RESEARCH STATEMENT

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My general research interest is the large scale geometry of finitely generated groups. A finitely generated group G can be considered as a metric space when endowed with a **word metric**

$$d_S(g,h) := \|g^{-1}h\|_S$$

where $\|\cdot\|_S$ measures the minimal word length with respect to a fixed finite generating set S. These metrics d_S, however, are not unique (they depend on S) nor do they determine G uniquely. In geometric group theory we are interested in properties that can be detected by (any) word metric on G. The notion of *quasi-isometric equivalence* (see definition below) captures this relation and it also allows one to study groups not only via their word metrics but also via nice *model* spaces on which these groups act.

A **quasi-isometry** between metric spaces is a map $f: X \to Y$ which is coarsely bilipschitz and coarsely onto: i.e. there exist $K, C \geq 0$ such that for all $x, y \in X$

$$-C + 1/Kd(x,y) \leq d(f(x), f(y)) \leq Kd(x,y) + C$$

and the C neighbourhood of $f(X)$ is all of Y.

A **model space** for a group G is a proper geodesic metric space X on which G acts properly discontinuously and cocompactly by isometries. In particular G is quasi-isometric to X.

A group property is said to be **geometric** if it is preserved under quasi-isometry. Gromov in [G1] initiated the program of studying geometric properties of finitely generated groups by showing that, up to finite index, nilpotence is a geometric property.

The problem of determining geometric properties of groups can be restated in the language of rigidity. A class of finitely generated groups $\mathcal{G}$ is said to be **rigid** if any group quasi-isometric to a group in $\mathcal{G}$ is *virtually* in $\mathcal{G}$. Here **virtually** means up to extensions of and by finite groups. The **classification** problem is to determine the smallest rigid subclasses.

Outline

As I will describe, my main theorems and projects are the following:

1. *The Eskin-Fisher-Whyte program on showing that polycyclicity is geometric.* My contribution to this program involves structure theorems about maps of boundaries of negatively curved homogenous spaces that occur in foliations of solvable Lie groups. (Solvable Lie groups are model spaces for polycyclic groups.)
2. *Comparison of quasi-isometric and bilipschitz equivalence.* I provide counterexamples to show that the two notions are not equivalent for finitely generated groups.
3. *The quasi-isometry group and Mostow-Margulis rigidity with locally compact targets.* For certain solvable groups, I determine which locally compact groups can contain them as lattices.
4. *Boundaries of hyperbolic spaces and groups.* I describe two projects: one on quasi-symmetric maps on boundaries of negatively curved homogeneous spaces and one on hyperbolic groups acting smoothly on their boundary.

1. BOUNDARY THEORY FOR POLYCYCLIC GROUPS

The main focus of my research has been on Eskin-Fisher-Whyte's program of the quasi-isometric classification of polycyclic groups. By a theorem of Mostow [M], polycyclic groups are those solvable groups that are virtually lattices in solvable Lie groups and conversely. As a consequence, solvable Lie groups are model spaces for polycyclic groups. One of the challenges of working with these solvable Lie groups is that their geometry has aspects of both positive and negative curvature. Traditionally, rigidity results have been most successful for spaces of nonpositive curvature.

To understand quasi-isometries of solvable Lie groups, Eskin-Fisher-Whyte developed a technique of "coarse differentiation" which allows them, in many cases, to show that a quasi-isometry preserves some geometric aspect. My work involves taking this structure, developing a boundary theory and completing the proof of rigidity. My first work in this area was a contribution to the following theorem (announced in [EFW1]).

