

COARSE HYPERBOLICITY AND CLOSED ORBITS FOR QUASIGEODESIC FLOWS

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ABSTRACT. We prove Calegari's conjecture that every quasigeodesic flow on a closed hyperbolic 3-manifold has closed orbits.

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1. INTRODUCTION

In 1950, Seifert asked whether every nonsingular flow on the 3-sphere has a closed orbit [20]. Schweitzer gave a counterexample in 1974 and showed more generally that every homotopy class of nonsingular flows on a 3-manifold contains a C^1 representative with no closed orbits [19]. Schweitzer's examples were generalized considerably and it is known that the flows can be taken to be smooth [17] or volume-preserving [16].

On the other hand, there are certain geometric constraints on flows that ensure the existence of closed orbits. Taubes' 2007 proof of the 3-dimensional Weinstein conjecture shows that every Reeb flow on a closed 3-manifold has a closed orbit [21]. Reeb flows are *geodesible*, i.e. there is a Riemannian metric in which the flowlines are geodesics. In 2010, Rechtman showed that real analytic geodesible flows on closed 3-manifolds have closed orbits, unless the manifold is a torus bundle with reducible monodromy [18].

Geodesibility is a global geometric condition. In contrast, a flow is said to be *quasigeodesic* if the flowlines lift to quasigeodesics in the universal cover, a local condition. In this paper we will show that every quasigeodesic flow on a closed hyperbolic 3-manifold has closed orbits.

1.1. Flows, transverse structures, and closing. Our proof works by studying the transverse structure of a quasigeodesic flow. For motivation we will outline a parallel picture that works for Anosov and pseudo-Anosov flows.

A flow Φ on a 3-manifold M is *Anosov* when the tangent bundle splits into three one-dimensional sub-bundles: the tangent bundle $T\Phi$ to the flow, a stable bundle E^s , and an unstable bundle E^u . The flow exponentially contracts the stable bundle and exponentially expands the unstable bundle. The two-dimensional sub-bundles $T\Phi \oplus E^s$ and $T\Phi \oplus E^u$ integrate to a pair of transverse two-dimensional foliations, the *weak stable* and *weak unstable foliations*. The flowlines in a weak stable leaf are all forward asymptotic, while the flowlines in a weak unstable leaf are all backwards asymptotic. Furthermore, the weak stable and unstable leaves are themselves foliated by *strong* stable and unstable leaves, obtained by integrating the one-dimensional sub-bundles E^s and E^u . Two flowlines lying in a single stable leaf are forwards asymptotic, and the points where these flowlines intersect a strong stable leaf are asymptotic on the nose.

Example 1.1. The simplest examples of Anosov flows are obtained as suspension flows of Anosov diffeomorphisms. An Anosov diffeomorphism $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ preserves a pair of 1-dimensional foliations, $\mathfrak{F}^{s/u}$, which are respectively contracted and expanded by ϕ .

Let M_ϕ be the 3-manifold obtained from $\mathbb{T}^2 \times [0, 1]$ by gluing the top face to the bottom using ϕ . The semi-flow on $\mathbb{T}^2 \times [0, 1]$ that moves points at unit speed in the interval direction glues up to a flow Φ on M_ϕ called the *suspension flow* of ϕ . This is an Anosov flow whose weak stable and unstable foliations are simply the suspensions of \mathfrak{F}^s and \mathfrak{F}^u .

More generally, a flow is *pseudo-Anosov* if it is Anosov everywhere except near some isolated closed orbits, where it is modeled on the suspension of a pseudo-Anosov diffeomorphism. Pseudo-Anosov flows have *singular* weak stable and unstable foliations, which look like Figure 1 near the singularities.

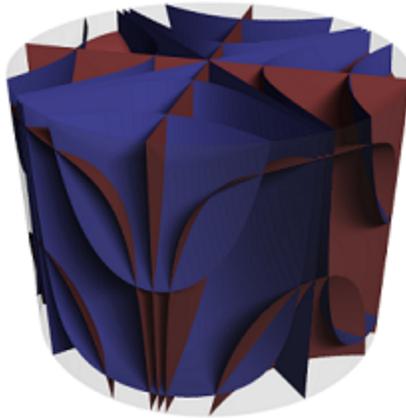


FIGURE 1. Singularities of stable and unstable foliations.

The Anosov Closing lemma leverages the transverse contracting-expanding behavior of a pseudo-Anosov flow to find closed orbits. An *almost-cycle* is a long flow segment whose endpoints are close together. The Anosov Closing Lemma says, roughly, that a sufficiently long almost-cycle whose endpoints are sufficiently close lies near a closed orbit.

The idea behind the Anosov Closing Lemma is illustrated in Figure 2. The left side of the figure depicts the local structure near the ends of an almost-cycle $[x_-, x_+]$, while the right side depicts the local structure near a point x in the middle. Since x_- is close to x_+ , the local stable/unstable leaf through x_- intersects the local unstable/stable leaf through x_+ .

Take a point y where the stable leaf through x_- intersects the unstable leaf through x_+ . Flowing forward, we arrive at a point y_+ which lies very close to x along its stable leaf. Flowing backwards, we arrive at a point y_- which lies very close to x along its unstable leaf. This produces an almost-cycle $[y_-, y_+]$ whose length is comparable to $[x_-, x_+]$, but whose ends are much closer. Repeating this, we obtain a sequence of better and better almost-cycles, which limit to a closed orbit.

Anosov and pseudo-Anosov flows are defined by their transverse structure. In contrast, a quasigeodesic flow is defined by a tangent condition. When the ambient manifold is hyperbolic, however, we will see that a quasigeodesic flow has a remarkably similar sort of transverse structure.

