Some Variational Problems from Image Processing *†

John B. Garnett ‡ Triet M. Le § Luminita A. Vese ¶

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Abstract

We consider in this paper a class of variational models introduced for image decomposition into cartoon and texture in [17] (see also [10]) of the form $\inf_u \left\{ \|u\|_{BV} + \lambda \|K * (f-u)\|^q_{L^p} \right\}$ where K is a real analytic integration kernel. We analyse and characterize the extremals of these functionals and list some of their properties.

1 Introduction and Motivations

A variational model for decomposing a given image-function f into $u + v$ can be given by

$$\inf_{(u,v)\in X_1\times X_2} \Big\{ F_1(u) + \lambda F_2(v) : f = u + v \Big\},$$

where $F_1, F_2 \geq 0$ are functionals and X_1, X_2 are function spaces such that $F_1(u) < \infty$, and $F_2(v) < \infty$, if and only if $(u, v) \in X_1 \times X_2$. The constant $\lambda > 0$ is a tuning (scale) parameter. A good model is given by a choice of X_1 and X_2 so that with the given desired properties of u and v, we have: $F_1(u) << F_1(v)$ and $F_2(u) >> F_2(v)$. The decomposition model is equivalent with:

$$\inf_{u\in X_1} \Big\{ F_1(u) + \lambda F_2(f-u) \Big\}$$

In this work we are interested in the analysis of a class of variational BV models arising in the decomposition of an image function f into cartoon or BV component, and a texture or oscillatory component. This topic has been of much interest in the recent years. We first recall the definition of BV functions.

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‡Department of Mathematics, University of California, Los Angeles, 405 Hilgard Ave., Los Angeles, CA 90095-1555, USA (jbg@math.ucla.edu).

§Department of Mathematics, Yale Unversity, New haven, CT 06500, USA (triet.le@yale.edu).

¶Department of Mathematics, University of California, Los Angeles, 405 Hilgard Ave., Los Angeles, CA 90095-1555, USA (lvese@math.ucla.edu).

Definition 1. Let $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ be real. We say $u \in BV$ if

$$\sup\Big\{\int u \text{div}\varphi dx : \varphi \in C^1_0(\mathbb{R}^d), \sup|\varphi(x)| \leq 1\Big\} = ||u||_{BV} < \infty.$$

If $u \in BV$ there is an $\mathbb{R}^d$ valued measure $\vec{\mu}$ such that $\frac{\partial u}{\partial x_j} = (\vec{\mu})_j$ as distributions, a positive measure μ, and a Borel function $\vec{\rho} : \mathbb{R}^d \to S^{d-1}$ such that

$$Du = \vec{\mu} = \vec{\rho}\mu$$

and

$$||u||_{BV} = \int d\mu.$$

(see Evans-Gariepy [16], for example).

1.1 History

Assume $f \in L^2(\mathbb{R}^d)$, f real. We list here several variational BV models that have been proposed as image decomposition models.

Rudin-Osher-Fatemi [23] (1992) proposed the minimization

$$\inf_{u\in BV}\Big\{\|u\|_{BV} + \lambda\int|f-u|^2dx\Big\}.$$

In this model, we call u a "cartoon" component, and $f-u$ a "noise+texture" component of f, with $f = u+v$. Note that there exists a unique minimizer u by the strict convexity of the functional.

A limitation of this model is illustrated by the following example [21, 13]: let $f = \alpha\chi_D$, $d = 2$, with D a disk centered at the origin and of radius R; if $\lambda R \geq 1/\alpha$, then $u = (\alpha - (\lambda R)^{-1})\chi_D$ and $v = f - u = (\lambda R)^{-1}\chi_D$; if $\lambda R \leq 1/\alpha$, then $u = 0$. Thus, although $f \in BV$ is without texture or noise, we do not have $u = f$.

Chan-Esedoglu [12] (2005) considered and analyzed the minimization (see also Alliney [5] for the discrete case)

$$\inf_{u\in BV}\Big\{\|u\|_{BV} + \lambda\int|f-u|dx\Big\}.$$

The minimizers of this problem exist, but they may not be unique. If $d = 2$, $\quad f = \chi_{B(0,R)}$, then $u = f$ if $R > \frac{2}{\lambda}$ and $u = 0$ if $R < \frac{2}{\lambda}$.

W. Allard [2, 3, 4] (2007) analyzed extremals of

$$\inf_{u\in BV}\Big\{\|u\|_{BV} + \lambda\int\gamma(u-f)dx\Big\}$$

where $\gamma(0) = 0$, $\gamma \geq 0$, γ locally Lipschitz. Then there exist minimizers u, perhaps not unique, and

$$\partial^*(\{u > t\}) \in C^{1+\alpha}, \quad \alpha \in (0,1)$$

where ∂^* denotes "measure theoretic boundary". Also, Allard gave mean curvature estimates on $\partial^*(\{u > t\})$.

Y. Meyer [21] (2001) in his book *Oscillatory Patterns in Image Processing* analysed further the R-O-F minimization and refined these models proposing

$$\inf_{u \in BV} \Big\{ \|u\|_{BV} + \lambda \|u - f\|_X \Big\}$$

where

$$X = (W^{1,1})^* = \Big\{ \mathrm{div}\vec{g} : \ \vec{g} \in L^\infty \Big\} = G, \quad X = \Big\{ \mathrm{div}\vec{g} : \ \vec{g} \in BMO \Big\} = F,$$

or

$$X = \Big\{ \triangle g : \ g \text{ Zygmund} \Big\} = E.$$

Inspired by the proposals of Y. Meyer, recently a rich literature of models has been developed and analyzed theoretically and computationally. We list the more relevant ones.

Osher-Vese [27] (2002) proposed

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu \|f - (u + \mathrm{div}\vec{g})\|_2^2 + \lambda \|\vec{g}\|_p \Big\}, \quad p \to \infty$$

to approximate the (BV, G) Meyer's model and make it computationally amenable. Osher-Solé-Vese [22] proposed the minimization

$$\inf_{u} \Big\{ \|u\|_{BV} + \lambda \|f - u\|_{H^{-1}} \Big\}$$

and later Linh-Lieu [20] generalized it to

$$\inf_{u} \Big\{ \|u\|_{BV} + \lambda \|f - u\|_{H^{-s}} \Big\}, \quad s > 0.$$

Similarly, Le-Vese [19] (2005) approximated (BV, F) Meyer's model by

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu \|f - (u + \mathrm{div}\vec{g})\|_2^2 + \lambda \|\vec{g}\|_{BMO} \Big\}.$$

Aujol et al. [7, 8] addressed the original (BV, G) Meyer's problem and proposed an alternate method to minimize

$$\inf_{u} \Big\{ \|u\|_{BV} + \lambda ||f - u - v||_2 \Big\},$$

subject to the constraint $||v||_G \leq \mu$.

Garnett-Le-Meyer-Vese [17] (2007) proposed reformulations and generalizations of Meyer's (BV, E) model (see also Aujol-Chambolle [10]), given by

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu \|f - (u + \triangle\vec{g})\|_2^2 + \lambda \|\vec{g}\|_{\dot{B}^{\alpha}_{p,q}} \Big\}$$

where $1 \leq p, q \leq \infty$, $0 < \alpha < 2$, and the exact decompositions are given by

$$\inf_{u} \Big\{ \|u\|_{BV} + \lambda \|f - u\|_{\dot{B}^{\alpha-2}_{p,q}} \Big\}.$$

In a subsequent work, Garnett-Jones-Le-Meyer [18] proposed different formulations,

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu \|f - (u + \triangle\vec{g})\|_2^2 + \lambda \|\vec{g}\|_{B\dot{M}O^{\alpha}} \Big\},$$

with $\dot{BMO}^{\alpha} = I_\alpha(BMO)$, $\|v\|_{\dot{BMO}^\alpha} = \|I_\alpha v\|_{BMO}$, and

$$\inf_{u,\vec{g}} \Big\{ \|u\|_{BV} + \mu\|f - (u + \triangle\vec{g})\|_2^2 + \lambda\|\vec{g}\|_{\dot{W}^{\alpha,p}} \Big\},$$

with $\|v\|_{\dot{W}^{\alpha,p}} = \|I_\alpha v\|_p$, $0 < \alpha < 2$.

