' & \leftarrow & A' \\ \uparrow & & \uparrow \\ B & \underset{\text{flat}}{\longleftarrow} & A. \end{array}$$

Any sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of A'-modules is also a sequence of A-modules. Suppose the sequence exact. The sequence

$$0 \longrightarrow (B \otimes_A A') \otimes_{A'} M' \longrightarrow (B \otimes_A A') \otimes_{A'} M \longrightarrow (B \otimes_A A') \otimes_{A'} M'' \longrightarrow 0$$

is isomorphic to the sequence $0 \longrightarrow B \otimes_A M' \longrightarrow B \otimes_A M \longrightarrow B \otimes_A M'' \longrightarrow 0$, and thus is exact since $A \longrightarrow B$ is flat by assumption. Thus $B \otimes_A A' \longleftarrow A'$ must be flat.

To verify claim **(ii)**, we can use Theorem 1.2.2 of the previous section and restrict our attention to the case that our étale map $U \longrightarrow X$ is of the form $U_x \cong \bigsqcup_i \operatorname{Spec} k_i \longrightarrow \operatorname{Spec} k(x)$, each k_i/k finite separable, where $Y \longrightarrow X$ is of the form $\operatorname{Spec} k(y) \longrightarrow \operatorname{Spec} k(x)$. But in this case, *every* composition

$$\operatorname{Spec} K \longrightarrow \operatorname{Spec} k(y) \longrightarrow \operatorname{Spec} k(x),$$

where $K/k(y)$ is separably closed, becomes a geometric point in $\operatorname{Spec} k(x)$. Thus by our geometric ramification criterion, our assumption that $U \longrightarrow \operatorname{Spec} k(x)$ is unramified implies that $K \otimes_{k(x)} U$ is unramified, and thus, again by our geometric ramification criterion, that $k(y) \otimes_{k(x)} U$ is. ■

Remark: One way to interpret this last theorem is by thinking of the pullback $U_Y := U \times_X Y$ of the étale $U \longrightarrow X$ along the arbitrary scheme map $f : Y \to X$ as being analogous to "$f^{-1}(U)$." From this perspective, the statement that $U_Y \longrightarrow Y$ remains étale becomes a *permanence property* comparable to those involved in statements like "f is continuous" (permanence of open immersions $U \hookrightarrow X$) or "f is measurable" (permanence of measurable subsets $U \subset X$).

Furthermore, since we've suggested from the very beginning that our interest in étale maps comes from an indication that they should replace Zariski open immersions, we can think of the stability of étale maps under base change as being something quite like the statement that "scheme maps aren't just continuous with respect to the Zariski topology, but are also 'étale-continuous'."

The next two observations regard a triple composition $Z \xrightarrow{f} Y \xrightarrow{g} X$ of scheme maps.

Observation 2.1.4 (Permanence Under Composition). — If g and f are étale, then gf is étale.

Proof: First, it's clear that if g and f are flat, then gf is itself flat: just reduce to the affine case of algebras $C \longleftarrow B \longleftarrow A$, and note that $C \otimes_A M \cong (C \otimes_B B) \otimes_A M \cong C \otimes_B (B \otimes_A M)$, etc.

So the main point is to establish that the composition of unramified maps is unramified. This is trivial if we use the "fiberwise separable" form of our ramification check (Theorem 1.2.2 of the previous section), noting that it gives us a pullback diagram

$$\begin{array}{ccc} \bigsqcup_{i,j} \operatorname{Spec} k_{ij} & \longrightarrow & Z \\ \downarrow & & \downarrow f \\ \bigsqcup_i \operatorname{Spec} k_i & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ \operatorname{Spec} k & \xrightarrow[x]{} & X \end{array}$$

where each k_i/k is finite separable and each $k_{ij}\big/k_i$ is finite separable. Since the compositions of finite separable extensions are finite separable, this implies that the composition gf is unramified. ■

The proof of the next observation is very similar to a proof we saw in Kapranov's class recently. We proved in his class that if g and gf are affine, then f is affine. Our observation in the current setting is instead:

Observation 2.1.5 — If g and gf are étale, then f is étale.