Given a quasigeodesic flow Φ on a closed hyperbolic manifold M , Calegari constructed a pair of flow-invariant decompositions of M into *positive leaves* and *negative leaves*. In the universal cover, all points in a positive/negative leaf are forwards/backwards asymptotic to a single point at infinity.

These decompositions are generally quite different from foliations. Leaves may have nontrivial interior, and may not be path-connected or even locally connected. Nevertheless, we can understand the separation properties of leaves by thinking of them as subsets of the *flowspace*, the orbit space of the lifted flow. This is a topological plane, and each leaf corresponds to a closed, connected, unbounded subset.

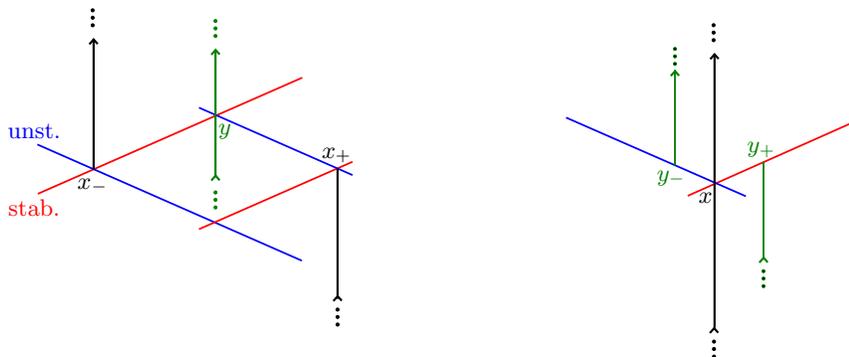


FIGURE 2. Anosov Closing Lemma.

We will see that the positive and negative leaves are *coarsely* contracted and expanded by the flow. Moreover, we will construct *strong* positive and negative decompositions, analogous to the strong stable and unstable foliations of a pseudo-Anosov flow.

Using coarse contraction-expansion, we will prove a Homotopy Closing Lemma. This allows us to approximate each recurrent orbit homotopically by closed orbits, provided that it lies in a reasonable topological configuration. We show that every quasigeodesic flow on a closed hyperbolic 3-manifold has a recurrent orbit in the appropriate topological configuration, and hence has closed orbits; many, in fact.

1.2. Organization. We review the basic theory of quasigeodesic flows in Section 2, and outline our main results in Section 2.5.

Section 3 is concerned with the transverse structure of a quasigeodesic flow. We show that quasigeodesic flows behave coarsely like pseudo-Anosov flows, and build the strong decompositions.

Section 4 studies the topological properties of the decompositions. In particular, we study Hausdorff limits and complementary regions of leaves. We motivate this by sketching a special case of our closing lemma in Section 4.1; the reader who is familiar with quasigeodesic flows may wish to look at this first.

Section 5 is the meat of the paper. We prove the closing lemma, and use this to show that quasigeodesic flows have closed orbits.

In Section 6 we ask whether our results can be extended to a larger class of *coarsely hyperbolic* flows. We propose an alternative method for finding closed orbits using ideas from geometric group theory.

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2. BACKGROUND

In this section we will review some of the basic topological theory around quasigeodesic flows. See [3], [8], and [9] for more details.

A *flow* on a manifold M is a continuous map

$$\Phi(\cdot) : \mathbb{R} \times M \rightarrow M$$

with the property that

$$\Phi_s(\Phi_t(x)) = \Phi_{t+s}(x)$$

for all $x \in M$ and $t, s \in \mathbb{R}$.

For a fixed $t \in \mathbb{R}$, the time- t map $\Phi_t : M \rightarrow M$ is a homeomorphism, since Φ_{-t} acts as its inverse. We can therefore think of a flow as an action of \mathbb{R} on M . When the flow is fixed, we will use the flow and action notation interchangeably, writing $x \cdot t = \Phi_t(x)$.

2.1. Quasigeodesic flows.

Definition 2.1. A curve γ in a metric space X is (k, ϵ) -*quasigeodesic* if

$$d_\gamma(x, y) \leq k \cdot d(x, y) + \epsilon$$

for all $x, y \in \gamma$, where d is the metric on X and d_γ is the distance along γ .

In \mathbb{H}^n , each quasigeodesic has well-defined and distinct endpoints in the Gromov boundary S_∞^{n-1} . Furthermore, each (k, ϵ) -quasigeodesic is contained in the C -neighborhood of the geodesic between its endpoints, where C is a constant depending only on k , ϵ , and n . See [12], [14], or [2].

Definition 2.2. A flow Φ on a manifold M is *quasigeodesic* if each flowline lifts to a quasigeodesic in the universal cover \widetilde{M} .

Calegari showed that every quasigeodesic flow on a closed hyperbolic 3-manifold is *uniformly quasigeodesic*, i.e. each lifted flowline is a (k, ϵ) -quasigeodesic for uniform k and ϵ .

On a closed hyperbolic 3-manifold, quasigeodesic flows are exactly those that can be studied “from infinity” in the following sense.

Proposition 2.3 ([7] Theorem B and [3] Lemma 4.3). *Let Φ be a flow on a closed hyperbolic 3-manifold M . Then Φ is quasigeodesic if and only if*

- (1) *each lifted flowline has well-defined and distinct endpoints in $S_\infty^2 = \partial_\infty \mathbb{H}^3$, and*
- (2) *the maps $e^\pm : \mathbb{H}^3 \rightarrow S_\infty^2$ that send each point to the positive/negative endpoint of its flowline are continuous.*

Quasigeodesic flows are quite common. Cannon and Thurston showed that the suspension flow of a pseudo-Anosov diffeomorphism is quasigeodesic [5]. Zeghib generalized this, showing that any flow transverse to a fibration of a 3-manifold is quasigeodesic [23]. Even more generally, Fenley and Mosher showed that any taut, finite-depth foliation on a closed hyperbolic 3-manifold admits a transverse or “almost-transverse” quasigeodesic flow [7].