Theorem 1 (Eskin-Fisher-Whyte). *Suppose M is a matrix with* $\det M = 1$ *and no eigenvalues on the unit circle. Let $G_M = \mathbb{R} \ltimes_M \mathbb{R}^n$. If Γ is a finitely generated group quasi-isometric to G_M then Γ is virtually a lattice in $\mathbb{R} \ltimes_{M^\alpha} \mathbb{R}^n$ for some $\alpha \in \mathbb{R}$.*

If $M \in SL(2, \mathbb{R})$ then G_M is the three dimensional geometry *Sol*. Proving rigidity of lattices in *Sol* was a major open problem. Using coarse differentiation, Eskin-Fisher-Whyte (in the case of *Sol*) and Peng (in higher dimensions) show that all quasi-isometries of G_M coarsely preserve *height*. (The first coordinate of G_M gives a notion of **height**.) Understanding the structure of quasi-isometries is important in the proof of rigidity because any group quasi-isometric to G_M (quasi) acts on G_M by quasi-isometries. In the case of *Sol* this characterization of quasi-isometries is combined with a theorem of Farb-Mosher [FM2] on bilipschitz maps of $\mathbb{R}$ to deduce rigidity. In the higher dimensional cases, a generalization of the Farb-Mosher result is necessary but not sufficient to complete the proof of rigidity. I provide the necessary generalization of [FM2] and complete the proof of rigidity for lattices in G_M when M is diagonalizable in [D2]. When M is nondiagonalizable this is joint with Peng [DP].

Theorem 2 (Dymarz, Dymarz-Peng). *Assuming that any quasi-isometry of G_M is height-respecting then any group quasi-isometric to G_M must be (virtually) a lattice in G_{M^α} for some $\alpha \in \mathbb{R}$.*

The proof of Theorem 2 relies on a theorem about bilipschitz maps of $\mathbb{R}^n$ with respect to certain metrics D_{M_i}. The metrics D_{M_i} can be viewed as metrics on certain *boundaries* of G_M.

Boundary Theory for $\mathbf{G_M}$. By decomposing the action of M into an expanding action (given by M_1) and a contracting action (given by M_2^{-1}) we can associate to $G_M = \mathbb{R} \ltimes_M \mathbb{R}^n$ two negatively curved homogeneous spaces G_{M_i}, $i = 1, 2$. These two G_{M_i} foliate G_M and since the norms of the eigenvalues of M_i are strictly greater than one, both G_{M_i} are negatively curved.

Like all negatively curved spaces, the G_{M_i} come equipped with a visual boundary ∂G_{M_i} where the natural metric to consider for our purposes is a visual metric (based at infinity) on $\partial G_{M_i} \backslash \{\infty\} \simeq \mathbb{R}^n$. In the diagonalizable case this metric is given by

$$D_{M_i}(x, y) = \max\{|x_1 - y_1|^{1/\alpha_1}, \cdots, |x_r - y_r|^{1/\alpha_r}\}$$

where the $\alpha_i > 1$ are determined by eigenvalues of M_i. In the nondiagonalizable case this metric is significantly more complicated to write down (see [DP]).

The boundary ∂G_M is the disjoint union of the visual boundaries of the G_{M_i}. Height-respecting isometries of G_M induce *similarities* (maps which scale distances by a fixed amount) of $\mathbb{R}^n$ with respect to D_{M_i}. Height-respecting quasi-isometries induce *quasisimilarities* (similarities composed with bilipschitz maps) with respect to D_{M_i}. We denote the group of all quasisimilarities with respect to D_{M_i} by $QSim_{D_{M_i}}(\mathbb{R}^n)$.

The first step in the proof of Theorem 2 above is to analyze the structure of maps in $QSim_{D_{M_i}}(\mathbb{R}^n)$. In the diagonalizable case this is straightforward and was also done by Tyson in [Ty]. In the non-diagonalizable case this is done by Peng and myself in [DP].

Proposition 3 (Dymarz-Peng)**.** *If $f \in QSim_{D_{M_i}}(\mathbb{R}^n)$ where D_{M_i} is a visual metric on the boundary of G_{M_i} then f preserves the flag of foliations defined by the generalized eigenspaces of M_i as well as a refined foliation defined by nilpotence degrees within the generalized eigenspaces.*

Using this structure of boundary maps we are able to provide the main ingredient in the proof of Theorem 2:

