Topology of Nonarchimedean Analytic Spaces

AMS Current Events Bulletin

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Complex algebraic geometry

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$$\{f_1, \ldots, f_r\} \subset \mathbb{C}[x_1, \ldots, x_n].$$

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- It may be a smooth complex manifold, like the surface

$$x_1^2 + x_2^2 + x_3^2 = 1,$$

- Or it may be singular, like the Whitney umbrella

$$x_1^2 - x_2^2 x_3 = 0.$$

Tameness

Although X may have complicated singularities, its topology is not pathological. Every complex algebraic set

- can be triangulated,
- admits a strong deformation retract onto a finite simplicial complex,
- and contains an open dense complex manifold whose complement is an algebraic set of smaller dimension.

In particular, X is a finite union of complex manifolds.

Beyond the complex numbers

We also study algebraic sets in K^n, the common solutions of a system of polynomial equations

$$\{f_1, \ldots, f_r\} \subset K[x_1, \ldots, x_n]$$

for fields K other than $\mathbb{C}$.

For instance, K could be

- the field of rational numbers $\mathbb{Q}$,
- the field of formal Laurent series $\mathbb{C}((t))$,
- the function field of an algebraic curve.

Norms

All of these fields can be equipped with norms.

Example

Consider the field of rational numbers, and fix a prime number p. Set

$$\left|\frac{p^a r}{s}\right|_p = p^{-a},$$

for p, r, s relatively prime.

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Example

Write each formal Laurent series uniquely as t^a times a power series with nonzero constant term. Set

$$\left|t^a \sum a_i t^i\right|_t = e^{-a}.$$

Naive analysis

Each norm induces a metric topology on K^n, and one can consider functions given locally by convergent series, but...

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Each norm induces a metric topology on K^n, and one can consider functions given locally by convergent series, but...

Even if K is complete with respect to its norm, K^n may be totally disconnected in its metric topology.

This happens whenever the norm is **nonarchimedean**.

Archimedean norms

Axiom of Archimedes (Axiom V, *On the Sphere and Cylinder*)

For any quantity x there is a natural number n such that $|nx| > 1$.

Up to rescaling, any **archimedean** norm on a field K is induced by an inclusion $K \subset \mathbb{C}$.

Corollary

The only complete archimedean fields are $\mathbb{R}$ and $\mathbb{C}$.

The rest of the zoo

Definition

A **nonarchimedean** field is any complete normed field other than $\mathbb{R}$ or $\mathbb{C}$.

Examples include:

- $\mathbb{C}((t))$
- $\mathbb{Q}_p$, the completion of $\mathbb{Q}$ with respect to $| \ |_p$.
- any K with the trivial norm, $|a| = 1$ for $a \in K^*$.

Algebraically closed examples include $\mathbb{C}_p$ and $\widehat{\mathbb{C}\{\{t\}\}}$, the completions of the algebraic closures of $\mathbb{Q}_p$ and $\mathbb{C}((t))$.

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Both $\mathbb{C}_p$ and $\widehat{\mathbb{C}\{\{t\}\}}$ are isomorphic to $\mathbb{C}$ as abstract fields.

Ultrametrics and clopen balls

In any nonarchimedean field, the triangle inequality can be strengthened to the **ultrametric inequality**:

$$|x + y| \leq \max\{|x|, |y|\}, \text{ with equality if } |x| \neq |y|.$$

Corollary

If y is a point in the closed ball

$$B(x, r) = \{y \in K \mid |y - x| \leq r\},$$

then $B(x, r) = B(y, r)$.

It follows that $B(x, r)$ is open in the metric topology.

Nonarchimedean analytic geometry

In the 1960s, Tate developed **rigid analytic spaces**. Two key steps:

- Replace the metric topology on K^n by a Grothendieck topology, and
- Study sheaves of rings in this Grothendieck topology, built from rings of convergent power series on closed balls.

Today we are talking about **nonarchimedean analytic spaces** (Berkovich, late 1980s–1990s). Two key features:

- Same algebraic foundations as rigid analytic geometry.
- New underlying space with additional points that fill in the gaps between the points of K^n.

Analytification

System of polynomials $\{f_1, \ldots, f_r\} \subset K[x_1, \ldots, x_n]$.

- Solution set $X = V(f_1, \ldots, f_r)$,
- Coordinate ring $K[X] = K[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$.

Definition

The analytification of X is

$$X^{\mathrm{an}} = \{\text{seminorms on } K[X] \text{ that extend the given norm on } K\},$$

equipped with the subspace topology from the inclusion in $\mathbb{R}_{\geq 0}^{K[X]}$.