Gabai showed that a closed hyperbolic 3-manifold with nontrivial second betti number has a taut, finite-depth foliation [11], so there are many such examples.

2.2. Flowspace and decompositions. Fix a quasigeodesic flow Φ on a closed hyperbolic 3-manifold M , and lift to a flow $\tilde{\Phi}$ on the universal cover $\tilde{M} \simeq \mathbb{H}^3$.

Definition 2.4. The *flowspace* P is the space of lifted flowlines, with the topology induced by the quotient map

$$\pi_P : \mathbb{H}^3 \rightarrow P = \mathbb{H}^3 / x \sim x \cdot t$$

that collapses each flowline to a point.

In other words, P is the orbit space of the lifted flow $\tilde{\Phi}$. Using uniform quasigeodesicity, Calegari showed that P is Hausdorff, and therefore homeomorphic to a plane. The action of $\pi_1(M)$ on \mathbb{H}^3 by deck transformations preserves the oriented foliation by flowlines, so it descends to an orientation-preserving action of $\pi_1(M)$ on P .

Each point $p \in P$ corresponds to a flowline, which we denote by $p \times \mathbb{R}$. Each subset $A \subset P$ corresponds to a union of flowlines, denoted $A \times \mathbb{R}$.

There are continuous *endpoint maps*

$$e^\pm : P \rightarrow S_\infty^2$$

that take each point to the positive/negative endpoint of the corresponding flowline.

Given subsets $A, B \subset S_\infty^2$, let $\{A \rightarrow\}$ and $\{\rightarrow B\}$ be the sets of flowlines that start in A and end in B , respectively. Similarly, $\{A \rightarrow B\} = \{A \rightarrow\} \cap \{\rightarrow B\}$.

As subsets of P ,

$$\begin{aligned} \{A \rightarrow\} &= (e^-)^{-1}(A), \\ \{\rightarrow B\} &= (e^+)^{-1}(B), \end{aligned}$$

and

$$\{A \rightarrow B\} = (e^-)^{-1}(A) \cap (e^+)^{-1}(B).$$

Consequently, if A and B are closed then each of these are closed. Moreover,

Lemma 2.5. *If $A, B \subset S_\infty^2$ are closed and disjoint then $\{A \rightarrow B\}$ is a compact subset of P .*

Proof. Each flowline in \mathbb{H}^3 is contained in a uniformly bounded neighborhood of its corresponding geodesic. Consequently, there is a compact set $C \subset \mathbb{H}^3$ that intersects every flowline with one end in A and the other in B . Then $\pi_P(C)$ is a compact subset of P that contains $\{A \rightarrow B\}$. \square

For each $z \in S_\infty^2$, each component of $\{\rightarrow z\}$ is called a *positive leaf*, and each component of $\{z \rightarrow\}$ is called a *negative leaf*. In contrast to the preceding lemma,

Lemma 2.6 ([3], Lemma 4.8). *Each positive or negative leaf is unbounded.*

If X is a space, a *decomposition* of X is a collection \mathcal{D} of closed subsets that fill X . The collections

$$\mathcal{D}^+ := \{\text{components of } \{\rightarrow z\} \subset P \mid z \in S_\infty^2\}$$

and

$$\mathcal{D}^- := \{\text{components of } \{z \rightarrow\} \subset P \mid z \in S_\infty^2\}$$

of positive and negative leaves are called the *positive* and *negative decompositions* of P . If $p \in P$, we write $\mathcal{D}^+(p)$ and $\mathcal{D}^-(p)$ for the positive and negative leaves through p .

By the preceding lemmas, our decompositions have two important properties.

- (1) Each leaf is closed, connected, and unbounded.
- (2) The intersection of a positive leaf with a negative leaf is compact.

Property (1) will allow us to treat the decompositions as if they were foliations. Property (2) is a weak form of transversality.

2.3. The compactified flowspace. In [8] we showed that P has a *universal compactification* to a closed disc $\widehat{P} = P \cup S_u^1$, whose boundary is Calegari’s *universal circle*.

Notation. If A is a subset of P or \widehat{P} , we will write \overline{A} for the closure of A in \widehat{P} . If B is a subset of \widehat{P} , we define $\partial_u B := B \cap S_u^1$.

The closure of each leaf $K \in \mathcal{D}^\pm$ intersects the universal circle in a totally disconnected set $\partial_u \overline{K}$ which we’ll call the *ends*¹ of K . Each of the sets

$$\bigcup_{K \in \mathcal{D}^\pm} \partial_u \overline{K},$$

consisting of all ends of positive/negative leaves are dense in S_u^1 . The compactification \widehat{P} is universal in the sense that any other disc compactification with these properties is a quotient of \widehat{P} .

The action of $\pi_1(M)$ on P extends naturally an orientation-preserving action on \widehat{P} . This restricts to Calegari’s universal circle action on the boundary.

Remark 2.7. Calegari-Dunfield showed that the fundamental group of the Weeks manifold admits no faithful orientation-preserving actions on the circle [4]. Consequently, the Weeks manifold has no quasigeodesic flows. This is the only way we know to show the non-existence of quasigeodesic flows.

Remark 2.8. Anosov and pseudo-Anosov flows also have universal circles, first constructed by Calegari-Dunfield [4]. The flowspace of a pseudo-Anosov flow is topologically a plane, and Fenley showed that one can use the universal circle compactify the flowspace [6]. In fact, this may be done in the same manner as for quasigeodesic flows.