Generalizing (BV, H^{-s}), $(BV, \dot{B}^{\alpha}_{p,q})$, and the $TV - Hilbert$ model [9], an easier cartoon+texture decomposition model can be defined using a smoothing convolution kernel K (previously introduced in [17]):

$$\inf_{u \in BV} \Big\{ \|u\|_{BV} + \lambda\|K * (f - u)\|^q_{L^p} \Big\}. \tag{1}$$

This can be seen as a simplified version of all the previous models.

2 The Variational Problems

In this paper we assume K is a positive, even, bounded and real analytic kernel on $\mathbb{R}^d$ such that $\int K dx = 1$ and such that the map $L^p \ni u \to K * u$ is injective. For example we may take K to be a Gaussian or a Poisson kernel. We fix $\lambda > 0$, $1 \leq p < \infty$ and $1 \leq q < \infty$. For real $f(x) \in L^1$ we consider the extremal problems

$$m_{p,q,\lambda} = \inf\{||u||_{BV} + \mathcal{F}_{p,q,\lambda}(f - u) : u \in BV\} \tag{2}$$

where

$$\mathcal{F}_{p,q,\lambda}(h) = \lambda||K * h||^q_{L^p}. \tag{3}$$

Since $BV \subset L^{\frac{d}{d-1}}$ and $K \in L^\infty$, a weak-star compactness argument shows that (2) has at least one minimizer u (see Section 3 below for a more detailed argument). Our objective is to describe, given f, the set $\mathcal{M}_{p,q,\lambda}(f)$ of minimizers u of (2) .

The papers of Chan-Esedoglu [12] and Allard [2, 3, 4] give very precise results about the minimizers for variations like (2) but without the real analytic kernel K, and this paper is intended to complement those works.

Remark 1. According to the definition of *admissibility* given in [2], the functional $\mathcal{F}_{p,q,\lambda}$ is admissible for an appropriate choice of K, for instance take K to be bounded (i.e. heat kernel K_t or Poisson kernel P_t for some $t > 0$). Thus the regularity results from section 1.5 in [2] holds for minimizers in $\mathcal{M}_{p,q,\lambda}(f)$. On the other hand, If K is not a Dirac delta function, then $\mathcal{F}_{p,q,\lambda}$ is not *local* as defined in [2].

2.1 Convexity

Since the functional in (2) is convex, the set of minimizers $\mathcal{M}_{p,q,\lambda}(f)$ is a convex subset of BV. If $p > 1$ or if $q > 1$, then the functional (3) is strictly convex and the problem (2) has a unique minimizer because $K * u$ determines u.

Lemma 1. *If $p = q = 1$ and if $u_1 \in \mathcal{M}_{p,q,\lambda}$ and $u_2 \in \mathcal{M}_{p,q,\lambda}$, then*

$$\frac{K * (f - u_1)}{|K * (f - u_1)|} = \frac{K * (f - u_2)}{|K * (f - u_2)|} \quad \textit{almost everywhere}, \tag{4}$$

and

$$\vec{\rho}_k \cdot \frac{d\vec{\mu}_j}{d\mu_k} = \left|\frac{d\vec{\mu}_j}{d\mu_k}\right|, \ j \neq k, \tag{5}$$

where for $j = 1, 2$,

$$Du_j = \vec{\mu}_j = \vec{\rho}_j \mu_j$$

with $|\vec{\rho}_j| = 1$ *and* $\mu_j \geq 0$.

Proof: Since $\mathcal{M}_{p,q,\lambda}(f)$ is a convex subset of BV, $\frac{u_1+u_2}{2}$ is also a minimizer. This implies,

$$\begin{aligned} \left\|\frac{u_1+u_2}{2}\right\|_{BV} + \lambda \left\|K * \left(f - \frac{u_1+u_2}{2}\right)\right\|_1 &= \frac{1}{2}\left[\|u_1\|_{BV} + \|u_2\|_{BV}\right] \\ &+ \frac{\lambda}{2}\left[||K*(f-u_1)||_1 + ||K*(f-u_2)||_1\right]. \end{aligned} \tag{6}$$

On the other hand, using convexity of $\|\cdot\|_{BV}$ and $\|\cdot\|_{L^1}$, we have

$$\begin{aligned} \left\|\frac{u_1+u_2}{2}\right\|_{BV} &\leq \frac{1}{2}\left[\|u_1\|_{BV} + \|u_2\|_{BV}\right], \text{ and} \\ \left\|K * \left(f - \frac{u_1+u_2}{2}\right)\right\|_1 &\leq \frac{1}{2}\left[||K*(f-u_1)||_1 + ||K*(f-u_2)||_1\right] \end{aligned} \tag{7}$$

Combining (6) and (7), we obtain

$$\left\|K * (f - \frac{u_1+u_2}{2})\right\|_1 = \frac{1}{2}\big(||K*(f-u_1)||_1 + ||K*(f-u_2)||_1\big),$$

which implies (4). Moreover,

$$\|u_1 + u_2\|_{BV} = \|u_1\|_{BV} + \|u_2\|_{BV}. \tag{8}$$

For $j = 1, 2$, let

$$Du_j = \vec{\mu}_j = \vec{\rho}_j \mu_j, \text{ with } |\vec{\rho}_j| = 1 \text{ and } \mu_j \geq 0.$$

Then for $k = 1, 2$, $k \neq j$, equation (8) implies

$$\int \left|\vec{\rho}_k + \frac{d\vec{\mu}_j}{\mu_k}\right| d\mu_k = \int d\mu_k + \int \left|\frac{d\vec{\mu}_j}{\mu_k}\right| d\mu_k,$$

which implies (5). □

2.2 Properties of $u \in \mathcal{M}_{p,q,\lambda}(f)$

Lemma 2. *Given an* $f \in L^1$. *Suppose* u *is a minimizer of (2) such that* $u \neq f$. *Let*

$$Du = \vec{\mu} = \vec{\rho} \cdot \mu.$$

For each real-valued $h \in BV$, *write* $Dh = \vec{\nu}$ *and* $\vec{\nu} = \frac{d\vec{\nu}}{d\mu}\mu + \vec{\nu}_s$ *as the Lebesgue decomposition of* $\vec{\nu}$ *with respect to* μ. *Then*

$$\left|\int \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu} d\mu - \lambda \int h(K * J_{p,q})dx\right| \leq ||\vec{\nu}_s||, \tag{9}$$

where

$$J_{p,q} = q\frac{F|F|^{p-2}}{||F||_p^{p-q}} \text{ with } F = K * (f-u) \tag{10}$$

and $||\vec{\nu}_s||$ *denotes the norm of the vector measure* $\vec{\nu}_s$*. Conversely, if* $u \in BV$*,* $u \neq f$ *and (9) and (10) hold, then* $u \in \mathcal{M}_{p,q,\lambda}(f)$*.*

Note that since $u \neq f$ and $K * (f-u)$ is real analytic, $J_{p,q}$ is defined almost everywhere.
Proof: Let $|\epsilon|$ be sufficiently small. Since u is extremal, we have

$$||u+\epsilon h||_{BV} - ||u||_{BV} + \mathcal{F}_{p,q,\lambda}(f-u-\epsilon h) - \mathcal{F}_{p,q,\lambda}(f-u) \geq 0. \tag{11}$$