Some points of X^{an}

Example

Evaluation at a point $x \in X(K)$ induces a seminorm $|\ |_x$, given by

$$|f|_x = |f(x)|$$

Example

If $L|K$ is a finite algebraic extension, then the norm on K extends uniquely to L (because K is complete). Composing with evaluation at points gives a natural inclusion

$$X(\overline{K})\,/\,\mathrm{Gal}\ \subset\ X^{\mathrm{an}}.$$

Topological properties of X^{an}

Theorem (Berkovich)

The topological space X^{an} is Hausdorff, locally compact, and locally path connected, of dimension equal to the algebraic dimension of X. Furthermore,

- *The induced topology on $X(K) \subset X^{\mathrm{an}}$ is the metric topology.*
- *If K is algebraically closed, then $X(K)$ is dense.*
- *In general, $X(\overline{K})/\operatorname{Gal}$ is dense.*

Projection to the scheme

There is a natural continuous projection onto the affine scheme

$$X^{\mathrm{an}} \xrightarrow{\pi} \operatorname{Spec} K[X],$$

taking a point $x \in X^{\mathrm{an}}$ to the prime ideal $\{f \in K[X] \mid |f|_x = 0\}$.

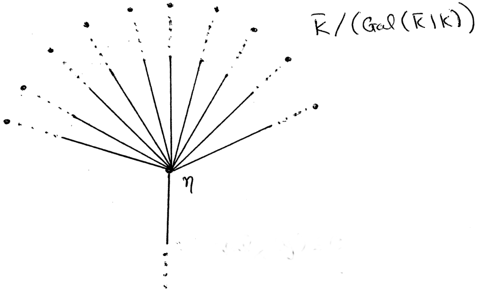
The fiber over a point $\mathfrak{p} \in \operatorname{Spec} K[X]$ is

$$\pi^{-1}(\mathfrak{p}) = \{\text{norms on } \kappa_{\mathfrak{p}} \text{ that extend } |\ | \text{ on } K\}.$$

- If $\mathfrak{p}$ is closed, then it comes from a point over a finite extension $L|K$, so the norm extends uniquely.

The affine line: trivial valuation

Consider the affine line $X = \operatorname{Spec} K[z]$, and assume the valuation on K is trivial.

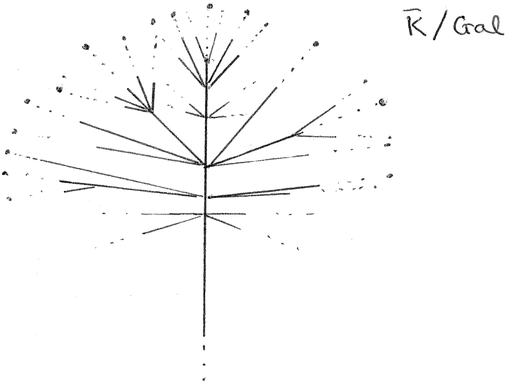


⚠ Every neighborhood of η contains all but finitely many branches.

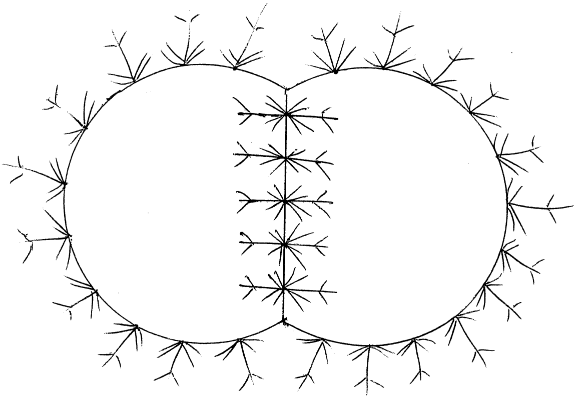
The affine line: nontrivial valuation

For each ball $B(x, r)$ in K, there is a seminorm given by

$$|f|_{x,r} = \max\{|f(y)| \mid y \in \overline{K} \text{ and } |y - x| \leq r\}$$



A genus 2 curve



The topology is an inverse limit over connected finite subgraphs that contain both loops.

Questions?

Toward the affine plane...

To get started,

1. Imagine taking the analytification of each curve in the plane.
2. Glue each pair of curves along the finitely many leaves corresponding to their points of intersections.
3. And fill in the space in the middle with two-dimensional "membrane" stretched between the analytifications of the curves.

This membrane is the space of norms on the function field $K(x_1, x_2)$ that extend the given norm on K.