Theorem 4 (Dymarz, Dymarz-Peng)**.** *If G is a uniform subgroup of the group $QSim_{D_{M_i}}(\mathbb{R}^n)$ that acts cocompactly on the space of distinct pairs of points of $\mathbb{R}^n$ then there exists an $f \in QSim_{D_{M_i}}(\mathbb{R}^n)$ such that fGf^{-1} is the composition of a similarity with respect to D_{M_i} and an almost translation:*

$$(x_1, x_2, \ldots, x_r) \mapsto (x_1 + B_1(x_2, \ldots, x_r), x_2 + B_2(x_3, \ldots, x_r), \ldots, x_r + B_r).$$

The uniformity and group structure impose additional constraints on the B_i. Theorem 4 alone does not finish the proof of quasi-isometric rigidity but by applying it to both boundaries of G_M we can use the structure of the B_i to deduce that any finitely generated group quasi-isometric to G_M is in fact polycyclic. Additional analysis and the structure theory for solvable Lie groups allow us to derive the final result as stated in Theorem 2.

Structure of solvable Lie groups. Eskin-Fisher-Whyte make the following conjecture for arbitrary solvable Lie groups:

Conjecture 5 (Eskin-Fisher-Whyte)**.** *Let G be a simply connected unimodular solvable Lie group. Then any self quasi-isometry of G coarsely preserves the cosets of the* exponential radical *$R(G)$.*

For a simply connected solvable Lie group G the **exponential radical** $R(G)$ is the smallest normal subgroup such that $G/R(G)$ has polynomial growth. Eskin, Fisher and Peng are currently working on Conjecture 5 by applying coarse differentiation techniques to G. In a joint project with Peng, we are working on developing a boundary theory for certain subclasses of solvable Lie groups.

The simplest extension of the abelian-by-cyclic case is the nilpotent-by-cyclic case $G_\phi = \mathbb{R} \ltimes_\phi N$ when N decomposes as a product of two nilpotent groups $N \simeq N_1 \times N_2$ and ϕ acts by an expanding map ϕ_1 on N_1 and a contracting map ϕ_2^{-1} on N_2. In this case, as before, the boundary theory reduces to the boundary theory of the negatively curved homogeneous spaces $G_{\phi_i} = \mathbb{R} \ltimes_{\phi_i} N_i$. Again, height-respecting quasi-isometries induce quasisimilarities of N_i with respect to visual metrics D_{ϕ_i} and height-respecting isometries induce similarities. Because of the additional structure that nilpotent groups carry, we now have that in certain cases $QSim_{D_{\phi_i}}(N_i) \simeq Sim_{D_{\phi_i}}(N_i)$ (see [Pa] for examples). When this is not the case we ask:

Question 6. *Is there an analogue of Theorem 4 for $QSim_{D_{\phi_i}}(N_i)$?*

For $G_\phi = \mathbb{R} \ltimes_\phi N$ when N does not split as a direct product as above, we have various obstacles to overcome. Now G_ϕ is not foliated by negatively curved homogeneous spaces. One can still define a foliation of G_ϕ depending on the expanding and contracting directions for the action of ϕ but now the leaves of these foliations are no longer isometric to each other. In particular each leaf can have a potentially different boundary metric.

Another class to consider is the split abelian-by-abelian groups $G_\phi = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$. These have already been studied by Peng in [Pe]. Peng extends the techniques of coarse differentiation to these groups and defines a notion of boundaries. To show rigidity for these groups, Peng applies the same theorems that are used in the proof of Theorem 2. The main difference is that now G_ϕ can have multiple boundaries. We expect that Peng's approach will also apply to split nilpotent-by-abelian groups.

2. BILIPSCHITZ EQUIVALENCE

As mentioned above, quasi-isometric equivalence is a coarse version of bilipschitz equivalence but for discrete spaces, a bijective quasi-isometry gives a bilipschitz equivalence. In [G2], Gromov proposed the problem of understanding when quasi-isometries are close to bijective quasi-isometries. In [W], Whyte proved that any quasi-isometry between *non-amenable* finitely generated groups is a bounded distance from a bijection. In [D1] I show that this is not the case for amenable groups.

