Joint work with

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Beginnings

It all started with a chessboard...
Beginnings

It all started with a chessboard...

...and a bottle of raki.
What’s the idea?

- Play chess as usual, except...
- Do NOT alternate moves.
- Instead, players bid for the right to make the next move.
How does it work?

- Each player starts with 100 bidding chips.
- Before each move, each player writes down a bid.
- The player who bids higher gives that many chips to the other player, and makes a move.

*How do you win?*

- By taking opponent’s king.
A mathematician’s quibble.

What if the bids are tied?
A mathematician’s quibble.

What if the bids are tied?

- The bids are NEVER tied.
- One player starts with an extra chip, with value \( \epsilon \).
- The player who has the \( \epsilon \)-chip is required to bid it.
A sample game

1015 Evans Hall, UC Berkeley

Tuesday, October 17, 2006
A sample game

Bidding chips

W: 100*
B: 100
A sample game

Bidding chips

W: 100*
B: 100

Bids

W: 13*
B: 12
A sample game

Bidding chips

W: 87
B: 113*
A sample game

Bidding chips

W: 87
B: 113*

Bids

W: 11
B: 11*
A sample game

Bidding chips

W: 98*
B: 102
A sample game

Bidding chips

W: 98*
B: 102

Bids

W: 9*
B: 15
A sample game

Bidding chips

W: 113*
B: 87
A sample game

Bidding chips

W: 113*
B: 87

Bids

W: 15*
B: 22
A sample game

Bidding chips

W: 135*
B: 65
A sample game

Bidding chips

W: 135*
B: 65

Bids

W: 65*
B: 65
A sample game

W: 70
B: 130*

Bidding chips

W: 70
B: 130*
A sample game

Bidding chips

W: 70
B: 130*

Bids

W: 30
B: 25*
A sample game

Bidding chips

W: 40
B: 160*
Agony of defeat

W: “That was a total mindf**k.”
Richman’s theory

Similar bidding games investigated by David Richman (mid-late 1980s). Features of Richman’s theory:

- Continuous (real-valued) bidding.
- Flip coin to break ties.
- Impartial play, no zugzwang.
- No ties or draws.
Every game $G$ has a critical threshold $R(G)$, between 0 and 1.

Let $b = \frac{B's\ resources}{B's\ resources + W's\ resources}$.

- If $b > R(G)$, then B wins.
- If $b < R(G)$, then B does not win.
- If $b = R(G)$, then the outcome may depend on coin flips.
How to compute \( R(G) \)?

Suppose directed graph is finite. Compute \( R(G) \) by working backward from ending positions.

- If \( v \) is a position from which only \( B \) can win, then \( R(G_v) = 0 \).
- If \( v \) is a position from which \( B \) cannot win, then \( R(G_v) = 1 \).
How to compute $R(G)$?

Otherwise, define

$$R^+(G_v) = \max_{v \rightarrow w} R(G_w) \quad \text{and} \quad R^-(G_v) = \min_{v \rightarrow w} R(G_w).$$

Then,

$$R(G_v) = \frac{R^+(G_v) + R^-(G_v)}{2}.$$
If the directed graph is infinite, then

\[ R(G) = \lim_{n \to \infty} R(G[n]), \]

where \( G[n] \) is the truncation of \( G \) after \( n \) moves.
Instead of alternating moves or bidding to move, just flip a coin.

Say $P(G)$ is the probability that $B$ wins, assuming optimal play.
Relations to random games

Instead of alternating moves or bidding to move, just flip a coin.

Say $P(G)$ is the probability that $B$ wins, assuming optimal play.

Theorem (Richman)

$P(G) = 1 - R(G)$. 
How to compute $P(G)$?

- If $B$ starts in a winning position, then $P(G) = 1$.
- If $B$ cannot win, then $P(G) = 0$.

Otherwise, compute $P(G)$ by working backward from ending positions.
How to compute $P(G)$?

If $v$ is not an ending position, define

$$P^+(G_v) = \max_{v \rightarrow w} P(G_w) \quad \text{and} \quad P^-(G_v) = \min_{v \rightarrow w} P(G_w).$$

Then,

$$P(G_v) = \frac{P^+(G_v) + P^-(G_v)}{2}.$$
Another approach to bidding games

Berlekamp’s “Economist’s view of combinatorial games”.

- The goal of the game is not just to win, but to accumulate as many points (or stones, or dollars) as possible.
- Players bid for the right to move, using these universal points as currency.
- Closely related to Conway’s *thermography*.
- Applications to classical combinatorial games, including Go endgames.
Richman games vs. discrete bidding

Continuous bidding
- Convenient for theoretical purposes.
- Impossible to play.

Discrete bidding chips.
- Theory is more subtle and difficult (two parameters).
- Playable and fun.
Can compute “discrete critical thresholds” just like $R(G)$.

- Start from ending positions and work backwards.
- At each step, round appropriately, taking $\epsilon$-chip into account.
- If there are few chips, effects of rounding may be severe.
Example: Tic-tac-toe

Tic-tac-toe, W starts with $N^*$ chips, B starts with $N$.

- Tie for $N = 0$.
- W wins for $N = 1, 2$. (Check by hand.)
- Tie for $3 \leq N \leq 7$. (Computer check.)

Conjecture

Tie for $N \geq 3$.

Open Problem

What is $R(G)$ for $G = \text{Tic-tac-toe}$?
Many chips

For large numbers of chips, discrete bidding roughly approximates continuous bidding.

- Fix $\epsilon > 0$.
- Let $b = \#\{B's\ \text{chips}\}/\#\{\text{Total chips}\}$.

**Theorem**

If $b > R(G) + \epsilon$ and $\#\{\text{Total chips}\} \gg 0$, then $B$ wins.

**Theorem**

If $b < R(G) - \epsilon$, $\#\{\text{Total chips}\} \gg 0$, and the directed graph is finite, then $B$ does not win.
Many chips

For large numbers of chips, discrete bidding roughly approximates continuous bidding.

- Fix $\epsilon > 0$.
- Let $b = \#\{\text{B's chips}\}/\#\{\text{Total chips}\}$.

Open Problem

If the directed graph of $G$ is infinite, is it possible that B can win for some $b < R(G) - \epsilon$ and $\#\{\text{Total Chips}\} \gg 0$?
At the critical threshold

What happens in discrete bidding games if $b = R(G)$?
At the critical threshold

What happens in discrete bidding games if \( b = R(G) \)?

**Theorem**

*If the directed graph of \( G \) is finite, then the outcome for \( b = R(G) \) is periodic, with period a power of 2.*
## Infinite games

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<thead>
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<th>Theorem</th>
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<td><em>For any N, there is a game whose outcome for</em> ( b = R(G) ) <em>is periodic, with period N.</em></td>
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Theorem

For any $N$, there is a game whose outcome for $b = R(G)$ is periodic, with period $N$.

Example

The path-of-length-$2N$ game with equal chips is first player win if and only if $N$ divides the number of chips.

Theorem

There exist games whose outcome for $b = R(G)$ is aperiodic.
Open Problem

What sequences of outcomes are possible at the critical threshold?
Further reading

E. Berlekamp.  
The economist’s view of combinatorial games.  
*Games of No Chance*, 365–405, MSRI 1996.

Richman games.  

Combinatorial games under auction play.  