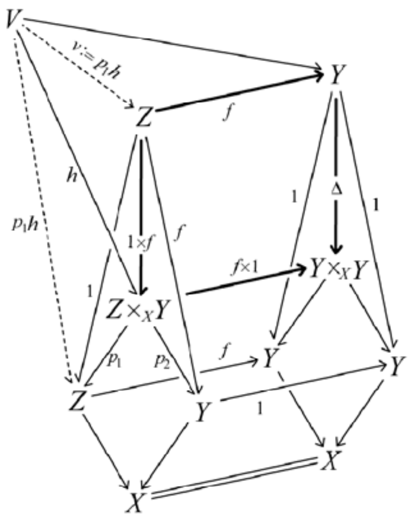
Proof: Recall that the *graph* $\Gamma_f : Z \to Z \times_X Y$ *of* f *over* X is simply $\Gamma_f := 1 \times f : Z \to Z \times_X Y$, so that we always have the factorization

$$f : Z \xrightarrow{\Gamma_f} Z \times_X Y \xrightarrow{p_2} Y.$$

If we can show that both Γ_f and p_2 are étale, then by the last Observation, we can conclude that f is étale. It's also already clear that p_2 is étale, since we already know that the pullback of any étale is étale, with p_2 being the pullback of of $g : Y \to X$ along $gf : Z \to X$ and .

Thus the claim rests on a demonstration that Γ_f is étale. We'll show that Γ_f is the pullback of $\Delta : Z \to Z \times_X Z$ along $f \times 1 : Z \times_X Y \to Z \times_X Z$, and that Δ is itself étale, from which étaleness of Γ_f follows by the last Observation.

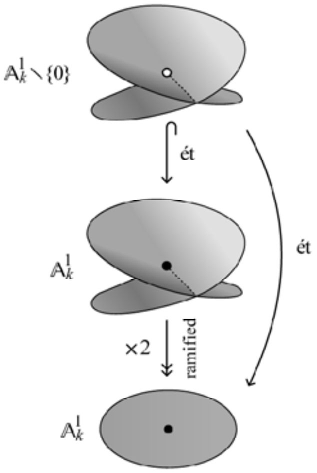
To see that Γ_f is the pullback of $\Delta : Z \to Z \times_X Z$ along $f \times 1 : Z \times_X Y \to Z \times_X Z$, consider the diagram



With a simple chase, one sees that the emboldened square in the center of the diagram is indeed cartesian.

Thus it only remains to argue that $\Delta : Z \to Z \times_X Z$ is étale. But $g\,f : Z \to X$ is étale by assumption, so by Observation 2.1.3 above combined with Corollary 1.2.8 of the previous section, the diagonal is a Zariski open immersion, and is thus étale (by the very first, trivial Observation we made in this §2). ■

The third observation that "if f and $g\,f$ are étale then g is étale" would give us a "two-out-of-three" axiom for étale maps fails. However it fails in general. Here's a counter example:



I'm not certain if there's a weaker version that is true. Perhaps it holds if one requires that $g\,f$ is surjective, or something along these lines.

Finally, although it won't become important until a bit later, we want to just collect some of these previous properties together and give a somewhat heuristic interpretation of each in familiar topological terms:

Proposition. — Let an *étale covering* of any given scheme X mean a set $\{p_i : U_i \longrightarrow X\}$ of étale maps such that $|X| = \bigcup_i p_i\,|U_i|$. Then the system of étale covers satisfies the following properties:

(i) Every singleton $\{U \xrightarrow{\sim} X\}$ consisting of an isomorphism is an étale cover (*objects themselves are "opens" in the same sense that the subset $X \subset X$ is always open in a topological space*);

(ii) If $\{U_i \longrightarrow X\}_{i\in I}$ is an étale cover, and $\left\{U_{ij} \longrightarrow U_i\right\}_{j\in J(i)}$ is an étale cover for each $i \in I$, then $\left\{U_{ij} \longrightarrow X\right\}_{i\in I,\, j\in J(i)}$ is an étale cover (*arbitrary accumulations of covers are covers, i.e., covers behave like open covers of a topological space*).

(iii) If $\{U_i \longrightarrow X\}$ is an étale cover and $f : Y \to X$ is an arbitrary scheme map, then $\{f^*U_i \longrightarrow Y\}$ is an étale cover (*scheme maps behave like "étale-continuous" functions in the same way that $f^{-1}\left(\bigcup_i U_i\right) = \bigcup_i f^{-1}(U_i)$ becomes an open cover for a continuous function between topological spaces*).

Proof: The proof is an easy application of the previous observations of this section. Work it out. ■