A finitely generated group is **amenable** if it contains a sequence of finite sets $\{S_i\}$ such that

$$\lim_{i\to\infty} |\partial_r S_i|/|S_i| = 0$$

where $\partial_r S$ is the set of points at distance at most r away from S in the cayley graph of G. It is non-amenable otherwise.

Theorem 7 (Dymarz). *For an amenable group G and a proper subgroup H, the inclusion $i : H \to G$ is never a bounded distance from a bijection.*

Nevertheless, the question remained whether there always exists *some* bijective quasi-isometry between quasi-isometric amenable groups. In [D3] I provide a counterexample by showing that certain quasi-isometric lamplighter groups do not admit a bijective quasi-isometry between them and hence are not bilipschitz equivalent.

Theorem 8 (Dymarz). *Let F and G be finite groups with $|F| = n$ and $|G| = n^k$ where $k > 1$. Then there does not exist a bijective quasi-isometry between the lamplighter groups $G \wr \mathbb{Z}$ and $F \wr \mathbb{Z}$ if k is not a product of prime factors appearing in n.*

Here $F \wr \mathbb{Z} = \mathbb{Z} \ltimes \oplus_{i\in\mathbb{Z}} F$. As before, we designate the first coordinate of the product to be *height*. A key ingredient of the proof of Theorem 8 is Eskin-Fisher-Whyte's theorem on the structure of quasi-isometries of lamplighter groups [EFW1] which says that, like in the case of *Sol*, all quasi-isometries are height-respecting. Another ingredient is the analysis of maps in $Bilip(\mathbb{Q}_n)$.

Question 9. *Are there examples of polycyclic groups that quasi-isometric but not bilipschitz equivalent? finitely presented groups?*

If F is a solvable finite group then $F \wr \mathbb{Z}$ is solvable but it is not polycyclic, nor is it finitely presented.

3. RIGIDITY WITH LOCALLY COMPACT TARGETS

A locally compact group H that contains a cocompact lattice Γ is called an **envelope** of Γ.

Problem 10. *Given a finitely generated group Γ, classify up to extensions of and by compact groups, all envelopes of Γ.*

In [Fu], Furman describes envelopes of lattices in semisimple Lie groups. This problem is a natural extension of the Mostow-Margulis Rigidity Theorem which states that a lattice in a semisimple Lie group cannot be a lattice in a different Lie group. For those finitely generated groups Γ that are not cocompact lattices in any Lie group there are often obvious locally compact groups H that do contain Γ as a cocompact lattices. For example, virtually free groups are lattices in automorphism groups of trees. In [MSW2], Mosher-Sageev-Whyte show that these are the only possible envelopes of a virtually free group. In [D4] we analyze envelopes of certain solvable groups. Our strategy is to utilize the quasi-isometry group:

$$QI(X) = \{f : X \to X \mid f \text{ a quasi-isometry}\}/\sim \text{ where } f \sim g \Leftrightarrow d_{sup}(f, g) < \infty.$$

This is fruitful since for the groups Γ that we consider, any envelope of Γ embeds, up to compact kernel, as a subgroup of $QI(X)$ for any model space X of Γ.

One of the classes of groups we can analyze is $BS(1,n) = \langle a, t \mid tat^{-1} = a^n \rangle$ the solvable Baumslag-Solitar groups. The model space we use for $BS(1,n)$ is a metric complex X_n that is tree-like but also contains many isometrically embedded copies of $\mathbb{H}^2$. By [FM1]

$$QI(X_n) \simeq Bilip(\mathbb{R}) \times Bilip(\mathbb{Q}_n).$$

We can then use results from [FM2] and [MSW1] that analyze subgroups of $Bilip(\mathbb{R})$ and $Bilip(\mathbb{Q}_n)$ to prove the following theorem:

Theorem 11 (Dymarz). *If $BS(1,n)$ is a cocompact lattice in a locally compact group H then, up to compact kernel, $H \subseteq Isom(X_m)$ where $m^i = n$ for some i.*

The key is identifying the quasi-isometry group with a group of maps on the boundaries $\mathbb{R}$ and $\mathbb{Q}_n$ of X_n. With this observation it is also possible to analyze lattices in the solvable Lie groups G_M (whose boundaries were described above) and the groups $F \wr \mathbb{Z}$ (whose boundaries are $\mathbb{Q}_n$.)

