# Notes on Étale Cohomology

Tyler Foster

## 0. Introduction

### ■ Introduction: Why *Étale* Maps.

When we say "étale map," think "local diffeomorphism."

Introductions to étale cohomology, Grothendieck topologies, etc., start by trying to gesture at why Grothendieck came up with the *étale site* in the first place. These introductions tend to be vague, and fail to explain what it is about étale maps that suggested to Grothendieck's school that they might be able to get good Weil cohomology groups using them. The first thing I'm going to try to do is give a short, and less vague explanation.

In Exposé VII of SGA 4, Grothendieck comments that

> We can say that in passing from Zariski cohomology to the étale topology, we "did what we had to" in order to obtain "the right" $H^1$ when our coefficients come from a constant, finite group.

Here *Zariski cohomology* mean's the familiar sheaf cohomology theory on schemes.

First of all, the reason the Zariski cohomology groups $H^n(X, G)$, with coefficients in a finite abelian group $G$, are the "wrong" groups is because they give no nontrivial $G$-valued information about $X$. Indeed, it's clear that the constant $G$-valued sheaf on $X$ is always flasque, so by [AG, §III, Prop. 2.5, p. 208] for instance, we realize that $H^n(X, G) = 0$ for all $n \geq 1$.

Of course, concerning $H^1$, this only tells us that Zariski cohomology gives the wrong $H^1$. It doesn't say anything about why étale maps might help us find "the right" $H^1$.

To understand why they might, conflate "$H^1$" with "$\check{H}^1$" a bit, keeping in mind that Čech cohomology groups tend to match the groups of other cohomology theories in non-pathological situations. In familiar categories of geometric objects, the proof that $\check{H}^1$ classifies principal bundles is formal enough that it goes through under very mild hypotheses, so it seems reasonable to imagine that, for a fixed, finite abelian group $G$:

$$\text{"the right" } H^1(X, G) \text{ must classify principal } G\text{-bundles on } X.$$

Thus, if whatever $H^1$ we happen to be using isn't giving "the right" group $H^1(X, G)$, then $H^1(X, G)$ must either include elements that shouldn't qualify as principal $G$-bundle classes, or fail to include objects that should, or both. Since our particular problem with Zariski cohomology is one of a vanishing $H^1(-, G)$, we conclude that on a general scheme $X$ and for a general finite abelian group $G$, there must be reasonable candidates for "principal $G$-bundles" over $X$ that the Zariski topology can't detect as being such.

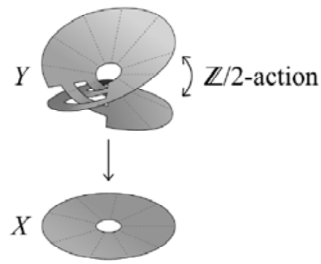
And this is where étale maps come in.

In a Chevalley seminar some time before 1958, Serre observed that by using a notion of local triviality based on étale maps (a notion he called *isotriviality*), he could recover at least a large family of bundles that from a geometric perspective should be principal, yet fail to be so with respect to the Zariski topology.

A very simple example of how this works is the following:

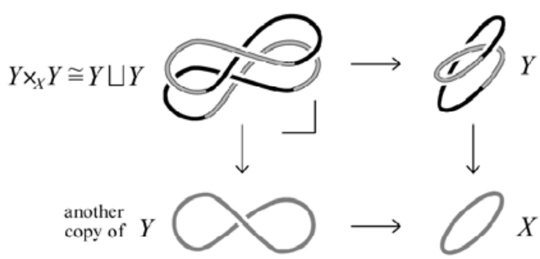
**Example: A "principal $\mathbb{Z}/2$-bundle" that isn't Zariski-locally trivial.** — Let $X := \operatorname{Spec} \mathbb{C}[x^{\pm 1}]$, let $Y := \operatorname{Spec} \mathbb{C}[x^{\pm 1}, y] / (y^2 - x)$, and let $Y \longrightarrow X$ be the obvious algebra. Then $Y$ is a degree-2 covering of $X$ with a $\mathbb{Z}/2$-action taking $y \mapsto -y$: it's just the punctured affine line wound twice over itself.

In the <u>complex</u> topology, the $\mathbb{Z}/2$-action of course turns $Y \longrightarrow X$ into a principal $\mathbb{Z}/2$-bundle over $X$:




But in the Zariski topology, this doesn't happen. A nonempty Zariski-open subset $U \subset X$ is simply the infinite compliment of a finite number of closed points in $X$, as is the restriction $Y_U$ of $Y$ to $U$, so that every Zariski-open set in $Y_U$ is itself the infinite compliment of a finite set of points in $Y$. On the other hand, to say that $Y_U$ becomes a trivial $\mathbb{Z}/2$-bundle over $U$ means that $Y_U \cong U \bigsqcup U$, each copy of $U$ open in $Y_U$, contradicting the fact that each copy of $U$ must be infinite. So $Y \longrightarrow X$ is "clearly supposed to be" a principal $\mathbb{Z}/2$-bundle, yet isn't one in the Zariski topology.

To see how étale maps (read "local diffeomorphisms") come into the picture, note first that since $Y(\mathbb{C}) \longrightarrow X(\mathbb{C})$ is a local diffeomorphism in the complex topology, it's easy to see from a purely complex-geometric perspective that pulling $Y(\mathbb{C})$ back along $Y(\mathbb{C}) \longrightarrow X(\mathbb{C})$ itself trivializes $Y$ as a $\mathbb{Z}/2$-bundle. I picture it like this:




In this way, we see that with a clever choice of a local diffeomorphic covering $U \twoheadrightarrow X(\mathbb{C})$, we can exhibit $Y(\mathbb{C})$'s locally trivial nature as a $\mathbb{C}$-manifold over $X(\mathbb{C})$ simply by pulling $Y(\mathbb{C})$ back along $U \twoheadrightarrow X(\mathbb{C})$, without any explicit recourse to open $\varepsilon$-balls or other topological machinery that we don't have in $\mathbf{Sch}_{/\mathbb{C}}$.

This suggests immediately that we try the same thing in the algebraic setting, i.e., that we ask if $Y \times_X Y$, the pullback of the bundle $Y$ along $Y \longrightarrow X$ itself as a $\mathbb{C}$-scheme, is also a trivial $\mathbb{Z}/2$-bundle over $Y$. Indeed it is. First, it's not hard to check that $Y \times_X Y = \operatorname{Spec} \mathbb{C}[u^{\pm 1}, v^{\pm 1}]/(u^2 = v^2)$, where $x = u^2 = y^2$. Let $A := \mathbb{C}[u^{\pm 1}, v^{\pm 1}]/(u^2 = v^2)$, let $\mathfrak{a} := (u + v)$, and let $\mathfrak{b} := (u - v)$. Then $\mathfrak{a} + \mathfrak{b} = A$, since $u$ and $v$ are invertible in $A$, and so we have a surjection $A \twoheadrightarrow A/\mathfrak{a} \times A/\mathfrak{b}$. Since $A/\mathfrak{a} \cong A/\mathfrak{b} \cong \Gamma(Y, \mathcal{O}_Y)$, this surjection induces a closed immersion $Y \amalg Y \hookrightarrow Y \times_X Y$ over $Y$. It's easy to see, investigating the situation fiberwise, that this immersion is in fact a bijection $|Y \amalg Y| \cong |Y \times_X Y|$ of the underlying sets.

We'll see in the next section that $Y \longrightarrow X$ satisfies the algebro-geometric version of being a local diffeomorphism, that of being *étale*, and we'll prove that one of the equivalent definitions of étale says, among other things, that the diagonal $\Delta : Y \to Y \times_X Y$ is an open immersion. If the reader looks more carefully at what's going on, he'll see that the diagonal is actually one component of our closed immersion $Y \amalg Y \hookrightarrow Y \times_X Y$, and thus a connected component of $Y \times_X Y$. Making use of the obvious $\mathbb{Z}/2$-action, we see then that both components of $Y \amalg Y$ must be connected components of $Y \times_X Y$, and thus the bijection $|Y \amalg Y| \cong |Y \times_X Y|$ implies that $Y \amalg Y \cong Y \times_X Y$ as schemes.

In particular, $Y$ pulls back to a trivial $\mathbb{Z}/2$-bundle over the étale covering $Y \twoheadrightarrow X$.

The argument is perhaps a little tedious at this early stage. The basic idea though is that this observation begins to justify a claim that $Y \longrightarrow X$ represents a class missing in the Zariski cohomology group $H^1(X, \mathbb{Z}/2) = 0$.

**Grothendieck's Angle:** Given these ideas, the first, most naive thing we might do is try to piece together "the right" $H^1$ of a given scheme $X$ directly using finite étale covers of $X$. This is roughly what Grothendieck does in his theory of the *Étale fundamental group* [SGA 1, book V]. We'll study the étale fundamental group later in these notes. Yet the group $\pi_1(X, \overline{x})$ Grothendieck obtained with this approach is homological rather than cohomological in nature.

Later on, Grothendieck's school took Serre's observation much further, interpreting it as a suggestion that étale maps should serve as the "open immersions" of a "topology" finer than the Zariski topology. This is quite strange because local diffeomorphisms need not be injective, so Grothendieck's "étale topology" can't be one of subset lattices.

Grothendieck's abstract generalization of the notion of a topology, its sheaves, and in particular the étale site, étale sheaves, and their resulting cohomology theory, étale cohomology, will be the larger goal of these notes, with a particular emphasis on the calculation of $H^1(X, \mathbb{Z}/n)$ for a $k$-curve $X$ when char $k \nmid n$.

# 1. Étale Maps

## ■ 1.0 Finiteness Conditions.

The first thing we're going to do is work our way through a few equivalent definitions of *étale*. We'll make significant use of several finiteness conditions for scheme maps in the process. In my opinion, this is the subtlest part of establishing the equivalences. For this reason, we put these finiteness conditions down here in a little glossary once-and-for-all.