On the other hand, we have

$$\left|\vec{\rho} + \epsilon\frac{d\vec{\nu}}{d\mu}\right| = \left(1 + 2\epsilon\vec{\rho}\cdot\frac{d\vec{\nu}}{d\mu} + \epsilon^2\left\|\frac{d\vec{\nu}}{d\mu}\right\|^2\right)^{1/2} = \left(1 + \epsilon\vec{\rho}\cdot\frac{d\vec{\nu}}{d\mu} + o(|\epsilon|)\right),$$

where in the last equality, we use the estimate $(1+\alpha)^{1/2} = 1 + \frac{\alpha}{2} + o(|\alpha|)$. This implies,

$$||u+\epsilon h||_{BV} - ||u||_{BV} = |\epsilon|||\vec{\nu}_s|| + \int\left(\left|\vec{\rho} + \epsilon\frac{d\vec{\nu}}{d\mu}\right| - 1\right)d\mu = |\epsilon|||\vec{\nu}_s|| + \epsilon\int\vec{\rho}\cdot\frac{d\vec{\nu}}{d\mu}d\mu + o(|\epsilon|).$$

Moreover,

$$\begin{aligned}\mathcal{F}_{p,q,\lambda}(f-u-\epsilon h) - \mathcal{F}_{p,q,\lambda}(f-u) &= -\lambda\epsilon\int(K*h)J_{p,q}dx + o(|\epsilon|)\\ &= -\lambda\epsilon\int h(K*J_{p,q})dx + o(|\epsilon|)\end{aligned}$$

since K is even (symmetric). By (11), we have

$$-\epsilon\left[\int\vec{\rho}\cdot\frac{d\vec{\nu}}{d\mu}d\mu - \lambda\int h(K*J_{p,q})dx\right] \leq |\epsilon|\|\vec{\nu}_s\| + o(|\epsilon|)$$

Taking $\pm\epsilon$ and since the right hand side of the above equation does not depend on the sign of ϵ, we see that (9) holds.

The converse holds because the functional (3) is convex. □

Following Meyer [21], we define

$$||v||_* = \inf\left\{|||u|||_\infty : v = \sum_{j=1}^{d}\frac{\partial u_j}{\partial x_j}, |u|^2 = \sum_{i=1}^{d}|u_j|^2\right\}$$

and note that $||v||_*$ is the norm of the dual of $W^{1,1} \subset BV$, when $W^{1,1}$ is given the norm of BV. By the weak-star density of $W^{1,1}$ in BV,

$$\left|\int hv dx\right| \leq ||h||_{BV}||v||_* \tag{12}$$

whenever $v \in L^2$.

Remark 2. Taking $h \in BV$ in Lemma 2 such that $\vec{\nu}_s = 0$, i.e. Dh is absolutely continuous with respect to Du, then (9) implies

$$\int \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu} d\mu - \lambda \int h(K * J_{p,q}) dx = 0. \tag{13}$$

In particular, for any $h \in W^{1,1}$, the above equation holds. I.e.

$$\int h(K * J_{p,q})\ dx = \frac{1}{\lambda} \int \vec{\rho} \cdot \frac{d\vec{\nu}}{d\mu} d\mu. \tag{14}$$

We have the following characterization of minimizers in terms of $\| \cdot \|_*$ (following Meyer [21]).

Lemma 3. *Let $u \in BV$ such that $u \neq f$, and let $J_{p,q}$ be defined as in Lemma 2. Then u is a minimizer for the problem (2) if and only if*

$$||K * J_{p,q}||_* = \frac{1}{\lambda} \tag{15}$$

and

$$\int u(K * J_{p,q}) dx = \frac{1}{\lambda} ||u||_{BV}. \tag{16}$$

Proof: If u is a minimizer, we use Lemma 2. For any $h \in W^{1,1}$, (14) yields

$$||K * J_{p,q}||_* \leq \frac{1}{\lambda}.$$

By (12)

$$\left| \int u(K * J_{p,q}) dx \right| \leq ||u||_{BV} ||K * J_{p,q}||_*,$$

and by setting $h = u$ in (13), we obtain

$$\lambda \int u(K * J_{p,q}) dx = ||u||_{BV}.$$

Therefore (15) and (16) hold.

Conversely, assume $u \in BV$ satisfies (15) and (16) and note that u determines $J_{p,q}$. Still following Meyer [21], we let $h \in BV$ be real. Then for small $\epsilon > 0$, (12), (15) and (16) give

$$\begin{aligned} ||u + \epsilon h||_{BV} + \lambda ||K * (f - u - \epsilon h)||_1 \ &\geq \ \lambda \int (u + \epsilon h)(K * J_{p,q}) dx + \lambda ||K * (f - u)||_1 \\ &- \ \epsilon \lambda \int h(K * J_{p,q}) dx + o(\epsilon) \\ &= \ ||u||_{BV} + \epsilon \lambda \int h(K * J_{p,q}) dx - \epsilon \lambda \int h(K * J_{p,q}) dx + o(\epsilon) \\ &\geq \ 0. \end{aligned}$$

Therefore u is a local minimizer for the functional (2), and by convexity that means u is a global minimizer.

2.3 Radial Functions

Assume K is radial, $K(x) = K(|x|)$. Also assume f is radial and $f \notin \mathcal{M}_{p,q,\lambda}(f)$. Then averaging over rotations shows that each $u \in \mathcal{M}_{p,q,\lambda}(f)$ is radial, so that

$$Du = \rho(|x|)\frac{x}{|x|}\mu$$

where μ is invariant under rotations and where $\rho(|x|) = \pm 1$ a.e. $d\mu$. Let $H \in L^1(\mu)$ be radial and satisfy $\int H d\mu = 0$ and $H = 0$ on $|x| < \epsilon$, and define

$$h(x) = \int_{B(0,|x|)} H(|y|)\frac{1}{|y|^{d-1}}d\mu.$$

Then $h \in BV$ is radial and

$$Dh = \vec{\nu} = H(|x|)\frac{x}{|x|}\mu.$$

Consequently $\vec{\nu}_s = 0$ and (9) gives

$$\int \rho H d\mu = \lambda \int K * J_{p,q}(x) \int_{B(0,|x|)} \frac{H(y)}{|y|^{d-1}}d\mu(y)dx = \lambda \int \Big(\int_{|x|>|y|} K * J_{p,q}(x)dx\Big)\frac{H(|y|)}{|y|^{d-1}}d\mu(y),$$

so that a.e. $d\mu$,

$$\rho(|y|) = \frac{\lambda}{|y|^{d-1}} \int_{|x|>|y|} K * J_{p,q}(x)dx. \tag{17}$$

But the right side of (17) is real analytic in $|y|$, with a possible pole at $|y| = 0$, and $\rho(|y|) = \pm 1$ almost everywhere μ. Therefore there is a finite set

$$\{r_1 < r_2 < \cdots < r_n\} \tag{18}$$

of radii such that

$$Du = \frac{x}{|x|}\sum_{j=1}^{n} c_j \Lambda_{d-1}|\{|x| = r_j\}|$$

for real constants $c_1, \ldots, c_n$, where Λ_{d-1} denotes $d-1$ dimensional Hausdorff measure. By Lemma 1, $J_{p,q}$ is uniquely determined by f, and hence the set (18) is also unique. Moreover, it follows from Lemma 1 that for each j, either $c_j \geq 0$ for all $u \in \mathcal{M}_{p,1,\lambda}(f)$ or $c_j \leq 0$ for all $u \in \mathcal{M}_{p,1,\lambda}(f)$. We have proved:

Theorem 1. *Suppose K and f are both radial. If $f \notin \mathcal{M}_{p,q,\lambda}(f)$, then there is a finite set (18) such that all $u \in \mathcal{M}_{p,q,\lambda}(f)$ have the form*

$$\sum_{j=1}^{n} c_j \chi_{B(0,r_j)}. \tag{19}$$

Moreover, there is $X^+ \subset \{1, 2, \ldots, n\}$ such that $c_j \geq 0$ if $j \in X^+$ while $c_j \leq 0$ if $j \notin X^+$.