More challenging are the non-polycyclic finitely presented abelian-by-cyclic groups. Like $BS(1,n)$, these groups have model spaces X_M that are tree-like but instead of copies of the hyperbolic plane they contain copies of the solvable Lie groups G_M. Quasi-isometries of these group were studied in [FM3]. If the eigenvalues of M are strictly greater than one in norm and $d = \det M$ then

$$QI(X_M) \simeq QSim_{D_M}(\mathbb{R}^n) \times Bilip(\mathbb{Q}_d).$$

When M has eigenvalues that are both greater and less than one in norm, $QI(X_M)$ is much harder to describe and to analyze. Also, as progress is made on Conjecture 5 it should be possible to describe envelopes of lattices in other solvable Lie groups as well as their non-polycyclic generalizations.

4. Boundaries of hyperbolic spaces and groups

My work on boundaries of solvable Lie groups relies heavily on ideas from δ-hyperbolic spaces and their boundaries. I will describe two projects that indirectly arise from this work.

4.1. Analysis on boundaries of negatively curved homogenous spaces.

Theorem 4 can be thought of as an analogue of Tukia's theorem [Tu] on quasiconformal maps of S^n and in fact the proof uses many of the same techniques.

Theorem 12 (Tukia). *Any uniform subgroup of $QC(S^n)$, the group of all quasiconformal homeomorphisms of S^n, that acts cocompactly on the space of distinct triples of S^n can be conjugated into the conformal group $Conf(S^n)$.*

For a general δ-hyperbolic space a quasi-isometry induces a *quasi-symmetric* map of the boundary. A map f is said to be **quasi-symmetric** if there exists a positive homeomorphism η such that for all $x, y, z \in X$ $\quad d(x,y) \leq t\ d(x,z) \Rightarrow d(f(x), f(y)) \leq \eta(t)\ d(f(x), f(z))$.

We write $QS_{D_{M_i}}(\mathbb{R}^n)$ to denote the group of all quasi-symmetric maps with respect to the metric D_{M_i}. Note that $QSim_{D_{M_i}}(\mathbb{R}^n) \subseteq QS_{D_{M_i}}(\mathbb{R}^n)$. Does the reverse inclusion hold? A positive answer would show that all quasi-isometries G_{M_i} that fix ∞ are in fact height-respecting. If not, we have the following project:

Problem 13. *Extend the methods of Theorem 4 to uniform subgroups of $QS_{D_{M_i}}(\mathbb{R}^n)$.*

4.2. Hyperbolic groups with manifold boundaries.

A new direction I have been exploring recently is hyperbolic groups acting smoothly on their boundaries. This work is motivated by a summer school I attended organized by Fisher, Kleiner and Pilgrim "Quasiconformal analysis and boundaries of hyperbolic groups" and is joint with Karin Melnick.

Conjecture 14. [G2] *A δ-hyperbolic group G with manifold boundary M that acts smoothly on its boundary is a lattice in $Isom(X)$ where X is a rank one symmetric space.*

As a simple consequence of the dynamics of the action of G on its boundary, M must be S^n. We have two possible strategies to approach this conjecture. Our first approach involves work of Benoist-Foublon-Labourie [BFL] on Anosov flows on negatively curved compact Riemannian manfolds with C^∞ central stable distributions. They prove that such a flow is conjugate to the geodesic flow on a rank one symmetric space. In our case, we can imitate the unit tangent bundle of a symmetric space by considering

$$X_G = (M \times M \setminus \Delta) \times \mathbb{R}$$

where Δ is the diagonal and constructing a geodesic flow on X_G/G. Recently, it was suggested to us by Gregory Margulis to bypass constructing the geodesic flow and instead to find a rigid geometric structure on M using the dynamics of the action of G on M. This geometric structure would come from using arguments in the spirit of [S, GK] to produce normal forms for the action.

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