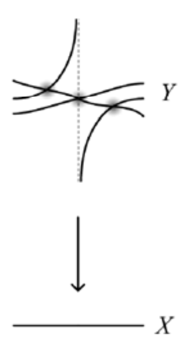
Fix a scheme map $f : Y \to X$. Actually, let me say something right here while I'm at it: since we're ultimately going to use étale maps as the "open immersions" of an abstract "topology," it will be the codomain of an étale map that we'll think of as the fixed space we're really interested in, so we're going to write étale maps at least as "$f : Y \to X$."

We'll make use of the following finiteness conditions:

**Finite.** — We say that an algebra $A \longrightarrow B$ is *finite* when $B$ is a finitely generated $A$-module (…that's $A$-module, not $A$-algebra: geometrically, this is the statement that Spec $B \longrightarrow$ Spec $A$ has finite order fibers).

We say that $f$ is *finite* if there exists a open affine covering $\bigcup_i U_i = X$ such that each $f^{-1}(U_i)$ is an open affine in $Y$ and such that each $\mathcal{O}_{U_i} \longrightarrow \mathcal{O}_{f^{-1}(U_i)}$ a finite algebra.

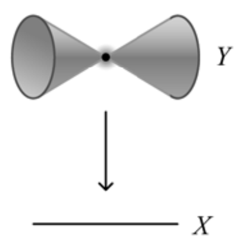
Cartoon of a finite scheme map:




**Finite Type.** — We say that an algebra $A \longrightarrow B$ is of *finite type* when $B$ is a finitely generated $A$-algebra (… geometrically, this is roughly the statement that Spec $B \longrightarrow$ Spec $A$ has finite dimensional fibers).

We say that $f$ is of *finite type* if there exists a open affine covering $\bigcup_i U_i = X$, and finite open affine coverings $\bigcup_j U_{ij} = f^{-1}(U_i)$ such that each algebra $\mathcal{O}_{U_i} \longrightarrow \mathcal{O}_{U_{ij}}$ is of *finite type*.

Cartoon of a finite type scheme map:



**Locally of Finite Type.** — We say that $f$ is *locally of finite type* if there exists a open affine covering $\bigcup_i U_i = X$, and (not necessarily finite) open affine coverings $\bigcup_j U_{ij} = f^{-1}(U_i)$, such that each algebra $\mathcal{O}_{U_i} \longrightarrow \mathcal{O}_{U_{ij}}$ is of finite type.

**Remark:** In general, the requirement of finiteness disallows for sudden jumps from non-empty to empty fibers, since the inversions required in the coordinate rings of "punctured" spaces wreck finite-generation as modules. For instance, the algebra $\mathbb{C}[t^{\pm 1}] \longleftarrow \mathbb{C}[t]$, dual to the Zariski open inclusion $\mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$, is not finite even though its every fiber is finite or empty. Indeed, given any Laurent polynomial

$$p(t) := a_{-n}\, t^{-n} + \cdots + a_{-1}\, t^{-1} + a_0 + a_1\, t + \cdots + a_m\, t^m,$$

the $\mathbb{C}[t]$-submodule $p(t)\,\mathbb{C}[t] \subset \mathbb{C}[t^{\pm 1}]$ will always lie in the proper $\mathbb{C}[t]$-submodule $t^{-n}\,\mathbb{C}[t] \subset \mathbb{C}[t^{\pm 1}]$, so that no finite set of Laurent polynomials $p_i(t) \in \mathbb{C}[t^{\pm 1}]$ can ever generate $\mathbb{C}[t^{\pm 1}]$ as a $\mathbb{C}[t]$-module.

Of course, $\mathbb{C}[t^{\pm 1}]$ is finitely as a $\mathbb{C}[t]$-algebra, so the inclusion $\mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$ is of finite type.

A very general observation along these lines is that if $f : Y \to X$ is not proper, then it is not finite. We prove this in §3.[_] below, where it becomes especially important. That non-proper maps can be locally of finite type but not finite is the main reason that we will require unramified maps to be locally of finite type rather than finite.

Finally, we will also make perpetual use of a certain finiteness condition on our topologies: **We will only work with *locally Noetherian* schemes.** Recall the definition:

**Locally Noetherian.** — A scheme $X$ is *locally Noetherian* if it admits a covering by open affines $U_i = \operatorname{Spec} A_i$ such that each $A_i$ is a Noetherian ring.

In this case, each $U_i$ is *quasi-compact*, though $X$ need not be quasi-compact globally (counter-example: an infinite, disjoint set of points isomorphic to some $\operatorname{Spec} k$).

## ■ 1.1 Separable Fields, Separable Algebras, and Ramification.

The formulation of the condition of a map $Y \longrightarrow X$ being *étale* that's easiest for me to understand intuitively focuses on the notion of *separability*, having to do with the multiplicity at points in the fibers of $Y \longrightarrow X$.

Recall that an algebraic field extension $K/k$ is *separable* if the minimal polynomial $p_a(t) \in k[t]$ of each element $a \in K$ has no roots of multiplicity higher than 1. It will become relevant later, to our geometric interests, that the condition that $p_a(t)$ have no multiple roots in $K$ is equivalent to the condition that the formal derivative $p_a'(t)$ of every minimal polynomial satisfies $p_a'(a) \neq 0$. This is the statement that the values $p_a(t)$ takes in $K$ pass linearly through $0 \in K$ at the point $(t - a) \in \mathbb{A}^1_K$.

For now we'll think about separability in terms of the multiplicity of roots. In the simple case that $K = k(a)$, we can consider the pullback diagram

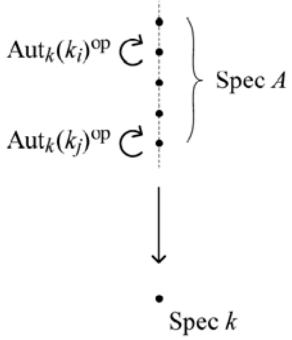
$$\begin{array}{ccc} \operatorname{Spec} \bar{k}[t]/(p_a) & \longrightarrow & \operatorname{Spec} K \\ \downarrow & & \downarrow \\ \operatorname{Spec} \bar{k} & \longrightarrow & \operatorname{Spec} k \end{array}$$

of affine schemes. From the geometric perspective, separability of $K$ says that $\operatorname{Spec} \bar{k}[t]/(p_a) \cong \bigsqcup_i \operatorname{Spec} k$, taken over the set $\{a_1, \ldots, a_n\}$ of $p_a$'s roots, whereas non-separability of $K$ says that $\operatorname{Spec} \bar{k}[t]/(p_a)$ contains unreduced points $\operatorname{Spec} k[\varepsilon]/\varepsilon^m$, with $m =$ the multiplicity of the root at which separability is failing.

To make this even more geometric, we introduce the notion of a finite *separable $k$-algebra*. We say that a finite $k$-algebra $A$ is *separable* if $A \cong k_1 \times \cdots \times k_n$, with each $k_i$ a (finite) separable field extension $k_i/k$. Geometrically, the statement of separability for $A$ is the statement that $Y := \operatorname{Spec} A$ is a $k$-scheme of the form

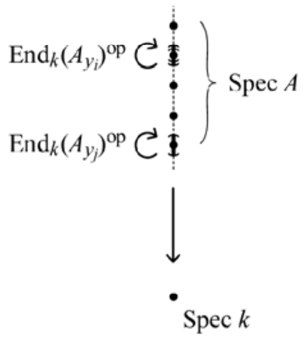
$$Y \cong \operatorname{Spec} k_1 \amalg \cdots \amalg \operatorname{Spec} k_n,$$

each $k_i/k$ finite separable. Thus $Y$ is just a finite number of points $y_i$ sitting over $\operatorname{Spec} k$, where each $y_i$ is allowed to contain the data of a finite set of points $\operatorname{Spec} \bar{k}$ equipped with an $\operatorname{Aut}_k(k_i)^{\mathrm{op}}$-action. As a cartoon, this looks something like



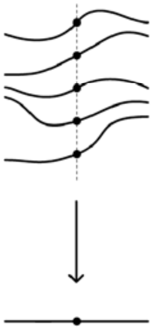

When $A$ is finite but not separable, then $Y$ still looks like a discrete set of points over Spec $k$ (indeed $A$ is Noetherian and Artinian, so use [A&M, §8, Prop.'s 8.1 & 8.3]). But now any given point $y$ in $Y$ can have a "vertical tangent neighborhood" of some order in $Y$, i.e., $y$ can lie in an unreduced component of $Y$. Indeed, if $A$'s stalk $A_y$ at $y \in \mathrm{Spec}\, A$ is not a field, and if we thus let $f \in A_y$ be any one of its non-invertible coordinate functions, and find $p(t) \in k[t]$ such that $p(f) = 0$, then $f$ must divide $p(f)$ since $f$ is not invertible, and thus $f$ is a torsion element in $A_y$. Consequently, the prime $\mathfrak{p} := (f) \in \mathrm{Spec}\, A_y$ is a "vertical tangent neighborhood" at $y$, i.e., a torsion neighborhood of $y$.

Every unreduced component of $Y$ will have a nontrivial $\mathrm{End}_k\left(A_y\right)^{\mathrm{op}}$-action. A prototypical example of such an action is $\varepsilon \mapsto 0$ in the case that $A_y = k[\varepsilon]\big/\varepsilon^2$. So the spectrum of every finite, non-separable $k$-algebra looks something like

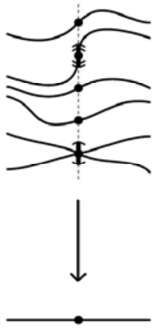



The picture can even go wrong at points without "vertical tangent neighborhoods" over $k$, since any point $y_i \in \mathrm{Spec}\, A$ underlying a non-separable extension $k_i/k$ will pull back, along $k$'s algebraic closure $\overline{k}$, to a whole new finite fiber that does contain one of these "vertical tangent neighborhoods." We say a little bit more about this in **Remark 3** at the end of this section.