⚠ There are many non-obvious norms on the function field in two variables, including "non-Abhyankar norms," such as those induced by order of contact with a transcendental germ of a curve.

The associated valuation

There is a valuation associated to the nonarchimedean norm on K.

- The valuation is given by $\mathrm{val}(a) = -\log |a|$.
- The valuation ring $R \subset K$ is the subring consisting of elements of norm less than or equal to 1.
- The maximal ideal $\mathfrak{m} \subset R$ consists of elements of norm strictly less than 1.
- The residue field is $k = R/\mathfrak{m}$.

Example

Suppose $K = \mathbb{C}((t))$. Then the valuation ring is $R = \mathbb{C}[[t]]$, the maximal ideal is $\mathfrak{m} = tR$, and the residue field is $k = \mathbb{C}$.

Models and skeletons

By choosing presentations for $K[X]$ and "clearing denominators," one can construct models of X defined over R, that have "special fibers" defined over the residue field k.

If X has a model with a nice special fiber, then the combinatorics of the special fiber can be used to control the topology of X^{an}.

Theorem (Berkovich 1990s)

If X has a semistable formal model then X^{an} admits a deformation retract onto the dual complex of the special fiber. In particular, X^{an} has the homotopy type of a finite simplicial complex.

Tameness?

Semistable models are difficult to produce in practice, and are not known to exist in general. There are major difficulties, related to resolution of singularities, if the residue field $k = R/\mathfrak{m}$ has positive characteristic.

⚠ A priori, even when a semistable model exists, the topological space X^{an} could still have local pathologies.

Berkovich's theorem implies that the analytification of a smooth variety with respect to the trivial norm is contractible, but local contractibility is **much** more difficult.

Semialgebraic sets

Definition

Let X be an affine algebraic variety over K. A semialgebraic subset $U \subset X^{\mathrm{an}}$ is a finite boolean combination of subsets of the form

$$\{x \in X^{\mathrm{an}} \mid |f|_x \bowtie \lambda |g|_x\},$$

with $f, g \in K[X]$, $\lambda \in \mathbb{R}$, and $\bowtie \in \{\leq, \geq, <, >\}$.

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with $f, g \in K[X]$, $\lambda \in \mathbb{R}$, and $\bowtie \in \{\leq, \geq, <, >\}$.

- Every point in X^{an} has a basis of neighborhoods consisting of semialgebraic sets.
- Semialgebraic subsets are *analytic domains*, and come equipped with canonical analytic structure sheaves induced from X.

The Tameness Theorem

Theorem (Hrushovski-Loeser 2010)

Let $U \subset X^{\mathrm{an}}$ be a semialgebraic subset. Then there is a finite simplicial complex $\Delta \subset U$, of dimension less than or equal to $\dim(X)$, and a strong deformation retraction $U \times [0,1] \to \Delta$.

Corollary

The topological space X^{an} is locally contractible.

Key Ingredients

The proof of the Tameness Theorem is long and difficult, involving:

1. A detailed study of spaces of stably dominated types (difficult model theory)
2. An induction on dimension, birationally fibering X by curves over a base of dimension $\dim X - 1$.
3. Proving a more subtle tameness statement controlling how the topology of "families" of lower dimensional semialgebraic sets vary over a lower-dimensional base.

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3. Proving a more subtle tameness statement controlling how the topology of "families" of lower dimensional semialgebraic sets vary over a lower-dimensional base.

The argument does **not** use resolution of singularities, alterations, or any construction of nice formal models.

Beyond tameness

Hrushovski and Loeser prove much more than the existence of a single simplicial complex $\Delta \subset U$ which is a strong deformation retract.

- There are infinitely many such complexes Δ_i, with natural projections between them.
- The inverse limit over these projections is $\varprojlim \Delta_i = U$.
- There are sections of these projections, and the union $\lim \Delta_i$ is the subset of U consisting of points corresponding to Abhyankar norms.

Relation to limits of tropicalizations

The topological space X^{an} can also be realized naturally as a limit of finite simplicial complexes using **tropical geometry** [P. 2009, Foster-Gross-P. 2012].

The construction of this tropical inverse system is elementary, but does not lead to a proof of tameness.

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The construction of this tropical inverse system is elementary, but does not lead to a proof of tameness.

Under suitable hypotheses, there are sections of the projections in the tropical inverse system, and the images of these sections is the subset of Abhyankar norms [Baker-P.-Rabinoff 2011].

⚠ The relation between these tropical inverse systems and [Hrushovski-Loeser 2010] is still unclear.