Note that by convexity $\mathcal{M}_{p,q,\lambda}(f)$ consists of a single function unless $p = q = 1$. In Section 2.6 we will say more about the solutions of the form (19).

2.4 Example

Unfortunately, Theorem 1 does not hold more generally. The reason is that when u is not radial it is difficult to produce BV functions satisfying $\vec{\nu} << \mu$. For simplicity we take $d = 2$ and $p = q = 1$. Let $J = J_{1,1} = \chi_{0<x\leq 1} - \chi_{-1<x\leq 0}$ and $J(x+2, y) = J(x, y)$. Choose $\lambda > 0$ so that $U = \lambda K * J$ satisfies $||U||_* = 1$, and note that $\frac{U}{|U|} = J$. Notice that $u \in C^2$ solves the curvature equation

$$\mathrm{div}\Big(\frac{\nabla u}{|\nabla u|}\Big) = U \tag{20}$$

if and only if the level sets $\{u = a\}$ are curves $y = y(x)$ that satisfy the simple ODE $y'' = U(x, 0)(1 + (y')^2)^{3/2}$ on the line. Consequently (20) has infinitely many solutions u and both u and J satisfy (15) and (16). Hence by Lemma 3 u is a minimizer for f provided that

$$J = \frac{K * (f - u)}{|K * (f - u)|} \tag{21}$$

and there are many f that satisfy (21). For example, one can choose u and f so that $f - u = J$. Note that in this example u can be real analytic except on $U^{-1}(0)$ and not piecewise constant. Similar examples can be made when $(p, q) \neq (1, 1)$.

2.5 Properties of Minimizers when q =1

Here we follow the paper of Strang [25].

Lemma 4. *If $q = 1$ and $u \in \mathcal{M}_{p,1,\lambda}(f)$, then $u \in \mathcal{M}_{p,1,\lambda}(u)$.*

Proof: If

$$||h||_{BV} + \lambda||K * (u - h)||_p < ||u||_{BV},$$

then by the triangle inequality

$$||h||_{BV} + \lambda||K * (f - h)||_p < ||u||_{BV} + \lambda||K * (f - u)||_p$$

so that u is not a minimizer for f. □

We write

$$\mathcal{M} = \mathcal{M}_{p,1,\lambda} = \bigcup_f \mathcal{M}_{p,1,\lambda}(f).$$

Lemma 5. *Let $u \in BV$. Then $u \in \mathcal{M}$ if and only if*

$$\Big|\int \rho \cdot \frac{d\vec{\nu}}{d\mu} d\mu\Big| \leq ||(\vec{\nu})_s|| + \lambda||K * h||_p \tag{22}$$

for all $h \in BV$, where $Dh = \vec{\nu}$.

Proof: This follows like the proof of Lemma 2. Let $a < b$ be such that

$$\mu(\{u = a\} \cup \{u = b\}) = 0. \tag{23}$$

Then $u_{a,b} = \mathrm{Min}\{(u - a)^+, (b - a)\} \in BV$ and $D(u_{a,b}) = \chi_{a<u<b}\vec{\rho}\mu$. □

Lemma 6. *Assume $q = 1$.*

(a) If $u \in \mathcal{M}$, then $u_{a,b} \in \mathcal{M}$.

(b) More generally, if $u \in \mathcal{M}$ and if $v \in BV$ satisfies $\mu_v << \mu_u$ and $\rho_v = \rho_u$ a.e. $d\mu_v$, then $v \in \mathcal{M}$.

Proof: To prove (a) we verify (22). Write $\mu_{a,b} = \chi_{(a,b)}\mu$ so that $D(u_{a,b}) = \vec{\rho}\mu_{a,b}$. Let $h \in BV$ and write $Dh = \vec{\nu}$. Then by (23)

$$\vec{\nu} = \chi_{a<u<b}\frac{d\vec{\nu}}{d\mu}\mu + \big((\vec{\nu})_s + \chi_{u(x)\notin[a,b]}\frac{d\vec{\nu}}{d\mu}\mu\big)$$

is the Lebesgue decomposition of $\vec{\nu}$ with respect to $\mu_{a,b}$, and

$$\int \vec{\rho}\cdot\frac{d\vec{\nu}}{d\mu_{a,b}}d\mu_{a,b} = \int \vec{\rho}\cdot\frac{d\vec{\nu}}{d\mu}d\mu - \int_{g(x)\notin[a,b]}\vec{\rho}\cdot\frac{d\vec{\nu}}{d\mu}d\mu.$$

Then (22) for ν and $\mu_{a,b}$ follows from (22) for μ and ν. The proof of (b) is similar. $\square$

For simplicity we assume $u \geq 0$. Write $E_t = \{x : u(x) > t\}$. Then by Evans-Gariepy [16], E_t has finite perimeter for almost every t,

$$||u||_{BV} = \int_0^\infty ||\chi_{E_t}||_{BV}dt, \tag{24}$$

and

$$u(x) = \int_0^\infty \chi_{E_t}(x)dt. \tag{25}$$

Moreover, almost every set E_t has a *measure theoretic boundary* $\partial_* E_t$ such that

$$\Lambda_{d-1}(\partial_* E_t) = ||\chi_{E_t}||_{BV} \tag{26}$$

and a *measure theoretic outer normal* $\vec{n}_t : \partial_* E_t \to S^{d-1}$ so that

$$D(\chi_{E_t}) = \vec{n}_t\Lambda_{d-1}\big|\partial_* E_t. \tag{27}$$

Theorem 2. *Assume $q = 1$.*

(a) If $u \in \mathcal{M}$, then for almost every t, $\chi_{E_t} \in \mathcal{M}$.

(b) If $u \in \mathcal{M}$ and $u \geq 0$, then for all nonnegative $c_1, ..., c_n$ and for almost all $t_1 < ... < t_n$, $\sum c_j\chi_{E_{t_j}} \in \mathcal{M}$.

Proof: Suppose (a) is false. Then there is $\beta < 1$, and a compact set $A \subset (0,\infty)$ with $|A| > 0$ such that for all $t \in A$ (26) and (27) hold and there exists $h_t \in BV$ such that

$$||\chi_{E_t} - h_t||_{BV} + \lambda||K * h_t||_p \leq \beta||\chi_{E_t}||_{BV}. \tag{28}$$

Choose an interval $I = (a,b)$ such that (23) holds and $|I \cap A| \geq \frac{|I|}{2}$. Define $h_t = 0$ for $t \in I \setminus A$, and take finite sums such that

$$\sum_{j=1}^{N_n} \chi_{E_{t_j^{(n)}}}\Delta t_j^{(n)} \to u_{a,b} \quad (n \to \infty), \tag{29}$$

$$\sum_{j=1}^{N_n} ||\chi_{E_{t_j^{(n)}}}||_{BV} \Delta t_j^{(n)} \to ||u_{a,b}|| \ \ (n \to \infty), \tag{30}$$

and $t_j^{(n)} \in A$ whenever possible. Write $h^{(n)} = \sum_{j=1}^{N_n} h_{t_j^{(n)}} \Delta t_j^{(n)}$. Then by (25) and (28) $\{h^{(n)}\}$ has a weak-star limit $h \in BV$, and by (28), (29) and (30),

$$||u_{a,b} - h||_{BV} + \lambda ||K * h||_p \leq \frac{1+\beta}{2} ||u_{a,b}||_{BV},$$

contradicting Lemma 6. The proof of (b) is similar. □

We believe that the converse of Theorem 2 is false, but we have no counterexample.