The larger idea in all of this is that finite separable and finite non-separable algebras are simply the two types of fibers that can appear in any finite scheme map. Global situations in which the two cartoon examples of such algebras above arise as fibers might look something like



and



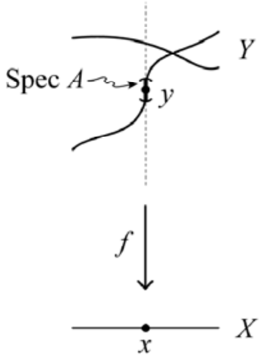
respectively. To isolate not just the fibers, but the points at which the picture at left differs from the picture at right, we define the notion of *ramification* of a scheme map $Y \longrightarrow X$.

**Definition.** — Let $f : Y \to X$ be a map of schemes locally of finite type, let $y$ be any point in $Y$, let $x := f(y)$ in $X$, and let $\mathfrak{a} \subset \mathcal{O}_{Y,y}$ be the fiber over $x$ in $Y$'s stalk at $y$. Then $A := \mathcal{O}_{Y,y} / \mathfrak{a}$ is the coordinate ring of this fiber in the stalk at $Y$, and we say that $f$ is *unramified at y* if the algebra $k(x) \longrightarrow A$ is a finite, separable, field extension.

We say that $f$ is *unramified* if it's unramified at every point $y \in Y$.

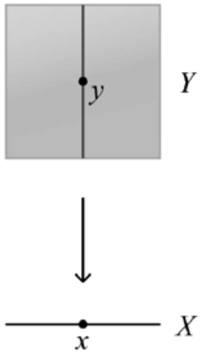
We'll also sometimes say that $f$ is *ramified at y* if $k(x) \longrightarrow A$ is finite but nonseparable.

The fiber $\mathfrak{a} \subset \mathcal{O}_{Y,y}$ here is the ideal $\mathfrak{a} := f^{\#}(\mathfrak{m}_x)\,\mathcal{O}_{Y,y}$, where $\mathfrak{m}_x$ is $x$'s ideal in its stalk $\mathcal{O}_{X,x}$. In general, $\mathfrak{a}$ is just an ideal, not necessarily a prime. It simply describes a closed subvariety of Spec $\mathcal{O}_{Y,y}$. The following drawing illustrates the situation in the definition, in a case where the map $f$ is ramified at $y$:




Several remarks:

**Remark 1:** Notice the role of finiteness in the definition. Intuitively, one thinks about the question of ramification coming up only when $f : Y \to X$ is finite, or only in fibers of $f$ that are locally finite. Maps as simple as the projection $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$ fail to be unramified at every point $y$ in their domains simply because the algebra $k(x) \longrightarrow A$ at $y$ appearing in the definition is of finite type, rather than finite. It doesn't make sense to check for a "vertical tangent neighborhood" at $y \in Y$ when $f$'s vertical fiber near $y$ is already a $d \geq 1$-dimensional scheme:




**Remark 2:** Don't confuse the notion of a scheme map being unramified at a point with that of a scheme map being *smooth* at a point.

Again, for the question of ramification to really even make sense, the map $f : Y \to X$ under consideration has to have a finite fiber in its stalk at $y$. Smoothness of scheme maps on the other hand, which is something like an algebro-geometric analog of the the condition of regularity combined with a condition of having nonsingular fibers, is a property that can hold for a map $f : Y \to X$ between schemes for which $\dim_y Y > \dim_x X$. Smoothness is a higher-dimensional generalization of the notion of being unramified.

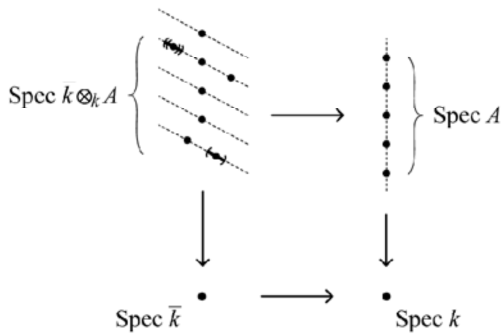
**Remark 3:** I want to say a little bit more about the geometric significance of a point $y$ over Spec $k$ for which $k(y)$ is a nontrivial, finite algebraic extension of $k$.

The basic idea is that such a point $y_i$ is in fact a larger finite set $\{y_{i\,1}, \ldots, y_{i\,m}\}$ of points (copies of Spec $\overline{k}$) glued into the single point $y$, but in a way that remembers some of

$$\mathrm{Aut}_{\overline{k}}(\{y_{i\,1}, \ldots, y_{i\,m}\}) \cong S_m,$$

relative to our choice of base field $k$.

In the general picture, given a finite algebra $k(x) \cong k \longrightarrow A \cong k_1 \times \cdots \times k_n$ appearing as the fiber $\operatorname{Spec} A = Y_x$ in some finite map $Y \longrightarrow X$ say, upon pulling $Y_x \cong \{y_1\} \amalg \cdots \amalg \{y_n\}$ back along $\operatorname{Spec} \overline{k} \longrightarrow \operatorname{Spec} k$, we'll discover that each of the points $y_i \in Y_x$ splits into any number of points over $\overline{k}$. When $k_i/k$ is nonseparable, the fiber along $\operatorname{Spec} \overline{k} \longrightarrow \operatorname{Spec} k$, over $y_i \in Y_x$ will contain points with "vertical tangent neighborhoods" over $y_i$ (or over $\operatorname{Spec} \overline{k}$, depending on your perspective):




In this way, finite nonseparable field extensions $k_i/k$ hide vertical tangent neighborhoods over $\operatorname{Spec} k$ that we can't see until we pull $k_i/k$ all the way back to $\overline{k}$.

The phenomena of points-with-automorphisms in algebraic geometry really intrigues me, which should explain the following example. I don't know how informative it is. The abstract situation just discussed might be enough. But I wanted to look at a concrete example, and this is the result:

**Example.** — I don't know how informative this example is. The abstract situation just discussed might be enough. But I wanted to look at a concrete example, and this is the result:

Let $X := \operatorname{Spec} \mathbb{Q}[x^{\pm 1}]$, let $Y := \operatorname{Spec} \mathbb{Q}[x^{\pm 1}, y]/(y^n - x)$, and let $Y \longrightarrow X$ be the obvious algebra.

Non-generic points in $X$ are of the form $\mathfrak{p} = (p(x)) \subset \mathbb{Q}[x^{\pm 1}]$ for irreducible $p(x)$. At any such $\mathfrak{p}$, we know that $k(\mathfrak{p}) = \mathbb{Q}[x]/(p(x))$, some finite separable extension of $\mathbb{Q}$. At points of the form $\mathfrak{p} = (x - a) \in \mathbb{Q}[x^{\pm 1}]$, with $a \in \mathbb{Q}$, we have $k(\mathfrak{p}) = \mathbb{Q}$. The number of points in fiber $Y_{\mathfrak{p}}$ of $Y$ over $\mathfrak{p} = (x - a) \in X$ is the number of irreducible factors in $y^n - a$'s prime factorization in $\mathbb{Q}[y]$.

For example, suppose $Y := \operatorname{Spec} \mathbb{Q}[x^{\pm 1}, y]/(y^3 - x)$, and consider the point $\mathfrak{p} := (x - 8) \in X$. Then $Y_{\mathfrak{p}} = \operatorname{Spec} \mathbb{Q}[y]/(y^3 - 8)$, and contains the closed points $y_1 := (y - 2)$ and $y_2 := (y^2 + 2y + 4)$ (the prime factors of $y^3 - 8$). It's not hard to see that $y_1$ and $y_2$ satisfy the hypotheses of the Chinese Remainder Theorem, and thus that

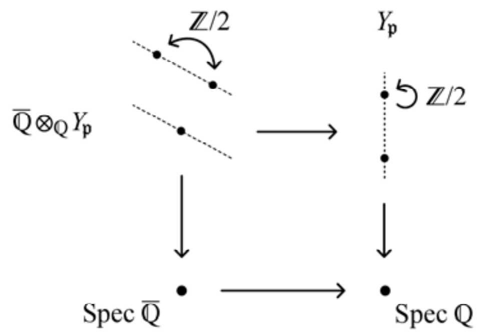
$$Y_{\mathfrak{p}} = \operatorname{Spec} \mathbb{Q} \amalg \operatorname{Spec} \mathbb{Q}(i\sqrt{3}),$$

where $\mathbb{Q} = k(y_1)$ and $\mathbb{Q}(i\sqrt{3}) = k(y_2)$. Note that the point $y_2 \in \operatorname{Spec} \mathbb{Q}(i\sqrt{3})$ has a nontrivial automorphism as a scheme, namely the one coming from the transformation $i\sqrt{3} \mapsto -i\sqrt{3}$.

On the other hand, it's easy to see that

$$\overline{\mathbb{Q}} \otimes Y_{\mathfrak{p}} \cong \operatorname{Spec} \overline{\mathbb{Q}} \amalg \operatorname{Spec} \overline{\mathbb{Q}} \amalg \operatorname{Spec} \overline{\mathbb{Q}},$$

since $\overline{\mathbb{Q}} \otimes Y_{\mathfrak{p}} \cong \operatorname{Spec} \overline{\mathbb{Q}}[y]/(y^3 - 8)$, so that pulling $Y \longrightarrow X$ back along $\overline{X} := \overline{\mathbb{Q}} \otimes X \longrightarrow X$ splits at least the single closed point $\operatorname{Spec} \mathbb{Q}(i\sqrt{3})$ of the fiber $Y_{\mathfrak{p}}$ into two separate points, both copies of $\operatorname{Spec} \overline{\mathbb{Q}}$. The $\mathbb{Z}/2$-action on $y_2$ becomes simply the transposition of the two points over $\overline{\mathbb{Q}}$ to which $y_2$ pulls back:




Of course, since every extension of $\mathbb{Q}$ is separable, the phenomena of "vertical tangent neighbors" in $Y_\mathfrak{p}$'s pullback to Spec $\overline{\mathbb{Q}}$ can't occur in this example.