A pseudo-Anosov flow Ψ on a closed 3-manifold M comes with 2-dimensional stable and unstable singular foliations. The lifts of these to the universal cover \widetilde{M} project to 1-dimensional singular foliations of the flowspace P_Ψ . The leaves of these foliations are properly embedded lines and n -prongs that intersect transversely. In particular, they are closed, connected, unbounded sets that intersect compactly, and we can use them to produce a universal compactification \widehat{P}_Ψ . The deck action induces an action on P_Ψ , which extends to \widehat{P}_Ψ .

If Ψ is both pseudo-Anosov and quasigeodesic then the stable and unstable foliations are exactly the positive and negative decompositions.

In [9] we showed that the endpoint maps e^\pm extend continuously to π_1 -equivariant maps

$$\widehat{e}^\pm : \widehat{P} \rightarrow S_\infty^2$$

¹Our usage of the word “end” differs slightly from that of [8] and [9], where it refers to a Freudenthal end. In fact, our set of ends $\partial_u \overline{K}$ is the closure of the image of K ’s Freudenthal ends (see [8], Lemma 7.8).

on the compactified flowspace. Furthermore, \hat{e}^+ agrees with \hat{e}^- on the boundary circle, where it restricts to a π_1 -equivariant sphere-filling curve

$$\hat{e} : S_u^1 \rightarrow S_\infty^2.$$

This generalizes the Cannon-Thurston Theorem, which produces such curves for suspension flows [5].

Notation. Given $A, B \subset S_u^2$ we define

$$\begin{aligned} \langle A \rightarrow \rangle &:= (\hat{e}^-)^{-1}(A), \\ \langle \rightarrow B \rangle &:= (\hat{e}^+)^{-1}(B), \end{aligned}$$

and

$$\langle A \rightarrow B \rangle := \langle A \rightarrow \rangle \cap \langle \rightarrow B \rangle.$$

Given $A \subset S_\infty^2$, we define

$$\langle A \rangle := \hat{e}^{-1}(A).$$

The endpoints of each flowline are distinct, so $e^+(p) \neq e^-(p)$ for each $p \in P$. Equivalently, $\langle z \rightarrow z \rangle = \langle z \rangle \subset S_u^1$ for each $z \in S_\infty^2$. Furthermore, if $A, B \subset S_\infty^2$ are disjoint, then $\langle A \rightarrow B \rangle = \{A \rightarrow B\} \subset P$.

2.4. Extended leaves. Distinct positive leaves are disjoint, but their closures may meet in S_u^1 . The extended endpoint maps provide a convenient way to organize such leaves.

The positive and negative *extended decompositions* are

$$\hat{\mathcal{D}}^+ := \{\text{components of } \langle \rightarrow z \rangle \subset \hat{P} \mid z \in S_\infty^2\}$$

and

$$\hat{\mathcal{D}}^- := \{\text{components of } \langle z \rightarrow \rangle \subset \hat{P} \mid z \in S_\infty^2\}.$$

The elements of $\hat{\mathcal{D}}^\pm$ is called a positive/negative *extended leaves*. If $p \in \hat{P}$, we write $\hat{\mathcal{D}}^+(p)$ and $\hat{\mathcal{D}}^-(p)$ for the positive and negative extended leaves through p

It would be convenient if the set of ends $\partial_u \hat{K}$ of an extended leaf \hat{K} were totally disconnected. We can take this to be true using the following construction.

Construction 2.9. Let

$$\mathcal{C} = \{\text{components of } \langle z \rangle \mid z \in S_\infty^2\}.$$

The quotient of S_u^1 obtained by collapsing the elements of \mathcal{C} is still a circle. Similarly, the quotient of \hat{P} obtained by collapsing the elements of \mathcal{C} is still a closed disc.

From now on, we will replace S_u^1 and \hat{P} by these quotients. For each $z \in S_\infty^2$, $\langle z \rangle$ is now totally disconnected.

An extended leaf is called *trivial* if it is contained entirely in S_u^1 . Each trivial extended leaf consists of a single point, since it is a connected component of a totally disconnected set $\langle z \rangle$.

On the other hand, each nontrivial extended leaf \hat{K} is the closure of a union of leaves, the *subleaves* of \hat{K} .

The following observation is ubiquitous in the sequel; we use it without further mention.

Lemma 2.10. *If \hat{K} and \hat{L} are the extended leaves through a point $p \in P$, then $\partial_u \hat{K}$ and $\partial_u \hat{L}$ are disjoint.*

Proof. If $\partial_u \hat{K} \cap \partial_u \hat{L} \neq \emptyset$, then $\hat{e}^+(\hat{K}) = \hat{e}^-(\hat{L})$, because the endpoint maps agree on S_u^1 . Then $e^+(p) = e^-(p)$, a contradiction. \square

2.5. Results. We will now summarize our main results.

Fix a quasigeodesic flow Φ on a closed hyperbolic 3-manifold M . If a point $x \in M$ is forward recurrent, then we can build a sequence of elements $g_i \in \pi_1(M)$ that approximates the homotopy class of its forward orbit. Simply take g_i to be the homotopy class of a long forward flow segment closed up with a short arc, chosen so that as i increases the flow segments get longer and the arcs get shorter. This is called an ω -sequence for x (see Section 3.4).

Homotopy Closing Lemma. *Let $(g_i)_{i=1}^\infty$ be an ω -sequence for a forward recurrent point $x \in M$. If the extended leaves through x are linked, then the g_i represent closed orbits when i is sufficiently large.*

By the extended leaves through x we really mean those through a lift of x . These are called linked when their ends are linked in the universal circle.

