Determinants

July 18, 2007

In class we showed that there exists a unique map

$$D : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ components}} \to \mathbb{R}, \quad (v_1, \ldots, v_n) \mapsto D(v_1, \ldots, v_n)$$

satisfying the properties

(1) D is **multilinear**:

$$D(\ldots, \lambda v + \mu w, \ldots) = \lambda D(\ldots, v, \ldots) + \mu D(\ldots, w, \ldots)$$

for all $\lambda, \mu \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$.

(2) D is **alternating**:

$$D(\ldots, v, \ldots, w, \ldots) = -D(\ldots, w, \ldots, v, \ldots)$$

for all $v, w \in \mathbb{R}^n$.

(3) D is **normalized**:

$$D(e_1, \ldots, e_n) = 1$$

We then defined the **determinant** of an nxn-matrix A with column vectors $a_1, \ldots, a_n$ to be

$$\det(A) := D(a_1, \ldots, a_n). \tag{1}$$

Therefore the determinant has properties (1)-(3) with respect to the columns of the matrix A.
We proved that for an nxn-matrix $A = (a_{ij})$ the determinant is given by the explicit formula

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{\sigma(1)\,1} \cdots a_{\sigma(n)n}. \tag{2}$$

This formula is sometimes called the **Leibniz formula**, named after the mathematician Gottfried Leibniz (1646-1716).

Problem 1. Use the Leibniz formula to give an explicit formula for the determinant of a 4x4-matrix. (This should be a sum of 24 products, since $\#S_4 = 24$).

Solution.

$$\begin{aligned}
\det(A) = {} & a_{11}a_{22}a_{33}a_{44} + a_{11}a_{32}a_{43}a_{24} + a_{11}a_{42}a_{23}a_{34} \\
& + a_{21}a_{12}a_{43}a_{34} + a_{21}a_{32}a_{13}a_{44} + a_{21}a_{42}a_{33}a_{14} \\
& + a_{31}a_{12}a_{23}a_{44} + a_{31}a_{22}a_{43}a_{14} + a_{31}a_{42}a_{13}a_{24} \\
& + a_{41}a_{12}a_{33}a_{24} + a_{41}a_{22}a_{13}a_{34} + a_{41}a_{32}a_{23}a_{14} \\
& - a_{11}a_{22}a_{43}a_{34} - a_{11}a_{32}a_{23}a_{44} - a_{11}a_{42}a_{33}a_{24} \\
& - a_{21}a_{12}a_{33}a_{44} - a_{21}a_{32}a_{43}a_{14} - a_{21}a_{42}a_{13}a_{34} \\
& - a_{31}a_{12}a_{43}a_{24} - a_{31}a_{22}a_{13}a_{44} - a_{31}a_{42}a_{23}a_{14} \\
& - a_{41}a_{12}a_{23}a_{34} - a_{41}a_{22}a_{33}a_{14} - a_{41}a_{32}a_{13}a_{24}
\end{aligned}$$

Problem 2. An nxn-matrix $A = (a_{ij})$ is called *diagonal* if $a_{ij} = 0$ for $i \neq j$. Compute the determinant of a diagonal matrix in two different ways. First use the Leibniz formula. Secondly, use the definition (1) and properties (1)-(3).

Solution. In the Leibniz formula the only product which does not involve a zero entry of the matrix A is the one corresponding to the identity permutation: $a_{11}a_{22}\dots a_{nn}$. This proves the claim.
For the second proof let $a_1, \dots, a_n$ denote the column vectors of A.

$$\begin{aligned}
\det(A) &= D(a_2, \dots, a_n) \\
&= D(a_{11}e_1, \dots, a_{nn}e_n) \\
&= a_{11}\dots a_{nn} D(e_1, \dots, e_n) && \text{by multilinearity} \\
&= a_{11}\dots a_{nn} && \text{by normalization}
\end{aligned}$$

Problem 3. An nxn-matrix $A = (a_{ij})$ is called *upper triangular* if $a_{ij} = 0$ for $i > j$. Show that the determinant of an upper triangular matrix is given by the product of the diagonal entries. Hint: Use the Leibniz formula and realize that only one permutation contributes a nonzero summand.

Solution. Same proof as above, the only permutation which leads to a nonzero product is the identity permutation.

Problem 4. Using properties (1)-(3) show that the determinant of a matrix does not change if we add a multiple of one column to another column.

Solution. Denote the columns of A by $a_1, \dots, a_n$. Let's say we add λa_i to the column a_j:

$$\begin{aligned} D(a_1, \dots, a_i, \dots, a_j + \lambda a_i, \dots, a_n) =& D(a_1, \dots, a_i, \dots, a_j, \dots, a_n) \\ &+ \lambda \underbrace{D(a_1, \dots, a_i, \dots, a_i, \dots, a_n)}_{0} \\ =& D(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = \det(A) \end{aligned}$$