**Quick Last Remark:** There is a more general definition of separability for $k$-algebras, one that does not require $k \longrightarrow A$ to be finite. Namely, $A$ is *separable* over $k$ if the *Jacobson radical* in $\overline{k} \otimes_k A$ vanishes. Milne [EC. Prop. I.3.1, p. 20] provides a proof that when $k \longrightarrow A$ is finite, the two definitions coincide. We're only going to be interested in separable algebras in relation to maps $f : Y \to X$ locally of finite type, and we'll see below that this ends up reducing our interest to separability of finite algebras $k \longrightarrow A$.

## ■ 1.2 *Étale*: Several Equivalent Definitions.

If the reader looks up "étale morphism" on Wikipedia, at the time of this writing he'll find nine equivalent definitions, plus a couple more in the case that the domain is locally Noetherian (which we're assuming here). The canonical definition of étale though is the following:

**Definition 1.2.1** — A map $Y \longrightarrow X$ of schemes is *étale* if it's flat and unramified.

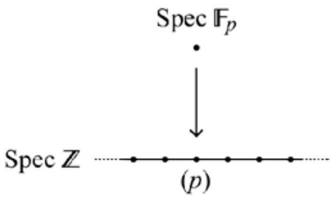
In this section, I want to pick this definition apart a bit, and then give three equivalent ones: a "fiberwise separable" definition, a "vanishing $\Omega^1_{Y/X}$" definition, and an "open diagonal" definition.

First though, I want to focus on the flatness condition in the canonical definition. Everyone should know what *flat* means: a map $f : Y \to X$ of schemes is *flat at* $y \in Y$ if $f_y^\# : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is a flat algebra. It's *flat* if it's so at every $y \in Y$. For affine maps, this reduces to the statement that $f$ is locally dual to a flat morphism $A \longrightarrow B$ of affine coordinate algebras at $y$ and $f(y)$.

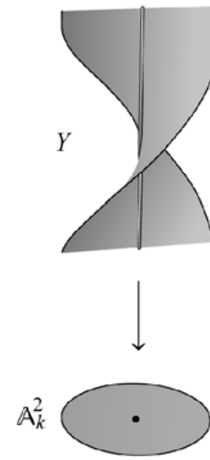
It was quite opaque to me at first why the condition of being étale, the supposed algebro-geometric generalization of the condition of being a local diffeomorphism, should be "flat + unramified." That the condition of being unramified is related to the idea of local diffeomorphism is clear enough, but it was not at all clear to me at first why flatness should come into play. There's is a perspective though by which the conditions of being flat and unramified begin to look more dual to one another in a way that calls to mind the pair of dual conditions that constitute the statement that $d_y f : T_y Y \to T_{f(y)} X$ is an isomorphism at $y$ for a map $f : Y \to X$ of manifolds, namely injectivity and surjectivity:

Flatness is a version of "co-continuity" in the sense that for a flat map, the codomain does a reasonable job of parametrizing the domain (we can view continuity of a map as the statement that the domain does a reasonable job of parametrizing the codomain). When we say "a flat map $Y \longrightarrow X$," think "a map with fibers that 'vary continuously' over $X$." In particular, since the dimension of a given fiber is discrete, the dimension of $Y$'s fibers shouldn't "jump" as they vary across $X$. The rigorous statement is that if $Y \longrightarrow X$ is flat and locally of finite type, then $\dim_y Y_x = \dim_x X - \dim_y Y$ at every point $y \in Y_x$ in every fiber over $X$ [AG, §III, Prop. 9.5, p. 256].

Two of the most basic examples of non-flat scheme maps where we can actually see this phenomena of dimension-jumping explicitly are the algebras $\mathbb{Z}/p \longleftarrow \mathbb{Z}$ and $\mathbb{C}[x, y, z]/(x z - y) \longleftarrow \mathbb{C}[x, y]$, whose dual geometric cartoons look like




and

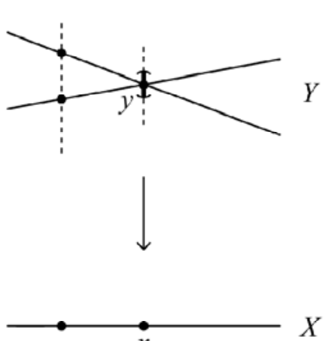



respectively. The jump in dimension at $(p)$ in the first case is $-1 \leftrightarrow 0$, that is, from empty fibers to a finite fiber. The jump in dimension at the vertical fiber over the origin in the latter case is $0 \leftrightarrow 1$. (To see that the latter isn't flat anywhere on its 1-dimensional fiber, use the fact that base change preserves flatness, and pullback along the $x$-axis $\mathbb{A}^1 \longrightarrow \mathbb{A}^2$ in the base: this reduces the problem to flatness of $\mathbb{C}[x, z]/(xz) \longleftarrow \mathbb{C}[x]$, which fails since $x$ is a torsion element in $\mathbb{C}[x, z]/(xz)$.)

If we restrict our attention to finite maps $Y \longrightarrow X$, then "morally," the only dimension jump we can expect must be of the form $-1 \leftrightarrow 0$, that is, from empty to nonempty fibers. This isn't strictly true, because the dimension of fibers in a flat map can jump at ramification points: for precise statements along these lines, see [AG, §III.9, Prop. 9.5, p. 256] and [AG, §III.9, Ex. 9.1, p. 266]. "Morally" though, the failure of flatness for a finite map $f : Y \to X$ at $y \in Y$ becomes the statement that the fibers over points near $x := f(y) \in X$ are empty, though the fiber over $x$ is non-empty. This is roughly the statement that $T_y Y \longrightarrow T_{f(y)} X$ is not surjective, whereas being ramified at $y$ is roughly the statement that $T_y Y \longrightarrow T_{f(y)}$ is not injective.

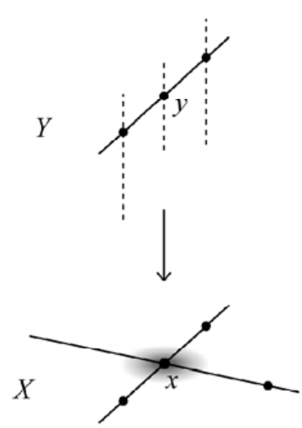
The "duality" here becomes more apparent perhaps when we draw pictures of simple examples of finite maps $Y \longrightarrow X$ that are (1) flat, but ramified at $y$, and (2) unramified, but not flat at $y$:

(1) flat, but ramified at $y$




"vertical failure" in the domain:
the "kernel of $df$" is nontrivial.

(2) unramified, but not flat at $y$




"horizontal failure" in the codomain:
the "cokernel of $df$" is nontrivial.

In this way, a map that's simultaneously flat and unramified begins to look very much like a multi-sheeted local diffeomorphism.

**Remark:** Let me take a moment to justify the claims that (1) is flat and ramified, whereas (2) is non-flat and unramified.

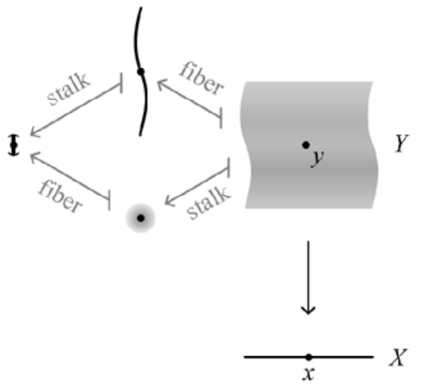
An instance of the first example is the algebra $B \longleftarrow A$ where $A = k[t]$ and $B = k[u, v]/(uv)$, with the morphism $A \longrightarrow B$ taking $t \mapsto u+v$. Since $B \cong k \oplus k[u]^+ \oplus k[v]^+$, the element $u+v$ is non-torsion in $B$ (every product is of the form $(u+v)\cdot(f(u)+g(v))$). Thus $u+v$ is non-torsion in $O_{Y,(u,v)}$, and so by the Local Criterion of Flatness (see for instance [AG, §III, Lemma 10.3.A, p. 270], and the fact that $t$ isn't torsion in $O_{X,x:=(t)}$, we need only check that $B/(u+v) \cong k[\varepsilon]/\varepsilon^2$ is flat over $A/(x) \cong k$, which is of course true. Regarding ramification, since $t \mapsto u+v$ in $B$, our fiber over the origin $x := (t)$ in $X$ becomes $k[\varepsilon]/\varepsilon^2$, which is our prototypical example of a finite algebra ramified at the only point in its spectrum.

An instance of the second example is the algebra $B \longleftarrow A$ where $A := k[u, v]/(uv)$ and $B := k[t]$, the morphism $A \longrightarrow B$ taking $u \mapsto 0$ and $v \mapsto t$. The failure of flatness appears in the pair of sequences

$$\begin{array}{c} 0 \longrightarrow A \xrightarrow{\times u} A \longrightarrow B \longrightarrow 0 \\ \Big\downarrow (-)\otimes_A B \\ 0 \longrightarrow B \xrightarrow{0} B \longrightarrow B \otimes_A B \longrightarrow 0. \end{array}$$

That it is unramified at the origin $y := (t)$ in $Y$ comes from the fact that its fiber over the origin $x := (u, v)$ in $X$ is Spec $k(x)$.

If the reader read through this last remark, then our method of checking ramification at $y \in Y$ by describing the form of the fiber $Y_x$ containing $y$ may have caught his attention. It makes implicit use of commutativity between the operators of "passing to the fiber over $x$" and of "taking the stalk at $y$:"




It's natural to ask if this works in general, that is, if we can define an étale map as any map $Y \longrightarrow X$ whose every fiber looks like a separable algebra, i.e., such that $Y_x \cong \bigsqcup_i \operatorname{Spec} k_i$ over every $x \in X$. This is so, and even more is true:

**The "Fiberwise Separable" Definition of *Étale*:** Checking that a map $f : Y \to X$ is unramified at $y \in Y$ entails a study of $f$ at all orders of locality near $y$. This is tedious in practice, because it requires us to work with the inductive limit $O_{Y,y}$.