2.6 Radial Minimizers

In this section we assume $q = 1$ and $p = 1$. For convenience we assume the kernel $K = K_t$ is Gaussian, so that K has the form

$$K_t(x) = t^{-d} K(\frac{x}{t}) \tag{31}$$

and

$$K_s * K_t = K_{\sqrt{s^2+t^2}}. \tag{32}$$

Note that (31) and (32) imply that

$$||K_t * f||_1 \text{ decreases in } t \tag{33}$$

and for $f \in L^1$ with compact support

$$\lim_{t \to \infty} ||K_t * f||_1 = |\int f dx|. \tag{34}$$

For fixed λ and t we set

$$R(\lambda, t) = \{r > 0 : \chi_{B(0,r)} \in \mathcal{M}\}.$$

By Theorem 1 and Theorem 2 we have $R(\lambda, t) \neq \emptyset$. For $t = 0$ and $K = I$ our problem (2) becomes the problem

$$\inf\{||u||_{BV} + \lambda ||f - u||_{L^1}\}$$

studied by Chan and Esedoglu in [12], and in that case Chan and Esedoglu showed $R(\lambda, 0) = [\frac{2}{\lambda}, \infty)$.

Theorem 3. *There exists $r_0 = r_0(\lambda, t)$ such that*

$$R(\lambda, t) = [r_0, \infty). \tag{35}$$

Moreover

$$[0, \infty) \ni t \to r_0(t) \ \ \textit{is} \ \ \textit{nondecreasing} \tag{36}$$

and

$$\lim_{t \to \infty} r_0(t) = \infty. \tag{37}$$

Proof: Assume $r \notin R(\lambda, t)$ and $0 < s < r$. Write $\alpha = \frac{r}{s} > 1$ and $f = \chi_{B(0,r)}$. By hypothesis there is $g \in BV$ such that

$$||g||_{BV} + \lambda||K_t * (f - g)||_1 < ||f||_{BV}. \tag{38}$$

We write $\tilde{g}(x) = g(\alpha x), \tilde{f}(x) = f(\alpha x) = \chi_{B(0,s)}(x)$, and change variables carefully in (38) to get

$$\alpha||\tilde{g}||_{BV} + \lambda||\frac{1}{t^d}\int K(\frac{x-y}{t})(\tilde{f} - \tilde{g})(\frac{y}{\alpha})dy||_{L^1(x)} < \alpha||\tilde{f}||_{BV}$$

so that

$$\alpha||\tilde{g}||_{BV} + \lambda||\frac{\alpha^d}{t^d}\int K(\frac{\alpha x' - \alpha y'}{t})(\tilde{f} - \tilde{g})(y')dy'||_{L^1(\alpha x')} < \alpha||\tilde{f}||_{BV}$$

and

$$\alpha||\tilde{g}||_{BV} + \lambda\alpha^d\int\Big|K_{\frac{t}{\alpha}} * (\tilde{f} - \tilde{g})(x')\Big|dx' < \alpha||\tilde{f}||_{BV}.$$

Since $\alpha > 1$, this and (33) show

$$||\tilde{g}||_{BV} + \lambda||K_t * (\tilde{f} - \tilde{g})||_1 < ||\tilde{f}||_{BV}$$

so that $s \notin R(\lambda, t)$. That proves (35), and (36) now follows easily from (33). To prove (37) take $g = \frac{r^d}{s^d}\chi_B(0, s),\ s > r$ and use (34). □

We note that not all radial minimizers have the form $\chi_{B(0,r)}$. This is seen by considering separately, for large fixed t and λ, the function $\chi_{B(0,r_2)} + \chi_{B(0,r_1)}$ with r_1 and $r_2 - r_1$ large.

2.7 Characteristic Functions

Still assuming $q = 1$ we let E be such that $\chi_E \in \mathcal{M}$. Then by Evans-Gariepy [16] $\partial_* E = N \cup \bigcup K_j$, where $D(\chi_E)(N) = \Lambda_{n-1}(N) = 0$, K_j is compact and $K_j \subset S_j$, where S_j is a C^1-hypersurface with continuous unit normal $\vec{n}_j(x), x \in S_j$, and $\vec{n}_j$ is the measure theoretic outer normal of E. After a coordinate change write $S_j = \{x_d = f_j(y)\}, y = (x_1, \ldots, x_{d-1})$ with ∇f_j continuous and $\vec{n}_j(y, f_j(y)) \perp (\nabla f_j, 1)$. Assume $y = 0$ is a point of Lebesgue density of $(f_j, 1)^{-1}(K_j)$, let $V \subset \mathbb{R}^{d-1}$ be a neighborhood of $y = 0$, let $g \in C_0^\infty(V)$ with $g \geq 0$, and consider the variation $u_\epsilon = \chi_{E_\epsilon}$ where $\epsilon > 0$ and

$$E_\epsilon = E \cup \{0 \leq x_d \leq \epsilon u(y), y \in V\}.$$

Then $E \subset E_\epsilon$, and writing $u_0 = \chi_E$, we have

$$||u_\epsilon||_{BV} - ||u_0||_{BV} = \int_V \sqrt{(1 + |\nabla(f_j + \epsilon g)|^2)} - \sqrt{(1 + |\nabla f_j|^2)}dy = o(\epsilon) \tag{39}$$

because by [16]

$$\Lambda_{d-1}((\partial_* E) \cup (E_\epsilon \setminus E)) = o(\epsilon)$$

Λ_{d-1} a.e. on K_j. Also, for a similar reason

$$\lambda||K * (u_\epsilon - u_0)||_p = \lambda|\epsilon|\int_V udy + o(\epsilon). \tag{40}$$

Together (39) and (39) show

$$\int_V \nabla u \cdot \Big(\frac{\nabla f_j}{\sqrt{1+|\nabla f_j|^2}}\Big) dy + \lambda \int_V u dy \geq 0. \tag{7.3}$$

Repeating this argument with $\epsilon < 0$, we obtain:

Theorem 4. *At Λ_{d-1} almost every $x \in \partial_* E$,*

$$\Big|\mathrm{div}\Big(\frac{\nabla f_j}{\sqrt{1+|\nabla f_j|^2}}\Big)\Big| \leq \lambda. \tag{41}$$

as a distribution on $\mathbb{R}^{d-1}$.

2.8 Smooth Extremals

For convenience we assume $d = 2$ and we take $p = q = 1$.

Theorem 5. *Let $u \in C^2 \cap \mathcal{M}_{1,1,\lambda}(f)$ and assume $u \neq f$. Set $E_t = \{u > t\}$ and $J = \frac{K*(f-u)}{|K*(f-u)|}$. Then*
(i) $\Lambda_1(\partial_* E_t) = \lambda \iint_{E_t} K * J dxdy$, (ii) *the level curve $\{u(z) = c\}$ has curvature $\lambda(K * J)(z)$, and*

(iii) *if $|\nabla u| \neq 0$, then*

$$\frac{d}{dt}\Lambda_1(\partial_* E_t) = -\int_{\partial E_t} \frac{\lambda(K * J)(z)}{|\nabla u(z)|} ds.$$

Theorem 5 is proved using the variation $u \to u + \epsilon h, h \in C_0^2$. It should be true in greater generality, but we have no proof at this time.

3 Existence of minimizers

Although the proof of the existence of minimizers of our problem can be seen as a generalization and application of more classical techniques [1], [11], [26], we include it here for completeness in several cases. We consider the cases of bounded domain Q and of the whole domain $\mathbb{R}^d$, with various kernel operators $Ku = K * u$. We recall that here, for $u \in BV(Q)$, $\|u\|_{BV(Q)}$ denotes the semi-norm

$$\|u\|_{BV(Q)} = \sup\Big\{\int u \mathrm{div}\varphi dx : \varphi \in C_0^1(Q, \mathbb{R}^d), \sup|\varphi(x)| \leq 1,\ x \in Q\Big\}.$$

3.1 Bounded domain, general operator K and general case $p \geq 1$, $1 \leq q < \infty$

We recall that $K(x)$ is non-negative and even on $\mathbb{R}^d$ with $\int K(x)dx = 1$, thus $K \in L^1(\mathbb{R}^d)$, $\|K\|_{L^1} = 1$, with $K1 = 1 \neq 0$. The linear and continuous operator $u \mapsto Ku = K * u$ is well defined on $L^1(\mathbb{R}^d)$. There are several ways to adapt linear and continuous convolution operators Ku to the case of bounded domains Q, e.g. as shown in [17].