Recurrent Links Lemma. *Every quasigeodesic flow on a closed hyperbolic 3-manifold has some recurrent point whose extended leaves are linked.*

Our main theorem follows immediately from these two lemmas.

Closed Orbits Theorem. *Every quasigeodesic flow on a closed hyperbolic 3-manifold has closed orbits.*

3. COARSE GEOMETRY

So far, we have only seen the topological picture of a quasigeodesic flow. In this section we will study its geometry, showing that a quasigeodesic flow has a coarsely hyperbolic transverse structure.

3.1. The comparison map. Fix a quasigeodesic flow Φ on a closed hyperbolic 3-manifold M . We will build a correspondence between the lifted flow on \mathbb{H}^3 and the geodesic flow on the unit tangent bundle $\mathbf{T}^1\mathbb{H}^3$.

Each lifted flowline $x \cdot \mathbb{R}$ is an oriented quasigeodesic that shares its endpoints with an oriented geodesic $(x \cdot \mathbb{R})_G$. The nearest-point projection $\rho_{(x \cdot \mathbb{R})} : (x \cdot \mathbb{R}) \rightarrow (x \cdot \mathbb{R})_G$ moves each point a bounded distance independent of $(x \cdot \mathbb{R})$. To see this, recall that $(x \cdot \mathbb{R})$ is contained in the C -neighborhood of $(x \cdot \mathbb{R})_G$ for some uniform constant C . This can be pictured as a “banana” foliated by the radius- C hyperbolic discs perpendicular to $(x \cdot \mathbb{R})_G$, and $\rho_{(x \cdot \mathbb{R})}$ simply slides points along these discs to $(x \cdot \mathbb{R})_G$.

Sending each point $x \in \mathbb{H}^3$ to the nearest point $\rho_{x \cdot \mathbb{R}}(x)$ in $(x \cdot \mathbb{R})_G$ we obtain a continuous, $\pi_1(M)$ -equivariant map

$$G' : \mathbb{H}^3 \rightarrow \mathbb{H}^3.$$

This takes each flowline to its associated geodesic, and moves points a uniformly bounded distance.

The map G' is not necessarily injective or even monotone along flowlines. To fix this, we work with a monotonized version

$$G : \mathbb{H}^3 \rightarrow \mathbb{H}^3$$

defined by

$$G(x) = \sup_{t \leq 0} G'(x \cdot t).$$

This still moves points a bounded distance, because flowlines have bounded backtracks. That is, if x and $x \cdot t$ intersect the same disc in the foliated banana around $(x \cdot \mathbb{R})_G$, then $|t|$ is less than some uniform bound depending only on the quasi-geodesic constants.

Think of $\mathbf{T}^1\mathbb{H}^3$ as the space of pairs (γ, x) where γ is an oriented geodesic and x is a point in γ . We can lift G to a map

$$F : \mathbb{H}^3 \rightarrow \mathbf{T}^1\mathbb{H}^3$$

by defining

$$F(x) = ((x \cdot \mathbb{R})_G, G(x)).$$

3.2. Strong leaves. In a pseudo-Anosov flow, each weak stable/unstable leaf is foliated by strong stable/unstable leaves. We can use the map F to build an analogous structure for quasigeodesic flows.

Let $K \in \mathcal{D}^+$ be a positive leaf, which corresponds to a 2-dimensional positive leaf $K \times \mathbb{R}$. Given a point $x \in K \times \mathbb{R}$, let $S^+(x)$ be the horosphere in $\mathbf{T}^1\mathbb{H}^3$ determined by the vector $F(x)$. The *strong positive leaf* through x is the preimage of this horosphere, intersected with $K \times \mathbb{R}$. That is,

$$k_x = \{z \in K \times \mathbb{R} \mid S^+(z) = S^+(x)\}.$$

Each flowline in $K \times \mathbb{R}$ intersects k_x nontrivially, so $\pi_P(k_x) = K$. Since F is monotone along flowlines, each flowline in $K \times \mathbb{R}$ intersects k_x in either a point or a closed interval. Each 2-dimensional positive leaf $K \times \mathbb{R}$ is decomposed into an \mathbb{R} 's worth of strong positive leaves, corresponding to the \mathbb{R} 's worth of horospheres centered at $e^+(K)$.

Similarly, each 2-dimensional negative leaf $L \times \mathbb{R}$ is decomposed into strong negative leaves (l_y) . For $y \in L \times \mathbb{R}$, let $S^-(y)$ be the horosphere determined by $-F(y)$. The strong negative leaf through y is

$$l_y = \{z \in L \times \mathbb{R} \mid S^-(z) = S^-(y)\}.$$

The collections of all strong positive and strong negative leaves form decompositions of \mathbb{H}^3 called the *strong decompositions*. These are preserved by deck transformations, so they project to decompositions of M .

3.3. Coarse transverse hyperbolicity. The strong decompositions are not necessarily invariant under the flow. This will not matter since we are interested in the flowlines themselves, not their parametrizations.

Let k and k' be two strong positive leaves in $K \times \mathbb{R}$, where k' lies in front of k . Given flowlines γ_1 and γ_2 in $K \times \mathbb{R}$, let $x_i \in \gamma_i \cap k$ and $x'_i \in \gamma_i \cap k'$ for $i = 1, 2$. The corresponding points $F(x'_i)$ are obtained by flowing the points $F(x_i)$ forward under the geodesic flow, so

$$d(F(x'_1), F(x'_2)) < \lambda \cdot d(F(x_1), F(x_2))$$

where $\lambda < 1$ is arbitrarily small when k and k' are sufficiently far apart. The comparison map F moves points by at most some fixed constant D , so

$$d(x'_1, x'_2) < \lambda \cdot d(x_1, x_2) + 2D.$$

Coarse expansion works similarly. Let l and l' be strong negative leaves in $L \times \mathbb{R}$, where l' lies behind l . Given flowlines γ_1 and γ_2 in $L \times \mathbb{R}$, let $y_i \in \gamma_i \cap l$ and $y'_i \in \gamma_i \cap l'$ for $i = 1, 2$. We have to be a little careful: if γ_1 and γ_2 have the same positive

endpoint then $F(y_1) = F(y_2)$ and $F(y'_1) = F(y'_2)$. However, if $e^+(\gamma_1) \neq e^+(\gamma_2)$ then

$$d(y'_1, y'_2) > \lambda \cdot d(y_1, y_2) - 2D$$

where λ is arbitrarily large when l and l' are sufficiently far apart.