On the other hand, since it's really $O_{Y,y}/\mathfrak{a}$ we're interested in, the question of ramification at $y$ is ultimately a question about the stalk at $y$ in the fiber $Y_x$, where $x := f(y)$. Thus it's natural to ask if there isn't a simpler way to check that $Y \longrightarrow X$ is étale, one that we can simply carry out fiberwise in $Y$.

Indeed there is, given by the following theorem:

**Theorem 1.2.2** — Given a scheme map $f : Y \to X$, the following are equivalent:

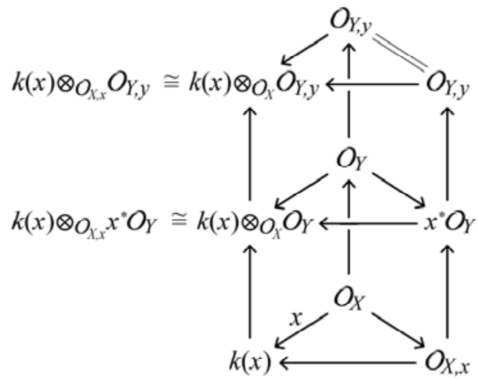
**(i)** $f$ is étale;

**(ii)** $f$ is flat and its every fiber $Y_x$ over $X$ is a disjoint union of spectra of finite separable field extensions $k_i/k$.

**Proof: (i)⇒(ii).** Let $f : Y \to X$ be étale. Then $f$ of is course flat. Fix a fiber $Y_x$. We claim that $O_{Y_x,y} \cong O_{Y,y}/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal showing up in the definition of ramification. Indeed, since $k(x) = O_X/\mathfrak{p}_x \cong O_{X,x}/\mathfrak{m}_x$, we have

$$\mathcal{O}_{Y_x,y} \;=\; \big((\mathcal{O}_{X,x}/\mathfrak{m}_x)\otimes_{\mathcal{O}_X}\mathcal{O}_Y\big)_y \;\cong\; (\mathcal{O}_{X,x}/\mathfrak{m}_x)\otimes_{\mathcal{O}_{X,x}}\mathcal{O}_{Y,y} \;\cong\; \mathcal{O}_{Y,y}\big/ f^{\#}(\mathfrak{m}_x)\,\mathcal{O}_{Y,y} \;=:\; \mathcal{O}_{Y,y}\big/\mathfrak{a},$$

where I get the second isomorphism at left by thinking about the diagram




for a while. Thus the hypothesis that $f$ is unramified tells us that $\mathcal{O}_{Y_x,y}$ is a finite, separable field extension $K/k(x)$. This shows at least that $Y_x \longrightarrow \operatorname{Spec} k(x)$ is unramified.

Since the property of being locally of finite type is stable under base change, we know that $Y_x \longrightarrow k(x)$ is locally of finite type. So cover $Y_x$ by open affines $U_i = \operatorname{Spec} B_i$ such that $B_i$ is a finitely generated $k(x)$-algebra. By [A&M, §8, Ex. 3], we know that

$$\text{finitely generated} + \text{Artinian} = \text{finite},$$

and since we assume $B_i \longleftarrow k(x)$ unramified, finiteness implies that $\operatorname{Spec} B_i = \coprod_j \operatorname{Spec} k_j$ where each $k_j\big/k(x)$ is finite separable [A&M. §8, Ex. 4, p. 92]. Thus all we need to show is that $B_i$ is necessarily Artinian.

Since $B_i \longleftarrow k(x)$ is finitely generated, $B_i$ is Noetherian, so by [A&M. §8, Prop. 8.5], $B_i$ will be Artinian if we can show that $\dim_{\text{Krull}} B_i = 0$, that is, that every prime $\mathfrak{p} \subset B_i$ is maximal. To this end, let $\mathfrak{p}$ be a prime in $B := B_i$. Then there's a tower

$$k(x) \hookrightarrow B/\mathfrak{p} \hookrightarrow B_{\mathfrak{p}}\big/\mathfrak{p}\,B_{\mathfrak{p}} =: k(\mathfrak{p})$$

of inclusions. Indeed, it's clear that $k(x) \hookrightarrow B/\mathfrak{p}$ is an inclusion, and to see that $B/\mathfrak{p} \hookrightarrow B_{\mathfrak{p}}\big/\mathfrak{p}\,B_{\mathfrak{p}}$ is an inclusion one makes use of $\mathfrak{p}$'s assumed primality in combination with the form of $\ker\big(B \to B_{\mathfrak{p}}\big)$ (see [A&M. §3, p. 37]). But now our assumption that $Y_x \longrightarrow k(x)$ is unramified implies in particular that $k(\mathfrak{p})/k(x)$ is a finite extension. Thus $B/\mathfrak{p}$ must be not just an intermediate ring, but an intermediate field, and so $\mathfrak{p}$ is maximal in $B$.

**(ii)⇒(i).** This follows immediately from the fact that $\mathcal{O}_{Y_x,y} \cong \mathcal{O}_{Y,y}\big/\mathfrak{a}$, which we established above. ■

Before we give the other two equivalent definitions of étale, we need to spend some time talking about the notion of the *geometric fiber* of $Y \longrightarrow X$ over a *geometric point* of $X$, and about geometric criteria for ramification. This will become important in establishing the equivalence with the remaining two definitions, which will in turn be important showing that étale maps satisfy Grothendieck's abstract axioms for a topology.

We begin with the following criterion. In its proof, we use a third familiar definition of *separability* of a field extension $k_i/k$ in the finite case, namely that $k_i/k$ is finite separable if $[k_i : k]_{\text{sep}} = [k_i : k]$, where

$$[k_i : k]_{\text{sep}} := \operatorname{card} \operatorname{Hom}_{k\textbf{-Alg}}(k_i, \overline{k}) \qquad \text{and} \qquad [k_i : k] := \dim_k k_i$$

(see for instance [Lang, *Algebra*. §V.4, p. 239]). We will also use a fourth formulation of finite separablility that's maybe not so familiar, but we describe it below, in the course of the proof.

**Theorem 1.2.3 (Geometric Separability Criterion for Finite Algebras).** — For any finite algebra $A \longleftarrow k$, the following are equivalent:

**(i)** $A$ is separable over $k$;

**(ii)** $\overline{A} := \overline{k} \otimes_k A$ is separable over $k$'s algebraic closure $\overline{k}$.

**Proof: (i)⇒(ii).** Recall that the *trace pairing* $\mathrm{Tr}_{A/k} : A \otimes_k A \to k$ associated to a given finite algebra $A \longleftarrow k$ is the $k$-bilinear function taking $ab \mapsto \mathrm{tr}(ab)$, the trace of $ab$ interpreted as an endomorphism $ab \in \mathrm{End}_{k\text{-}\mathbf{Vec}}(A)$ via its left-regular multiplicative action. We also associate to $A \longleftarrow k$ a number $D_{A/k} \in k$, called the *discriminant*. We define $D_{A/k}$ to be the square of the determinant of the matrix giving the trace pairing in any given basis for $A$ over $k$. The importance of the discriminant is that it vanishes if and only if the trace pairing is non-degenerate.

The relevancy of all this to separability is the fact that a finite field extension $K/k$ is separable if and only if the trace pairing $\mathrm{Tr}_{K/k} : K \otimes_k K \to k$ is non-degenerate, and thus if and only if $D_{A/k} \neq 0$ [Roman, *Field Theory*. §8, Thm. 8.2.2, p. 204].

So suppose $A \cong k_1 \times \cdots \times k_n$. Since each $\mathrm{Spec}\, k_i$ is a component of $\mathrm{Spec}\, A$, it is enough to consider the case where we have only one factor $k_i$.

Let $K := k_i$, and define $\overline{B} := \overline{k} \otimes_k K$. Since $K/k$ is separable, we know that $\mathrm{Tr}_{K/k} : K \otimes_k K \to k$ is non-degenerate, and $D_{K/k} \neq 0$. Now extending our the scalars of our $k$-vector space $K$ doesn't change the matrix defining $K \otimes_k K \to k$ with respect to a given basis, so in particular

$$D_{\overline{B}/\overline{k}} = D_{K/k} \neq 0,$$

and the trace pairing $\mathrm{Tr}_{\overline{B}/\overline{k}} : \overline{B} \otimes_{\overline{k}} \overline{B} \to \overline{k}$ is non-degenerate as well.

Since $\overline{B}$ is finite dimensional over $\overline{k}$, it is Artinian. In particular, $\mathfrak{J} = \mathfrak{N}$, that is, $\overline{B}$'s Jacobson radical equals its nilradical [A&M. §8, Cor. 8.2, p. 89]. This means that if $\mathfrak{J} \neq 0$, we have some $f \neq 0 \in \mathfrak{N}$ such that $fb$ is nilpotent for all $b \in B$. The general fact that a nilpotent matrix always has trace = 0 implies thus that $\overline{B}$'s trace pairing is degenerate, a contradiction.

Thus $\mathfrak{J} = 0$. Since $\mathfrak{J}$ is the intersection of the maximal ideals in $\overline{B}$, it being clear that $\overline{B}$ is the sum of its maximal ideals, this implies that $\overline{B} = \overline{B}/\mathfrak{m}_1 \times \cdots \times \overline{B}/\mathfrak{m}_m$. Each $\overline{B}/\mathfrak{m}_i$ is a finite field extension of $\overline{k}$, so must be isomorphic to $\overline{k}$.

**(ii)⇒(i).** Let $\mathfrak{J}$ be $A$'s Jacobson radical. As we just explained, $A/\mathfrak{J} \cong k_1 \times \cdots \times k_n$ where each $k_i/k$ is a finite field extension. (Note: since $A$ is Artinian, the Jacobson radical equals the nilradical, so $\mathrm{Spec}\, A/\mathfrak{J} = (\mathrm{Spec}\, A)_{\mathrm{red}}$, i.e., is $\mathrm{Spec}\, A$ without out any of its torsion neighborhoods over $k$).