Theorem 6. *Assume $p \geq 1$, $1 \leq q < \infty$, $\lambda > 0$, Q open, bounded and connected subset of $\mathbb{R}^d$, with Lipschitz boundary ∂Q. If $f \in L^p(Q)$ and $K : L^1(Q) \to L^p(Q)$ is linear and continuous, such that $\|K\chi_Q\|_{L^1(Q)} > 0$, then the minimization problem*

$$\inf_{u \in BV(Q)} \|u\|_{BV(Q)} + \lambda \|K(f-u)\|^q_{L^p(Q)} \tag{42}$$

has an extremal $u \in BV(Q)$.

Proof: Let $E(u) = \|u\|_{BV} + \lambda\|K(f-u)\|_p^q$. Infimum of E is finite since $E(u) \geq 0$, and $E(0) = \lambda\|Kf\|_p^q < \infty$. Let u_n be a minimizing sequence, thus $\inf_v E(v) = \lim_{n\to\infty} E(u_n)$. Then $E(u_n) \leq C < \infty$, $\forall n \geq 1$. Poincaré-Wirtinger inequality implies that there is a constant $C' = C'(d, Q) > 0$ such that for all $n \geq 1$, we have $\|u_n - u_{n,Q}\|_1 \leq C'\|u_n\|_{BV}$, where $u_{n,Q}$ is the mean of u_n over Q. Let $v_n = u_n - u_{n,Q}$, thus $v_{n,Q} = 0$ and $Dv_n = Du_n$. Similarly, we have $\|v_n\|_1 \leq C'\|v_n\|_{BV}$.

Since Q is bounded, we have for some constant $C_1 > 0$,

$$(C/\lambda)^{2/q} \geq \|K(f-u_n)\|_p^2 \geq C_1\|K(f-u_n)\|_1^2$$

$$= C_1\|Ku_n - Kf\|_1^2 = C_1\|(Kv_n - Kf) + Ku_{n,Q}\|_1^2$$

$$\geq C_1\Big|\|Kv_n - Kf\|_1 - \|Ku_{n,Q}\|_1\Big|^2$$

$$\geq C_1\|Ku_{n,Q}\|_1(\|Ku_{n,Q}\|_1 - 2\|Kv_n - Kf\|_1)$$

$$\geq C_1\|Ku_{n,Q}\|_1\Big(\|Ku_{n,Q}\|_1 - 2\|K\|(\|v_n\|_1 + \|f\|_1)\Big).$$

Let $x_n = \|Ku_{n,Q}\|_1$ and $a_n = \|K\|(\|v_n\|_1 + \|f\|_1)$. Then $x_n(x_n - 2a_n) \leq \frac{(C/\lambda)^{2/q}}{C_1} = c$, with $0 \leq a_n \leq \|K\|(CC' + \|f\|_1)$, thus we obtain $0 \leq x_n \leq a_n + \sqrt{a_n^2 + c^2} \leq C_2$ for some constant $C_2 > 0$, which implies

$$\|Ku_{n,Q}\|_1 = \frac{|\int_Q u_n dx|}{|Q|}\|K\chi_Q\|_1 \leq C_2.$$

Thanks to assumptions on K, we deduce that the sequence $|u_{n,Q}|$ is uniformly bounded. By Poincaré-Wirtinger inequality we obtain $\|u_n\|_1 \leq$ *constant*. Thus, $\|u_n\|_{BV(Q)} + \|u_n\|_{L^1(Q)}$ is uniformly bounded. Following e.g. [16], we deduce that there is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$, and $u \in BV(Q)$, such that u_{n_j} converges to u strongly in $L^1(Q)$. Then we also have $\|u\|_{BV(Q)} \leq \liminf_{n_j\to\infty}\|u_{n_j}\|_{BV(Q)}$. Since $(u_{n_j} - f) \to (u - f)$ in $L^1(Q)$, and K is continuous from $L^1(Q)$ to $L^p(Q)$, we deduce that $\|K(u_{n_j} - f)\|_p \to \|K(u-f)\|_p$ as $n_j \to \infty$. We conclude that

$$E(u) \leq \liminf_{n_j\to\infty} E(u_{n_j}) = \inf_v E(v),$$

thus u is extremal. □

3.2 Convolution operator K and particular case $p = q = 1$

In this section, we study the existence of minimizers for different choices of convolution kernels K, in the particular case $p = q = 1$.

3.2.1 Smooth Kernels

Suppose $Kv = K_t * v$, where for example K_t is the Poisson kernel of scale $t > 0$. We have $\hat{K}_t(\xi) = e^{-2\pi t|\xi|}$. Let f be a distribution such that $\|K_t * f\|_{L^1} < \infty$. We recall our minimization problem,

$$\inf_{u \in BV} \{\mathcal{J}(u) = \|u\|_{BV} + \lambda \|K_t * (f - u)\|_{L^1}\}. \tag{43}$$

To motivate the proposed minimization model (43) with $t > 0$ for the decomposition of an image f into a BV component u and an oscillatory component $f - u$ (rather than taking $t = 0$), we consider the following two examples of functions or distributions f with $\|K_t * f\|_{L^1}$ small while $\|f\|_{L^1}$ is large.

Example 1. Suppose $f(x) = \sin(2\pi nx)$, $x \in \mathbb{R}$, is an oscillatory function. Then $K_t * f = 2\sin(2\pi nx)e^{-2\pi tn}$. For $Q = [-m/n, m/n]$, we have

$$\|K_t * f\|_{L^1(Q)} = \frac{8m}{\pi n} e^{-2\pi tn}.$$

On the other hand, $\|f\|_{L^1(Q)} = \frac{4m}{\pi n}$. Clearly, $\|K_t * f\|_{L^1} \ll \|f\|_{L^1}$ when n is large.

Example 2. Suppose we are in $\mathbb{R}$ and $f = \sum_{i=0}^{\infty} a_i \delta_{x_i}$ with $\sum_{i=0}^{\infty} |a_i| < \infty$ can also be seen as a (generalized) oscillatory distribution. Note that $f \notin L^1(\mathbb{R})$. However,

$$\|K_t * f\|_{L^1} \leq \sum_{i=0}^{\infty} |a_i| < \infty.$$

Recall that by using the standard property of convolution (Young's inequality), we have for all $v \in L^p$, $1 \leq p \leq \infty$,

$$\|K_t * v\|_{L^p} \leq \|K_t\|_{L^1} \|v\|_{L^p} = \|v\|_{L^p}.$$

Also, using the same arguments as the ones from Lemma 3.24 in Chapter 3 of [6], one obtains the following result

Lemma 7. *Let $u \in BV(\mathbb{R}^d)$. Then*

$$\|K_t * u - u\|_{L^1} \leq t\|u\|_{BV}. \tag{44}$$

Theorem 7. *Let $Q = (0,1)^d$ or $Q = \mathbb{R}^d$, $\lambda > 0$. For each distribution f such that $\|K_t * f\|_{L^1} < \infty$, the variational problem (43) has a minimizer.*

Proof. Let $\{u_n\}$ be a minimizing sequence for (43). This minimizing sequence exists because $\mathcal{J}(u) \geq 0$ for all $u \in BV(Q)$ and $\mathcal{J}(0) = \|K_t * f\|_{L^1(Q)} < \infty$. We have the following uniform bounds,

$$\|u_n\|_{BV(Q)} \leq C, \tag{45}$$

$$\|K_t * (f - u_n)\|_{L^1(Q)} \leq C. \tag{46}$$

Suppose $Q = \mathbb{R}^d$, then

$$\|u_n\|_{L^1(Q)} \leq \|u_n - K_t * u_n\|_{L^1(Q)} + \|K_t * u_n\|_{L^1(Q)} \leq t\|u_n\|_{BV(Q)} + \|K_t * u_n\|_{L^1(Q)}. \tag{47}$$