Connections to complex algebraic geometry

Let $X \subset \mathbb{C}^n$ be a closed algebraic set, and consider $\mathbb{C}$ with the **trivial** valuation.

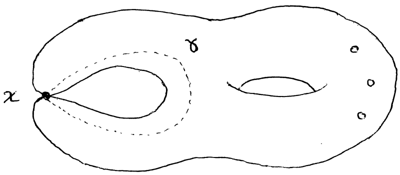
Theorem (Berkovich 2000)

There is a canonical isomorphism

$$H^*(X^{\mathrm{an}}, \mathbb{Q}) \cong W_0 H^*(X(\mathbb{C}), \mathbb{Q})).$$

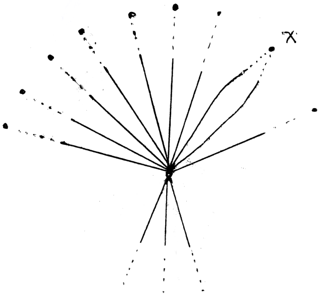
Example: A nodal curve

Consider an affine curve of geometric genus 1, with three punctures and one node.



Example: A nodal curve

The nonarchimedean analytification looks like this:



Some interesting semialgebraic sets

Each point of X^{an} corresponds to a point of the scheme X over a valued field;

- a point $x \in X^{\mathrm{an}}$ corresponds to $\pi(x) = \mathfrak{p}$ in X,
- which is defined over the field $\kappa_{\mathfrak{p}}$,
- and equipped with the valuation $-\log|\ |_x$.

Any constructible condition on the specialization of $\pi(x)$, with respect to the valuation $-\log|\ |_x$, is semialgebraic on X^{an}.

Example

If $z \in X$ is a point, then the **link**

$$\mathcal{L}_z = \{x \in X^{\mathrm{an}} \mid \pi(x) \text{ specializes to } z\}$$

is semialgebraic.

Resolution complexes

Let $f : \widetilde{X} \to X$ be a log resolution of the pair (X, z).

Theorem (Thuillier 2007)

The dual complex of $f^{-1}(z)$ embeds naturally in $\mathcal{L}_z$ as a strong deformation retract.

Theorem (Arapura-Bakhtary-Włodarczyk 2010)

If (X, z) is an isolated rational singularity, then $\mathcal{L}_z$ has the rational homology of a point.

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⚠ There are many examples of isolated rational singularities (X, z) such that $\mathcal{L}_z$ is not contractible.

Contractibility

Theorem

Let (X, z) be an isolated rational singularity, and let Δ be the dual complex of the exceptional divisor of a log resolution. Then Δ is contractible if (X, z) is a

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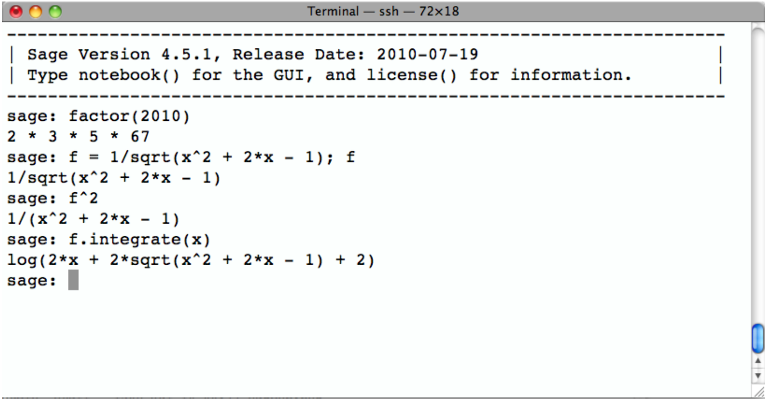
1. toric singularity [Stepanov 2006]
2. finite quotient singularity [Kerz-Saito 2011]
3. log terminal singularity [de Fernex-Kollár-Xu December 2012]

A log terminal singularity

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| Type notebook() for the GUI, and license() for information.          |
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sage: factor(2010)
2 * 3 * 5 * 67
sage: f = 1/sqrt(x^2 + 2*x - 1); f
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A log terminal singularity

Terminal — ssh — 72×18

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sage:
```

Or just a terminal log?

Further reading

1. E. Hrushovski and F. Loeser, *Nonarchimedean tame topology and stably dominated types*, arXiv:1009.0252v3.
2. A. Ducros, *Les espaces de Berkovich sont modérés, d'après E. Hrushovski et F. Loeser*, arXiv:1210.4336.
3. T. de Fernex, J. Kollár, and C. Xu, *The dual complex of singularities*, arXiv:1212.1675.