Suppose $\overline{A}$ separable over $\overline{k}$, that is, $\overline{A} \cong \overline{k} \times \cdots \times \overline{k}$. Since the maximal ideals in $\overline{A}$ are simply the kernels of the coordinate projections, and since every ring morphism $\overline{A} \longrightarrow \overline{k}$ must lift $0 \subset \overline{k}$ to one of these maximal ideals, we know that

$$\mathrm{card}\, \mathrm{Hom}_{\overline{k}\text{-}\mathbf{Alg}}(\overline{A}, \overline{k}) = \text{number of } \overline{k}\text{-factors in } \overline{A} = \dim_{\overline{k}} \overline{A}.$$

Simultaneously, the universal property of $\overline{A}$ as a $\overline{k}$-algebra, described by the diagram

$$\begin{array}{ccccc} \overline{k} & \longrightarrow & \overline{k} \otimes_k A & \longleftarrow & A, \\ & {\scriptstyle 1}\searrow & \downarrow & \swarrow & \\ & & \overline{k} & & \end{array}$$

puts

$$\mathrm{Hom}_{\overline{k}\text{-}\mathbf{Alg}}(\overline{A}, \overline{k}) \cong \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, \overline{k}).$$

Thus $\mathrm{card}\, \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, \overline{k}) = \dim_{\overline{k}} \overline{A}$. Yet we can also count $\mathrm{Hom}_{k\text{-}\mathbf{Alg}}(A, \overline{k})$ as

$$\sum_{i=1}^{n} [k_i : k]_{\mathrm{sep}},$$

where $[k_i : k]_{\text{sep}}$ is the separable degree of $k_i/k$. Indeed, $[k_i : k]_{\text{sep}} := \text{Hom}_{k\text{-}\mathbf{Alg}}(k_i, \overline{k})$, the Jacobson radical $\mathfrak{J} \subset A$ lies in the kernel of every morphism $A \longrightarrow \overline{k}$, and the maximal ideals of $A/\mathfrak{J} \cong k_1 \times \cdots \times k_n$ are the kernels of the coordinate projections. Since $[k_i : k]_{\text{sep}} \le [k_i : k]$, this gives us the long string of inequalities

$$\dim_{\overline{k}} \overline{A} = \text{card}\,\text{Hom}_{k\text{-}\mathbf{Alg}}(A, \overline{k}) = \sum_{i=1}^{n} [k_i : k]_{\text{sep}} \le \sum_{i=1}^{n} [k_i : k] = \dim_k A/\mathfrak{J} \le \dim_k A$$

(the identity $\sum_i [k_i : k] = \dim_k A/\mathfrak{J}$ comes from the fact that $[k_i : k] := \dim_k k_i$). But finally,

$$\dim_{\overline{k}} \overline{k} \otimes_k A = \dim_k A,$$

so we see (i) that $A/\mathfrak{J} = A$, thus $A \cong k_1 \times \cdots \times k_n$, and (ii) that $[k_i : k]_{\text{sep}} = [k_i : k]$ for each $1 \le i \le n$. Thus each $k_i/k$ is separable, and so $A$ is separable. ■

We can extend this last equivalence to the point of making rigorous our prior observations that non-separability of a field extension $k_i/k$ seems to hide a more geometric condition in the fiber

$$\overline{k} \otimes_k k_i \longleftarrow \overline{k}.$$

To formulate this observation, we need the definition of a geometric point and its geometric fiber.

**Definition 1.2.4** — By a *geometric point* $\overline{x}$ in a scheme $X$, we mean any map $\overline{x} : \text{Spec}\, K \to X$ of the spectrum of a separably closed field $K$ into $X$.

Given a scheme map $f : Y \to X$, by a *geometric fiber* $Y_{\overline{x}}$ of $f$ we'll mean the pullback of $f$ along any geometric point $\overline{x} : \text{Spec}\, K \to X$.

We will sometimes denote the base field $K$ of a geometric fiber $Y_{\overline{x}}$ as $k(\overline{x})$, conflating the map $\overline{x} : \text{Spec}\, K \to X$ with the singleton $\text{Spec}\, K$.

The geometric fiber $Y_{\overline{x}}$ tends to be much simpler than $Y_x$, more "geometric," i.e., locally more similar to $\text{Spec}\, K$ than $Y_x$ is to $\text{Spec}\, k$. Just think of the $\mathbb{Q}$-versus-$\overline{\mathbb{Q}}$ example we worked through further above. Any geometric point $\overline{x} : \text{Spec}\, K \to X$ takes the lone point $\mathfrak{m}$ in $\text{Spec}\, K$ to a point $x \in X$, so that we can factor $\overline{x}$ as

$$\overline{x} : \text{Spec}\, K \longrightarrow \text{Spec}\, k(x) \xrightarrow{x} X.$$

This lets us think, loosely, of a point in a geometric fiber $Y_{\overline{x}}$ of $Y \longrightarrow X$ as being one of the points in $\overline{k} \otimes_{k(x)} Y_x = Y_{\overline{x}}$ over $\text{Spec}\, \overline{k}$ we discussed above, a point that possibly gets glued with other points in $Y_{\overline{x}}$ to form a single point in $Y_x$. As we saw in our discussion there, separability of a finite extension $k_i/k(x)$, corresponding to the point $y_i \in Y_x$ say, is related to the existence of "vertical tangent neighborhoods" in $\overline{k} \otimes_{k(x)} Y_x$, that is, to the presence of ramification in $Y_{\overline{x}} \longrightarrow \text{Spec}\, k(x)$.

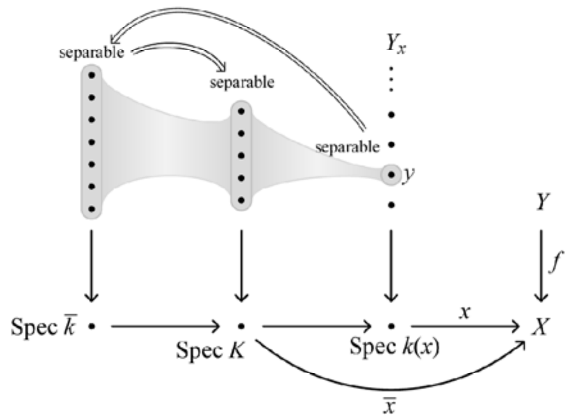
What becomes quite beautiful is that this is really all ramification is:

**Theorem 1.2.5 (Geometric Ramification Criterion).** — Given any scheme map $f : Y \to X$, the following are equivalent:

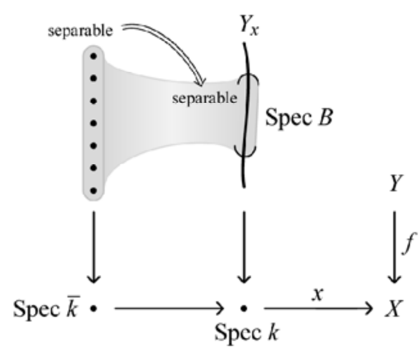
**(i)** $f$ is unramified;

**(ii)** Every geometric fiber $Y_{\overline{x}}$ is a disjoint union of schemes isomorphic to $\text{Spec}\, k(\overline{x})$.

**Proof: (i)⇒(ii).** Fix a geometric point $\xi : \text{Spec}\, K \to X$ factoring through $x \in X$, with $k := k(x)$, and suppose $f : Y \to X$ unramified. Then by Theorem 1.2.2 above, $Y_x \cong \bigsqcup_i \text{Spec}\, k_i$, each $k_i/k$ finite separable. Let $\{y\} \subset Y$ be any one of these components $\text{Spec}\, k_i$. Since the pullbacks of the components $\text{Spec}\, k_i$ cover $Y_\xi$, it is enough to verify that each $A := K \otimes_k k_i$ is a separable $K$-algebra. Our geometric separability criterion tells us that this will hold if $\overline{A} := \overline{k} \otimes_K A$ is a separable $\overline{k}$-algebra. But $\overline{k} \otimes_K A \cong \overline{k} \otimes_k k_i$, so separability of $k_i/k$ implies, again by our geometric separability criterion, $\overline{A}$ is indeed separable: **[FIX DRAWING, there's a serious flaw in this, because we need it to be more general]**




**(ii)⇒(i).** By Theorem 1.2.2, if is enough to prove that $Y_x \cong \bigsqcup_i \operatorname{Spec} k_i$, with $k_i/k$ finite separable, for each $x \in X$. Fix $x \in X$, let $k := k(x)$, and fix a geometric point $\overline{x} : \operatorname{Spec} \overline{k} \to X$ factoring through $x \in X$ by choosing a specific algebraic closure $\overline{k}/k$. Since $f : Y \to X$ is locally of finite type, we can cover $Y_x$ by open affines finitely generated over $k$. By our geometric separability criterion, it becomes enough to verify that each of these affines pulls back to a separable $\overline{k}$-algebra. But this follows immediately from Theorem 1.2.2 by our geometric separability criterion:




■

**Corollary 1.2.6** — The following are equivalent:

**(i)** $f$ is étale;

**(ii)** $f$ is locally of finite type, flat, and its every geometric fiber $Y_{\overline{x}}$ is a disjoint union of schemes isomorphic to Spec $k(\overline{x})$. ■

**Remark:** This last Theorem is important in a rather heuristic way as well as its specifics. In indicates a general principle that when we work with étale maps $Y \longrightarrow X$ over a $k$-scheme $X$, we're implicitly, or may as well be, working will the maps' pullbacks over $\overline{k}$ or $k^{\text{sep}}$, or over any separably closed $K/k$. Different riffs of this principle should show up repeatedly.

**The "Vanishing $\Omega^1_{Y/X}$," and "Open Diagonal" Definitions of Étale:** Recall that a locally Noetherian $k$-scheme $X$ admits an important coherent sheaf $\Omega^1_X$, or $\Omega^1_{X/k}$, called $X$'s *sheaf of Kähler differentials*. We also call it $X$'s *cotangent sheaf*, and in cases where $X$ actually has a cotangent bundle $T^*X$, say when $k = \mathbb{C}$ and $X$ is smooth, then I believe $\Omega^1_X$ is isomorphic to the sheaf of regular sections $X \longrightarrow T^*X$.