3.4. Dynamics in the flowspace. In the introduction we sketched a proof of the Anosov Closing Lemma that worked directly in the manifold M . This idea is difficult to generalize to a quasigeodesic flow, since our contraction-expansion is only coarse. Instead, we will work in the universal cover, translating some dynamical properties of the flow to the flowspace.

Let x be a point in the manifold M . If $x \cdot t_i \rightarrow y$ for a sequence of times $t_i \rightarrow \infty$, then y is called an ω -limit point of x . The ω -limit set $\omega(x)$ is the set of all ω -limit points of x . Since our manifold is compact, every point has a nontrivial ω -limit set.

For each $x \in M$, the ω -limit set $\omega(x)$ is clearly flow-invariant. Furthermore, it is an invariant of the orbit $x \cdot \mathbb{R}$. Therefore, it makes sense to write $y \cdot \mathbb{R} \in \omega(x \cdot \mathbb{R})$ whenever $y \in \omega(x)$.

An orbit $x \cdot \mathbb{R}$ is ω -recurrent if it is in its own ω -limit set. In particular, closed orbits are ω -recurrent.

We will work with ω -limit sets on the universal cover. Given points $p, q \in P$, we'll say that $q \in \omega(p)$ if this holds for the corresponding orbits in M . Notice that $q \in \omega(p)$ if and only if there is a sequence of points $x_i \in p \times \mathbb{R}$ that escape to $e^+(p)$ and a sequence of deck transformations g_i such that $\lim_{i \rightarrow \infty} g_i(x_i) = x_\infty \in q \times \mathbb{R}$. Such a sequence (g_i) is called an ω -sequence for $q \in \omega(p)$.

A point $p \in P$ corresponds to a closed orbit in M if and only if it is fixed by some nontrivial element of $\pi_1(M)$. If $g(p) = p$, then g represents a power of the free homotopy class of the corresponding orbit, and we will say that g represent this closed orbit. In this case, either $(g^i)_{i=1}^\infty$ or $(g^{-i})_{i=1}^\infty$ is an ω -sequence for $p \in \omega(p)$.

3.5. Dynamics of ω -sequences.

Lemma 3.1. *Let (g_i) be an ω -sequence for a recurrent point $p \in P$. For each i let α_i and ρ_i be the attracting and repelling fixed points for g_i in S_∞^2 . Then $\lim_{i \rightarrow \infty} \alpha_i = e^-(p)$ and $\lim_{i \rightarrow \infty} \rho_i = e^+(p)$.*

Proof. Since (g_i) is an ω -sequence, there are points $x_i \in p \times \mathbb{R}$ that escape to $e^+(p)$ such that $\lim_{i \rightarrow \infty} g_i(x_i) = x_\infty \in p \times \mathbb{R}$. For each $i = 1, 2, \dots, \infty$, let $x'_i = F(x_i)$.

Let H_i be the hyperplane perpendicular to x'_i for each i . The closure of H_i in $\overline{\mathbb{H}^3}$ separates S_∞^2 into two discs, $D_i^+ \ni e^+(p)$ and $D_i^- \ni e^-(p)$. When i is large, D_i^+ is a small disc near $e^+(p)$, and $g_i(H_i)$ is close to H_∞ , so $g_i(D_i) \supset D_i$. It follows that $\rho_i \in D_i$ and $\lim_{i \rightarrow \infty} \rho_i = e^+(p)$ as desired.

On the other hand, when i is large, $g_i(D_\infty^-)$ is a small ball around $g_i(e^-(p))$, which lies near $e^-(p)$. Then $g_i(D_\infty^-) \subset D_\infty^-$, so $\alpha_i \in g_i(D_\infty^-)$. Then $\lim_{i \rightarrow \infty} \alpha_i = e^-(p)$ as desired. \square

Our proof of the Homotopy Closing Lemma will use the following version of coarse contraction-expansion, which takes place entirely in the flowspace.

Proposition 3.2 (Coarse contraction-expansion). *Let (g_i) be an ω -sequence for $q \in \omega(p)$.*

- (1) If r is a point in P with $e^+(r) = e^+(p)$, then the accumulation points of $g_i(r)$ lie in the compact subset

$$\{e^-(q) \rightarrow e^+(q)\} \subset P.$$

- (2) If r is a point in P with $e^-(r) = e^-(p)$ and $e^+(r) \neq e^+(p)$, then the accumulation points of $g_i(r)$ lie in

$$\langle e^-(q) \rangle \subset S_u^1.$$

Proof. Since (g_i) is an ω -sequence, there are points $x_i \in p \times \mathbb{R}$ that escape to $e^+(p)$ such that $\lim_{i \rightarrow \infty} g_i(x_i) = x_\infty \in q \times \mathbb{R}$.

Claim (1): For each i , let $y_i \in r \times \mathbb{R}$ be a point that lies in the same strong positive leaf as $x_i \in p \times \mathbb{R}$. Then $x'_i := F(x_i)$ and $y'_i := F(y_i)$ are both contained in the horosphere $S_i^+ := S^+(x_i) = S^+(y_i)$. See Figure 3.