This shows that $\|u_n\|_{L^1(Q)}$ is uniformly bounded. On the other hand, if $Q = (0,1)^d$, then (45) and (46) imply that $\|u_n\|_{L^1(Q)}$ is uniformly bound. Indeed, suppose (45) and (46) hold. Let $w_n = u_n - u_{n,Q}$ as before, then

$$\|w_n\|_{BV(Q)} \leq C.$$

By Poincare's inequality, we have

$$\|w_n\|_{L^1(Q)} = \|w_n - w_{n,Q}\|_{L^1(Q)} \leq C_Q \|w_n\|_{BV} \leq C.$$

But,

$$|u_{n,Q}| = \|K_t * u_{n,Q}\|_{L^1} \leq \|K_t * u_n\|_{L^1(Q)} + \|K_t * w_n\|_{L^1(Q)} \leq \|K_t * u_n\|_{L^1(Q)} + \|w_n\|_{L^1(Q)} \leq C,$$

thus $u_{n,Q}$ is uniformly bounded. Moreover, by applying Poincare's inequality to u_n, we have

$$\|u_n\|_{L^1} \leq |Q||u_{n,Q}| + \|u_n - u_{n,Q}\|_{L^1(Q)} \leq |Q||u_{n,Q}| + C_Q \|u_n\|_{BV(Q)} \leq C.$$

Therefore, $\|u_n\|_{L^1(Q)}$ is uniformly bounded.

Now, using the compactness property in BV and the lower semicontinuity property of the map $u \to \|u\|_{BV(Q)}$ [16, 6], there exists $u \in BV(Q)$ such that, up to a subsequence (which we still denote by u_n), $u_n \to u$ in $L^1(Q)$ and

$$\|u\|_{BV(Q)} \leq \liminf_{n\to\infty} \|u_n\|_{BV(Q)}. \tag{48}$$

Moreover,

$$\|K_t * (u_n - u)\|_{L^1(Q)} \leq \|u_n - u\|_{L^1(Q)} \to 0, \text{ as } n \to \infty. \tag{49}$$

This together with the asumption that $K_t * f \in L^1(Q)$, we have

$$\|K_t * (f - u)\|_{L^1(Q)} \leq \lim_{n\to\infty} \|K_t * (f - u_n)\|_{L^1(Q)}. \tag{50}$$

Combining (48) and (50), one obtains

$$\mathcal{J}(u) \leq \liminf_{n\to\infty} \mathcal{J}(u_n),$$

which shows that u is a minimizer. □

3.2.2 Riesz Potential

Recall the Riesz potential I_α, $0 < \alpha < d$, defined as [24]

$$I_\alpha f = (-\Delta)^{-\alpha/2} f = K_\alpha * f,$$

where $\hat{K}_\alpha(\xi) = (2\pi|\xi|)^{-\alpha}$. For each $\alpha \in (0, d)$, the homogeneous Sobolev potential space $\dot{W}^{-\alpha,1}$ is defined as

$$\dot{W}^{-\alpha,1} = \{f : \|I_\alpha f\|_{L^1} < \infty\}.$$

Equipped with the norm $\|f\|_{\dot{W}^{-\alpha,1}} = \|I_\alpha f\|_{L^1}$, $\dot{W}^{-\alpha,1}$ becomes a Banach space. From Stein [24] (Chapter V, Section 1.2), if $1 < p < \infty$ and $1/q = 1/p - \alpha/d$, then

$$\|I_\alpha f\|_{L^q(\mathbb{R}^d)} \leq A_{p,q} \|f\|_{L^p(\mathbb{R}^d)}. \tag{51}$$

Here we would like to model the oscillatory component using I_α, $0 < \alpha < d$. Thus the variational problem (43) can be rewritten as

$$\inf_{u \in BV} \{\mathcal{J}(u) = \|u\|_{BV} + \lambda \|K_\alpha * (f - u)\|_{L^1}\}. \tag{52}$$

Theorem 8. *Let $Q = (0,1)^d$. For each $0 < \alpha < d$ and a distribution f such that $\|K_\alpha * f\|_{L^1(\Omega)} < \infty$, the above variational problem (52) has a minimizer.*

Proof. Again, as before, let $\{u_n\}$ be a minimizing sequence for (52). We have

$$\|u_n\|_{BV(Q)} \leq C, \tag{53}$$

$$\|K_\alpha * (f - u_n)\|_{L^1(Q)} \leq C. \tag{54}$$

As in the proof of Thm. 7, condition (54) implies that $u_{n,\Omega}$ is uniformly bounded, and so by Poincare's inequality, $\|u_n\|_{L^1} \leq C$, for all n. This implies that the BV-norm of u_n is uniformly bounded. Thus, there exists $u \in BV$ such that, up to a subsequence, $u_n \to u$ in L^1 and

$$\|u\|_{BV} \leq \liminf_{n \to \infty} \|u_n\|_{BV}.$$

By the compactness of BV in L^p, $1 \leq p < d/(d-1)$, we have up to a subsequence, $u_n \to u$ in L^p, $1 \leq p < d/(d-1)$. Now for a fixed $p \in (1, d/(d-1))$, we have

$$\|K_\alpha * (u_n - u)\|_{L^1} \leq C_q \|K_\alpha * (u_n - u)\|_{L^q} \leq C_{p,q} \|u_n - u\|_{L^p} \to 0 \text{ as } n \to \infty.$$

This implies, up to a subsequence,

$$\|K_\alpha * (f - u)\|_{L^1} = \lim_{n \to \infty} \|K_\alpha * (f - u_n)\|_{L^1}.$$

Therefore, u is a minimizer. □

4 Characterization of Minimizers 2

In this section, we apply the general duality techniques of Ekeland-Temam [15] and in particular of Demengel-Temam [14] to our minimization problem. We note that these results may be seen as related with the other characterization of minimizers from Lemma 2 and Lemma 3, but expressed and proven here in a different language.

4.1 Dual problem and optimality conditions $p = q = 1$

Let $f : L^1(Q)$ be the given data, with $Q \subset \mathbb{R}^d$ open, bounded, connected, and K a smoothing (analytic) convolution kernel, such as the Gaussian kernel or the Poisson kernel. The minimization problem for $p = 1$, $q = 1$ is

$$(P_1) \qquad \inf_{u \in BV(Q)} E(u) = \|u\|_{BV(Q)} + \lambda \|K * (u - f)\|_{L^1(Q)},$$

using the notation $\|u\|_{BV(Q)} = \int_Q |Du|$ for the semi-norm of u in $BV(Q)$. As we have seen, this problem has a solution $u \in BV(Q) \subset L^2(Q)$. For $u \in L^1(Q)$, we will also use the operator notation $Ku = K * u$ to be the corresponding linear and continuous operator from L^1 to L^1, with adjoint K^* (with radially symmetric kernel K, then the operator K is self-adjoint). We wish to characterize the solution u of (P_1) using duality techniques.

We have

$$\inf_{u \in BV(Q)} E(u) = \inf_{u \in W^{1,1}(Q)} F(u),$$

since for any $u \in BV(Q)$, we can find $u_n \in W^{1,1}(Q)$ such that $u_n \to u$ strongly in $L^1(Q)$ and $\|u_n\|_{BV(Q)} \to \|u\|_{BV(Q)}$. Thus let's first consider the simpler problem

$$(P_2) \qquad \inf_{u \in W^{1,1}(Q)} F(u) = \int_Q |\nabla u| dx + \lambda \|K * (u - f)\|_{L^1(Q)}.$$

We now write (P_2^*), the dual of (P_2), in the sense of Ekeland-Temam [15]. We first recall the definition of the Legendre transform (or polar) of a function: let V and V^* be two normed vector spaces in duality by a bilinear pairing denoted $\langle \cdot, \cdot \rangle$. Let $\phi : V \to \bar{\mathbb{R}}$ be a function. Then the Legendre transform $\phi^* : V^* \to \bar{\mathbb{R}}$ is defined by

$$\phi^*(u^*) = \sup_{u \in V} \Big\{ \langle u^*, u \rangle - \phi(u) \Big\}.$$

We let $G_1(w_0) = \lambda \int_Q |w_0 - K * f| dx$ and $G_2(\bar{w}) = \int_Q |\bar{w}| dx$, with $G_1 : L^1(Q) \to \mathbb{R}$, $G_2 : L^1(Q)^d \to \mathbb{R}$, and using $w = (w_0, w_1, w_2, ..., w_d) \in L^1(Q)^{d+1}$, we define $G(w) = G_1(w_0) + G_2(\bar{w})$.