We can also define the cotangent sheaf in the relative context, that is, we define it with respect to maps rather than schemes. Given a map $f : Y \to X$ of schemes, we define $f$'s *sheaf of relative Kähler differentials* to be the pullback along the diagonal $\Delta : Y \to Y \times_X Y$ of a certain quotient sheaf:

$$\Omega^1_{Y/X} := \Delta^* \left(\mathcal{J} \big/ \mathcal{J}^2\right).$$

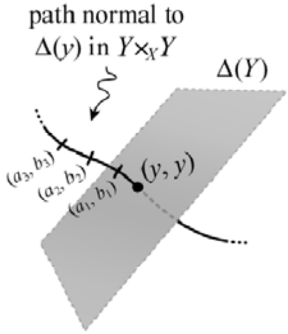
The sheaf $\mathcal{J}$ on $Y \times_X Y$ here is the sheaf of ideals of the image of the diagonal. It is the subsheaf

$$\mathcal{J} \hookrightarrow \mathcal{O}_{Y \times_X Y}$$

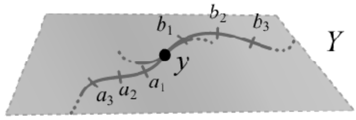
consisting, locally at least, of those sections vanishing on the diagonal's image. The well definedness of $\mathcal{J}$ comes from the fact that $Y \cong \Delta(Y)$ with $\Delta(Y)$ locally closed in $Y \times_X Y$ [AG, §II.8, p. 175] (the fact that $\Delta : Y \to Y \times_X Y$ is always locally a closed immersion will become important in the proof below).

The definition of $\Omega^1_{Y/X}$ is rather abstract, and I want to spend a moment explaining what $\Omega^1_{Y/X}$ measures from a geometric perspective:

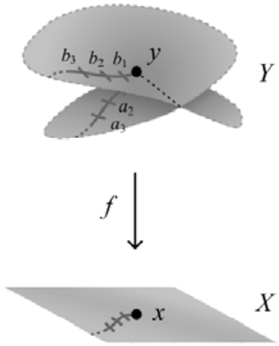
We focus on the sheaf $\mathcal{J}/\mathcal{J}^2$ over $Y \times_X Y$. The local sections in $\mathcal{J}$ are functions on $Y \times_X Y$ that vanish on $\Delta(Y)$, so to a first order approximation, they are (co)vectors "normal" to $\Delta(Y)$ in $Y \times_X Y$. For ease of thought, let $X$ and $Y$ be smooth manifolds (say submanifolds of affine space) so that the notion of a path in $Y \times_X Y$ makes sense. A path in $Y \times_X Y$ normal to $\Delta(Y)$ at the point $(y, y) \in \Delta(y)$ is the same thing as pair of paths in $Y$, both passing through $y \in Y$ at time 0:




gives us



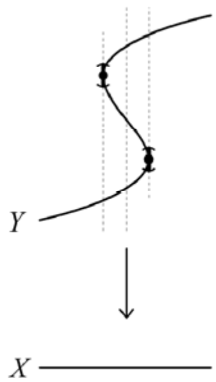

Using the set-theoretical definition of $Y \times_X Y$, note that pairs of points $a_i$, $b_i$ on these two paths in $Y$ that correspond under the two coordinate projections $Y \times_X Y \longrightarrow Y$ get mapped to the same point in $X$ under $f : Y \to X$. We can picture this as follows by "branching" or "winding" a copy of $Y$ up over $X$ so that the two paths converging to $y \in Y$ at time 0 lie over their shared image in $X$:



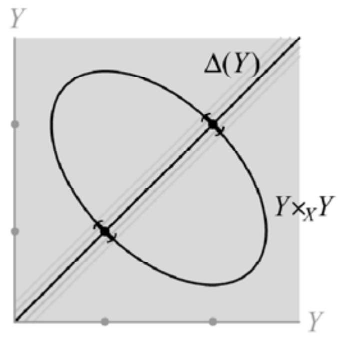

So paths normal to $\Delta(Y)$ in $Y \times_X Y$ describe pairs of paths in $Y$ that converge, over a single path in $X$, to a point $y \in Y$. Very roughly, the sheaf $\Omega^1_{Y/X}$ measures the convergence, at points in $Y$, of pairs of "$1^{st}$-order paths" or "$1^{st}$-order arcs" in $Y$ over single paths in $X$. Of course, paths don't really make sense in a scheme — we actually have to replace them with "copaths," that is, maps $Y \longrightarrow \mathbb{A}^1$, or even localizations or formal completions of such maps — but this doesn't effect the intuition.

In particular, what $\Omega^1_{Y/X}$ measures is quite similar to our notion of ramification, insofar as ramification describes points of high multiplicity, i.e., "points that have converged" in fibers over $X$. In the theorem below, we'll show that indeed "unramified" ≡ "$\Omega^1_{Y/X} = 0$" when $Y \longrightarrow X$ is locally of finite type. But I want to make one more brief geometric observation first.

In the case of a smooth manifold $Y$, we have a canonical isomorphism $TY \cong N\Delta(Y)$ between $Y$'s tangent bundle $TY$ and the normal bundle to $Y$'s diagonal in $Y \times Y$ (the argument is simple; see for instance [Milnor & Stasheff, *Characteristic Classes*. Lemma 11.5, p. 121]). This suggests that, at least when $Y$ is a smooth, quasi-projective scheme say, we might identify the nonvanishing stalks of $\Omega^1_{Y/X}$ with "tangent stalks" on $Y$. We can take this as a rough justification for the heuristic idea that $\Omega^1_{Y/X}$ measures the "vertical component of $TY$ over $X$," or the "vertical fibers in $TY$" etc. For a cartoon example, we provide the following. The two normal directions to $\Delta(Y)$ in $Y \times_X Y$ (in the drawing at right), plus the corresponding points of $Y$ at which $\Omega^1_{Y/X} \neq 0$ is identified with vertical tangent stalks (in the drawing at left), plus the relation to ramification, are all apparent. The only tricky thing is convincing one's self that the figure at right really is $Y \times_X Y$:




whose fiber square over $X$ is




Now the theorem:

**Theorem 1.2.7** — Given a scheme map $f : Y \to X$ locally of finite type, and a point $y \in Y$, the following are equivalent:

**(i)** $f$ is unramified at $y$;

**(ii)** $\Omega^1_{Y/X} = 0$ at $y$;

**(iii)** The diagonal $\Delta : Y \to Y \times_X Y$ is a local isomorphism at $y$.

**Proof:** **(i)⇒(ii).** Assume $f : Y \to X$ unramified at $y$. We proceed in two parts. The first part of the argument consists of running around like mad in Hartshorne's AG. Moreover, the first part only makes use of our finiteness assumptions on $X$, $Y$, and $f$, whereas the second part is based entirely on the finite separablility part of non-ramification, and I suggest beginning with **Part 2 of (i)⇒(ii)** below, and then going back to **Part 1** once **Part 2** is clear:

**Part 1 of (i)⇒(ii). Reduction to the case that $f$ is the algebra $O_{Y_x,y} \longleftarrow k(x)$:** We can carry out the construction of $\Omega^1_{Y/X}$ affine-locally [AG, §II, Remark 8.9.2, p. 175]. Since $Y$ and $X$ are locally Noetherian to begin with, while $f$ is locally of finite type, this implies that we can take suitable affine neighborhoods of $y$ and $f(y)$, reducing ourselves to the case that $f : Y \to X$ is a map of Noetherian affine schemes with $Y$ a finitely generated $X$ algebra.

Since $f : Y \to X$ is then a map of affine schemes, we know that it must be separated [AG, §II, Prop. 4.1, p. 96], i.e., $\Delta : Y \to Y \times_X Y$ must be a closed immersion. Since $X$ is Noetherian and $Y$ is a finitely generated $X$-algebra, we know that $Y \times_X Y$ must be Noetherian (a finitely generated algebra over a Noetherian ring is always Noetherian). Thus the ideal sheaf $\mathcal{J}$ of the diagonal in $Y \times_X Y$ is coherent [AG, §II, Prop. 5.9, p. 116], and so also is $\Omega^1_{Y/X}$ (Question: is an easier way to see this, simply using affine properties?). This lets us apply Nakayama's Lemma: we conclude that to prove that $\Omega^1_{Y/X}$'s stalk at $y$ vanishes, it is enough to prove that $\Omega^1_{Y/X}$'s fiber $k(y) \otimes_Y \Omega^1_{Y/X}$ at $y$ vanishes.

Let $x := f(y)$. Since $\Omega^1_{Y_x/k(x)}$ is the pullback of $\Omega^1_{Y/X}$ along $Y_x \longrightarrow Y$ [AG, §II, Prop. 8.10, p. 175], this fiber $k(y) \otimes_Y \Omega^1_{Y/X}$ will certainly vanish if we can show that $\Omega^1_{Y_x/k(x)}$'s stalk at $y$ vanishes. Thus we can finally take the fiber $Y_x$ we're interested in to be Spec $A$, where $A := \mathcal{O}_{Y,y}\big/\mathfrak{a}$ is the $k(x)$-algebra appearing in the definition of ramification (our ability to do this depends on [AG, §II, Prop. 8.2A, p. 173], which tells us that talking stalks in $Y$ commutes with calculating $\Omega^1_{Y/X}$). Thus we need only prove the special case:

**Part 2 of (i)⇒(ii). Proof in the special case:** Suppose $f$ is merely the algebra $\mathcal{O}_{Y_x,y} \longleftarrow k$, where $k := k(x)$. Then the very definition of $f$ being unramified at $y$ says that $\mathcal{O}_{Y_x,y}$ is a finite, separable field extension $K/k$.