Notice that $\lim_{i \rightarrow \infty} g_i(x'_i) = x'_\infty$. We will show that $\lim_{i \rightarrow \infty} g_i(y'_i) = x'_\infty$ and the result follows.

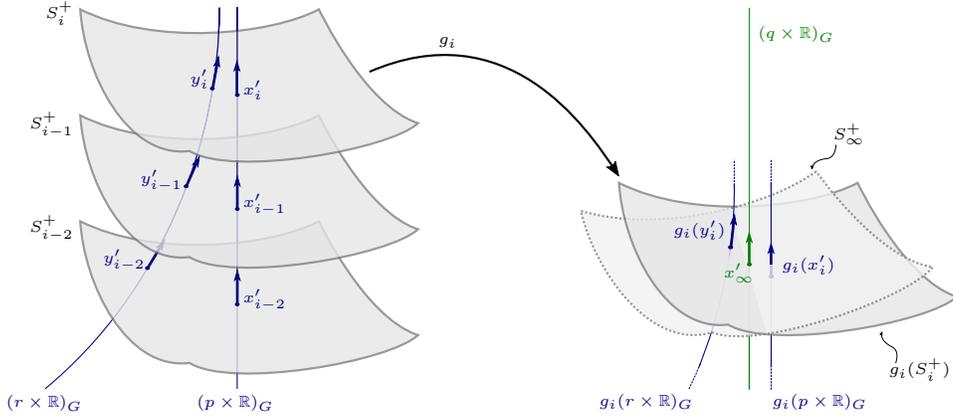


FIGURE 3. Coarse contraction.

When i is large, $d(x'_i, y'_i)$ is small, and g_i is an isometry, so $d(g_i(x'_i), g_i(y'_i))$ is small (here, d denotes the distance along the appropriate horosphere). Furthermore, $g_i(S_i^+)$ is close to S_∞^+ , and $g_i(x'_i)$ is close to x'_∞ so

$$\lim g_i(y'_i) = \lim g_i(x'_i) = x'_\infty$$

as desired.

Claim (2): We need to show that $\lim_{i \rightarrow \infty} e^\pm(g_i(p')) = e^-(q)$.

For each i , let $y_i \in r \times \mathbb{R}$ be a point in the same strong negative leaf as $x_i \in p \times \mathbb{R}$. Then $x'_i := -F(x_i)$ and $y'_i := -F(y_i)$ are both contained in the horosphere $S_i^- := S^-(x_i) = S^-(y_i)$.

When i is large, $d(g_i(x'_i), g_i(y'_i))$ is large. Furthermore, $g_i(x'_i)$ is close to x'_∞ , and S_i^- is close to S_∞^- , so

$$\lim_{i \rightarrow \infty} e^\pm(g_i(r)) = e^-(q)$$

as desired. □

4. TOPOLOGY IN THE FLOWSPACE

Our search for closed orbits will use three major ingredients:

- (1) coarse contraction and expansion,
- (2) the “foliation-like” behavior of the positive and negative decompositions, and
- (3) the “pseudo-Anosov-like” character of the universal circle action.

The latter two points will require a considerable amount of topological work. For motivation, we will sketch a special case of the Homotopy Closing Lemma.

4.1. Homotopy Closing for 3-prongs. Let $p \in P$ be an ω -recurrent point, and let \hat{K} and \hat{L} be the positive and negative extended leaves through p . Suppose that these are topological 3-prongs in standard position; see Figure 4(a). Let (g_i) be an ω -sequence for $p \in \omega(p)$. We will show that each g_i represents a closed orbit for i sufficiently large.

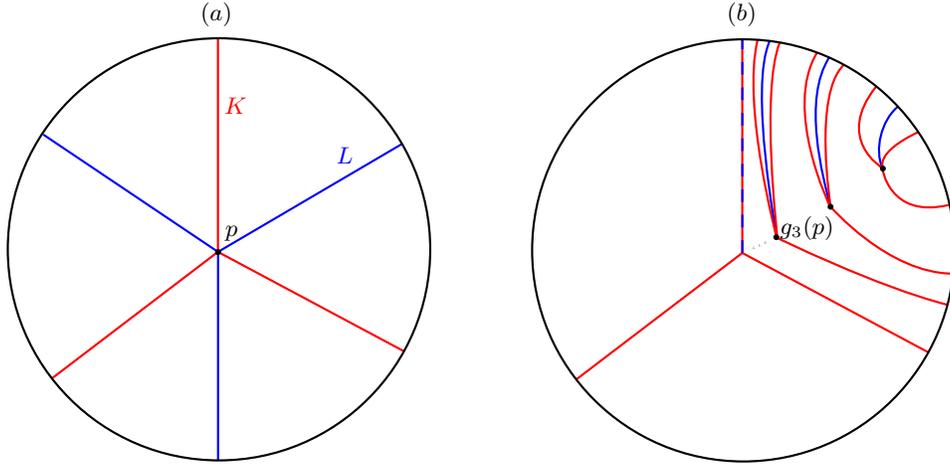


FIGURE 4. Sketch: homotopy closing for 3-prongs.

Sketch. Let (\hat{J}_i) be a Hausdorff convergent sequence of, say, positive extended leaves. In Section 4.2 we will see that $\lim \hat{J}_i$ is contained in a single positive extended leaf.

If $g_i(\hat{K}) = \hat{K}$ then $g_i(p) = p$ and g_i represents a closed orbit. If only finitely many g_i represent closed orbits, then after taking a subsequence we can assume that $g_i(\hat{K}) \neq \hat{K}$ for each i .