Let $\Lambda = (Ku, \nabla u) : W^{1,1}(Q) \to L^1(Q)^{d+1}$, and Λ^* be its adjoint. Then $E(u) = F(u) + G(\Lambda u)$, with $F(u) \equiv 0$.

Then (P_2^*) is ([15], Chapter III, Section 4):

$$(P_2^*) \qquad \sup_{p^* \in L^\infty(Q)^{d+1}} -F^*(\Lambda^* p^*) - G^*(-p^*).$$

We have $F^*(\Lambda^* p^*) = 0$ if $\Lambda^* p^* = 0$, and $F^*(\Lambda^* p^*) = +\infty$ otherwise. It is easy to see that

$$G^*(p^*) = G_1^*(p_0^*) + G_2^*(\bar{p}^*), \text{ for } p^* = (p_0^*, \bar{p}^*).$$

We have that,

$$G_1^*(p_0^*) = \int_\Omega p_0^*(K * f) dx$$

if $|p_0^*| \le \lambda$ a.e., $G_1^*(p_0^*) = +\infty$ otherwise, and

$$G_2^*(\bar{p}^*) = 0$$

if $|\bar{p}^*| \le 1$ a.e., $G_2^*(\bar{p}^*) = +\infty$ otherwise.

Thus we have

$$(P_2^*) \qquad \sup_{p^* \in X} - \int_\Omega (-p_0^*)(K * f) dx,$$

where $X = \{(p_0^*, p_1^*, ..., p_d^*) = (p_0^*, \bar{p}^*) \in L^\infty(\Omega)^{d+1},\ |p_0^*| \le \lambda,\ |\bar{p}^*| \le 1,\ \Lambda^* p^* = 0\}$.

Under the satisfied assumptions, we have that $\inf(P_1) = \inf(P_2) = \sup(P_2)^*$ and (P_2^*) has at least one solution p^*.

Using the definition of Λ, we can show that [26]

$$X = \{(p_0^*, p_1^*, ..., p_d^*) = (p_0^*, \bar{p}^*) \in L^\infty(\Omega)^{d+1},\ |p_0^*| \leq \lambda,\ |\bar{p}^*| \leq 1,\ K^* p_0^* - \text{div}\bar{p}^* = 0,\ \bar{p}^* \cdot \nu = 0 \text{ on } \partial Q\}.$$

Now let $u \in BV(Q)$ be the solution of (P_1) and $p = (p_0, \bar{p}) \in X$ be the solution of (P_2^*). We must have the extremality relation

$$\|u\|_{BV(Q)} + \|K * u - K * f\|_{L^1(Q)} = \int_Q p_0 (K * f) dx.$$

We have that $Du \cdot \bar{p}$ is an unsigned measure, satisfying a Generalized Green's formula

$$\int_Q Du \cdot \bar{p} = -\int_Q u \text{div} \bar{p} dx + \int_{\partial Q} u(\bar{p} \cdot \nu) ds.$$

Since $\bar{p} \cdot \nu = 0$ $\partial\Omega$ a.e., we have

$$\int_Q |Du| + \int_Q |K * u - K * f| dx + \int_Q p_0 K u dx + \int_Q Du \cdot \bar{p} - \int_Q p_0 (K * f) dx = 0,$$

or using the decomposition $Du = \nabla u dx + D_s u = \nabla u dx + C_u + J_u = \nabla u dx + C_u + (u^+ - u^-)\nu d\mathcal{H}^{d-1}|_{S_u}$ [16], with S_u the support of the jump measure J_u, we get

$$\int_Q |\nabla u| dx + \int_{Q \setminus S_u} |C_u| + \int_{S_u} (u^+ - u^-) d\mathcal{H}^1 + \int_Q \nabla u \cdot \bar{p} dx + \int_{Q \setminus S_u} \bar{p} \cdot C_u$$

$$+ \int_{S_u} (u^+ - u^-) \bar{p} \cdot \nu d\mathcal{H}^{d-1} + \int_Q |K * u - K * f| dx + \int_Q p_0 K u dx - \int_Q p_0 (K * f) dx = 0.$$

Since for any function ϕ and its polar ϕ^* we must have $\phi^*(u^*) - \langle u^*, u \rangle + \phi(u) \geq 0$ for any $u \in V$ and $u^* \in V^*$, we obtain:

1. $|K * u - K * f| - (-p_0)(K * u) + (-p_0)(K * f) \geq 0$ for dx a.e. in Ω
2. $|\nabla u| - \nabla u \cdot (-\bar{p}) + 0 \geq 0$ for dx a.e. in Ω where $\nabla u(x)$ is defined
3. $0 - (-\bar{p}) \cdot C_u + |C_u| = (1 + \bar{p} \cdot h)|C_u| \geq 0$, since $|\bar{p}| \leq 1$ (letting $C_u = h \cdot |C_u|$, $h \in L^1(|C_u|)^d$, $|h| = 1$)
4. $0 - (-\bar{p} \cdot \nu)(u^+ - u^-) + (u^+ - u^-) = (u^+ - u^-)(1 + \bar{p} \cdot \nu) \geq 0$ for $d\mathcal{H}^{d-1}$ a.e. in S_u (again since $|\bar{p}| \leq 1$).

Therefore, each expression in 1-4 must be exactly 0 and we obtain another characterization of extremals u:

Theorem 9. *u is a minimizer of (P_1) if and only if there is $(p_0, p_1, ..., p_d) = (p_0, \bar{p}) \in (L^\infty)^{d+1}$ such that*

$$|p_0| \leq \lambda, \quad |\bar{p}| \leq 1,$$

$$\bar{p} \cdot \nu = 0 \text{ on } \partial Q,$$

$$K^* p_0 - div \bar{p} = 0,$$

$$|K * (u - f)| + p_0 (K * u - K * f) = 0, \tag{55}$$

$$|\nabla u| + \nabla u \cdot \bar{p} = 0,$$

$$1 + \bar{p} \cdot \nu = 0 \ on\ S_u \ and\ |\bar{p}| = 1 \ on\ S_u,$$

and

$$supp|C_u| \subset \{x \in \Omega \setminus S_u, 1 + \bar{p}(x) \cdot h(x) = 0,\ h \in L^1(|C_u|)^d,\ |h| = 1,\ C_u = h|C_u|\}.$$

4.2 Case $1 \leq p, q < \infty$

A similar statement as Thm. 9 can be shown for the general case $1 \leq p, q < \infty$. The main change is in the definition of $\mathcal{G}_1$, which becomes $\mathcal{G}_1(w_0) = \lambda \|w_0 - K * f\|_p^q = \lambda \Big(\int_Q |w_0 - K * f|^p dx \Big)^{q/p}$ for $w_0 \in L^p(Q)$. For example, if $1 < q < \infty$, then $\mathcal{G}_1^*(p_0^*) = \lambda q \Big[\frac{1}{q'} \| \frac{p_0^*}{\lambda q} \|_{p'}^{q'} + \int_Q (K * f) \frac{p_0^*}{\lambda q} dx \Big]$ and (55) changes accordingly, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ (similarly in the case $1 \leq p < \infty$, $q = 1$).

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