Fix $a \in K$, letting $p_a(t) \in k[t]$ be $a$'s minimal polynomial. We repeat one more time that one formulation of separability of $K/k$ says that although $p_a(a) = 0$, we must have $p'_a(a) \neq 0$, where $p'_a(t)$ is the formal derivative. Now, the $K$-module $\Omega^1_{K/k}$ comes with a universal $k$-derivation $\delta : K \to \Omega^1_{K/k}$ [AG, §II, Prop 8.1A, p. 173]. In particular, $\delta$ is additive and satisfies the Leibniz rule (for constants in $k$), which gives us the formal chain rule

$$0 = \delta(0) = \delta(p_a(a)) = p'_a(a)\,\delta a.$$

Since $p'_a(a) \neq 0 \in K$, this implies that $\delta a = 0$. But the image of $\delta : K \to \Omega^1_{K/k}$ generates $\Omega^1_{K/k}$ (again [AG, §II, Prop 8.1A, p. 173], combined with the direct construction at the bottom of the preceding p. 172 therein), so $\Omega^1_{K/k} = 0$.

**(ii)⇒(iii).** First, the diagonal $\Delta : Y \to Y \times_X Y$ is always locally closed (we cited this fact above, it comes from [AG, §II.8, p. 175]). Since we're only interested in the local picture here, we can thus reduce to the case that $\Delta : Y \to Y \times_X Y$ is a closed immersion.

Now as a general remark, suppose that (1) $\mathcal{M}$ is some coherent sheaf on a scheme $Z$ and that (2) the stalk $\mathcal{M}_z = 0$ at some point $z \in Z$. Then $\mathcal{M}|_U = 0$ in some open neighborhood $U \ni z$ in $Z$. Indeed, fix an affine neighborhood Spec $A \ni z$ over which $\mathcal{M}$ is an $A$-module $M$, with generators $\{s_1, \ldots, s_n\}$. If $M_z = 0$, then each generator satisfies $s_i|_z = 0$, thus restricts to 0 in some open neighborhood $U_i \ni z$. Thus

$$M|_{U_1 \cap \cdots \cap U_n} = 0,$$

due to right exactness of $A_\mathfrak{p} \otimes_A (-)$.

Our particular interest is in the coherent sheaf $\mathcal{J}$ on $Y \times_X Y$. If we can prove that the vanishing of $\Omega^1_{Y/X}$ at $y$ implies that $\mathcal{J}_y = 0$, then by the observation of the previous paragraph, we can conclude that $\mathcal{J}$ actually vanishes over a whole open neighborhood of $y$ in $Y \times_X Y$ (indeed, $\mathcal{J}$ is coherent: [AG. §II.5, Prop. 5.9, p.116]), implying that this whole open neighborhood is in the support of $\mathcal{O}_{Y \times_X Y}/\mathcal{J}$. But since we've reduced to the case that $\Delta : Y \to Y \times_X Y$ is a closed immersion, we know that $Y \cong \Delta(Y)$ is the support of $\mathcal{O}_{Y \times_X Y}/\mathcal{J}$ **[citation?]**. Thus $\Delta : Y \to Y \times_X Y$ is an open immersion in a neighborhood of $y$.

The whole proof rest therefore on a demonstration that

$$\Omega^1_{Y/X} = 0 \text{ at } y \qquad \text{implies} \qquad \mathcal{J}_y = 0.$$

To this end, let $\mathfrak{m}_y$ and $I$ be the ideals generated by $y$ and $\mathcal{J}$, respectively, in $\mathcal{O}_{Y \times_X Y, y}$. Then the statement that $\Omega^1_{Y/X} = 0$ at $y$ implies that $I\big/I^2 = 0$. But since $y \in \Delta(Y)$, we have $\mathfrak{m}_y \supset I$, and thus $I\big/\mathfrak{m}_y I = 0$. By Nakayama's Lemma (whose hypotheses, in particular coherency of $I$, we've already established above), this implies that $I = 0$, that is, $\mathcal{J}_y = 0$.

**(iii)⇒(i).** Let $\Delta : Y \to Y \times_X Y$ be a local isomorphism at $y \in Y$. Then $\Delta : Y_x \to Y_x \times_{\{x\}} Y_x$ is a local isomorphism at $y \in Y_x$, since it is the pullback of $\Delta : Y \to Y \times_X Y$ along $Y_x \times_{\{x\}} Y_x \longrightarrow Y \times_X Y$, pullbacks of open immersions being open immersions themselves (see the proof of [AG, §II, Thm 3.3, p. 87]). In fact, letting $A := \mathcal{O}_{Y,y}\big/\mathfrak{a}$, the algebra showing up in the definition of ramification, we have that $\Delta : \operatorname{Spec} A \to \operatorname{Spec} A \times_{\{x\}} \operatorname{Spec} A$ is the pullback of $\Delta : Y_x \to Y_x \times_{\{x\}} Y_x$ along

$$\operatorname{Spec} A \times_{\{x\}} \operatorname{Spec} A \longrightarrow Y_x \times_{\{x\}} Y_x,$$

and thus $\Delta : \operatorname{Spec} A \to \operatorname{Spec} A \times_{\{x\}} \operatorname{Spec} A$ is an open immersion. Thus we can reduce to the case where $f : Y \to X$ is dual to a finite type local algebra $A \longleftarrow k$.

We want to show simply that $A \longleftarrow k := k(x)$ is unramified. But now we can employ our geometric ramification criterion, and observe that it is enough to demonstrate that $\overline{A} \longleftarrow \overline{k}$ is of the form $\operatorname{Spec} \overline{A} \cong \bigsqcup_i \operatorname{Spec} \overline{k}$. In fact, it's easy to see that $\Delta : \operatorname{Spec} \overline{A} \to \operatorname{Spec} \overline{A} \times_{\{\overline{x}\}} \operatorname{Spec} \overline{A}$ is the pullback of $\Delta : \operatorname{Spec} A \to \operatorname{Spec} A \times_{\{x\}} \operatorname{Spec} A$ along

$$\operatorname{Spec} A \times_{\{x\}} \operatorname{Spec} A \longrightarrow \operatorname{Spec} \overline{A} \times_{\{\overline{x}\}} \operatorname{Spec} \overline{A},$$

so $\Delta : \operatorname{Spec} \overline{A} \to \operatorname{Spec} \overline{A} \times_{\{\overline{x}\}} \operatorname{Spec} \overline{A}$ is another open immersion.

Let $Y_{\overline{x}} := \operatorname{Spec} \overline{A}$.

Then for each <u>closed</u> $\overline{y} \in Y_{\overline{x}}$, we have a section $\overline{y} : \operatorname{Spec} \overline{k} \to Y_{\overline{x}}$ whose image is $\{\overline{y}\}$. To see this, one has to think for a moment about finite-versus-transcendental extensions. It's not hard to see though that the resulting square

$$\begin{array}{ccc} \{\overline{y}\} & \longrightarrow & Y_{\overline{x}} \\ \downarrow & & \downarrow \Delta \\ Y_{\overline{x}} & \underset{(\overline{y}f,1)}{\longrightarrow} & Y_{\overline{x}} \times_{\overline{x}} Y_{\overline{x}} \end{array}$$

is Cartesian. Since open immersions are preserved under base change, and since $\Delta$ <u>is</u> an open immersion, this means that $\{\overline{y}\}$ must be open in $Y_{\overline{x}}$. Thus $\{\overline{y}\}$ is a component of $Y_{\overline{x}}$, and $Y_{\overline{x}}$ is a disjoint union of spectra $\operatorname{Spec} \overline{k}$. In short, $Y_{\overline{x}} \longrightarrow \operatorname{Spec} \overline{k}$ is unramified, and thus our original $Y \longrightarrow X$ is unramified at $y \in Y$. ■

**Corollary 1.2.8** — Let $f : Y \to X$ be a scheme map. Then the following are equivalent:

**(i)** $f$ is étale;

**(ii)** $f$ is locally of finite type, flat, and $\Omega^1_{Y/X} = 0$;

**(iii)** $f$ is locally of finite type, flat, and the diagonal $\Delta_{Y/X} : Y \to Y \times_X Y$ is an open immersion. ■

**Other Interesting Topics that We've Neglected in this Section:** First, the reader should look up *étale morphism* on Wikipedia and read all the equivalent definitions. That list touches on many of the subjects we've neglected here. The main ones I'm aware of are:

• *Standard étale maps*: when $Y \to X$ is just an algebra $B \longleftarrow A$, we say that $Y$ is *standard étale* if we can find polynomials $f, g \in A[t]$, with $f$ monic, such that $f'$ is a unit in $A[t]_g$ and such that $B \cong (A[t]/(f))_g$. This generalizes examples like

$$\mathbb{C}[t^{\pm 1}] \longleftarrow \mathbb{C}[t^2],$$

the double sheeted cover of the punctured affine line. The basic theorem here is that a map is étale if and only if it is locally of finite type and locally standard étale;

• *Étale maps and the Implicit Function Theorem*: a map $Y \longrightarrow X$ between <u>smooth</u> schemes is étale at $y$ if and only if the induced map $\mathfrak{m}_y / \mathfrak{m}_y^2 \longrightarrow \mathfrak{m}_x / \mathfrak{m}_x^2$ of "tangent spaces" is an isomorphism;

• *Functorial characterization of étale maps*: one can characterize the condition that $Y \longrightarrow X$ is étale in terms of properties of the representable functor $\operatorname{Hom}_{\mathbf{Sch}/X}(-, Y) : \mathbf{Sch}^{\mathrm{op}}_{/X} \to \mathbf{Sets}$. When $f$ is étale, $\operatorname{Hom}_{\mathbf{Sch}/X}(-, Y)$ can't distinction between an affine scheme $Z \longrightarrow X$ and any subscheme $Z_0 \hookrightarrow Z$ of it determined by a square-zero ideal $I \subset \Gamma(Z, \mathcal{O}_Z)$. This is roughly the statement that "an affine-local section of $Y$ over $X$ takes its entire determination from its behavior on the $1^{\mathrm{st}}$-order tangent neighborhood of any point in $X$." The connection to Riemann surfaces and analytic continuation is apparent.