After taking a further subsequence, we can assume that all of the $g_i(\hat{K})$ are contained on one side of \hat{K} , $\lim g_i(\hat{K}) \subset \hat{K}$, and $\lim g_i(\hat{L}) \subset \hat{L}$. See Figure 4(b). In the limit, two ends of $g_i(\hat{K})$ must collapse to a single point. Some end of $g_i(\hat{L})$ is trapped between these two, so $g_i(\hat{L})$ must accumulate on this same point. Then $\partial_u \hat{K}$ and $\partial_u \hat{L}$ intersect at this point, a contradiction. Thus the g_i must eventually represent closed orbits. \square

Notice that we haven't used coarse contraction-expansion. This only comes into play when \hat{K} and \hat{L} are 2-prongs, i.e. lines (see Proposition 5.7).

4.2. Hausdorff limits of extended leaves. Let $(A_i)_{i=1}^{\infty}$ be a sequence of subsets of a space X . Then

$$\underline{\lim} A_i = \{x \in X \mid \text{every neighborhood of } x \text{ intersects all but finitely many } A_i\}$$

and

$$\overline{\lim} A_i = \{x \in X \mid \text{every neighborhood of } x \text{ intersects infinitely many } A_i\}.$$

If these limits agree then (A_i) is *Hausdorff convergent* and

$$\lim A_i = \underline{\lim} A_i = \overline{\lim} A_i$$

is its *Hausdorff limit*. See [13].

If X is a compact metric space, then every sequence (A_i) has a Hausdorff convergent subsequence. Furthermore, if each A_i is closed and connected then $\lim A_i$ is closed and connected.

The following lemma is stated for positive extended leaves, but it holds as well for negative extended leaves.

Lemma 4.1. *Let $(\hat{K}_i)_{i=1}^{\infty}$ be a sequence of positive extended leaves. If $\underline{\lim} \hat{K}_i$ intersects an positive extended leaf \hat{K}_{∞} , then $\overline{\lim} \hat{K}_i \subset \hat{K}_{\infty}$.*

In particular, if $p_i \in \hat{P}$ are points with $\lim p_i = p_{\infty}$, then $\overline{\lim} \hat{D}^{\pm}(p_i) \subset \hat{D}^{\pm}(p_{\infty})$.

Proof. Let X be a compact metric space. A decomposition \mathcal{D} of X is said to be *upper semicontinuous* if it satisfies any of the following equivalent conditions (see Section 3.6 in [13]):

- (1) If $U \subset X$ is an open set containing a decomposition element $K \in \mathcal{D}$, then there is an open set $V \supset K$ such that each decomposition element that intersects V is contained in U .
- (2) If $U \subset X$ is open, then the union of all decomposition elements that are contained in U is open.
- (3) If $A \subset X$ is closed, then the union of all decomposition elements that intersect A is closed.
- (4) If (K_i) is a sequence of decomposition elements, and $\underline{\lim} K_i$ intersects a decomposition element K_{∞} , then $\overline{\lim} K_i \subset K_{\infty}$.

If $f : X \rightarrow Y$ is a continuous, then the decomposition of X by point preimages is upper semicontinuous. If \mathcal{D} is upper semicontinuous then the decomposition \mathcal{D}' consisting of connected components of elements of \mathcal{D} is upper semicontinuous. It follows that \hat{D}^+ and \hat{D}^- are upper semicontinuous, and our lemma is condition (4). \square

If (g_i) is an ω -sequence for $q \in \omega(p)$, and $r \in P$ is contained in the positive extended leaf through p , then by coarse contraction, together with this lemma,

$$\overline{\lim} g_i(r) \subset \hat{D}^+(q) \cap \{e^-(q) \rightarrow e^+(q)\}.$$

If $r \in \hat{P}$ is contained in the negative extended leaf through p , and $e^+(r) \neq e^+(p)$, then

$$\overline{\lim} g_i(r) \subset \partial_u \hat{D}^-(q).$$

4.3. Complementary components. If $A \subset \hat{P}$, each component of $\hat{P} \setminus A$ is called a *complementary component*. If $B \subset S_u^1$, each component of $S_u^1 \setminus B$ is called a *complementary interval*.

Lemma 4.2. *If U is a complementary component of an extended leaf \hat{K} , then $\partial_u U$ is a complementary interval of $\partial_u \hat{K}$. Furthermore, the correspondence $U \mapsto \partial_u U$ is a bijection between the complementary components of \hat{K} and complementary intervals of $\partial_u \hat{K}$.*

Proof. For concreteness, let \hat{K} be positive.

Let U be a complementary component of \hat{K} . We will start by showing that $\partial_u U \neq \emptyset$. Choose a point $p \in U$ with $e^+(p) \neq \hat{e}^+(\hat{K})$, and let \hat{L} be the extended positive leaf through p . Then \hat{L} is contained in U , so $\partial_u U \supset \partial_u \hat{L} \neq \emptyset$. Clearly, $\partial_u U$ is a union of complementary intervals of \hat{K} .

If $\partial_u U$ were disconnected, then we could find an arc $\gamma \subset U$ with endpoints in distinct components of $\partial_u U$. This would disconnect \hat{K} , so $\partial_u U$ is a single complementary interval.

Each complementary interval of $\partial_u \hat{K}$ is contained in some complementary component of \hat{K} , so $U \mapsto \partial_u U$ is surjective. If U and V are complementary components of \hat{K} , and $\partial_u U = \partial_u V$, then $U = V$, so $U \mapsto \partial_u U$ is injective. \square

Given $a, b \in S_u^1$, the oriented interval between a and b is denoted (a, b) .

The following two lemmas are stated for positive leaves, but they work as well for negative leaves.

Lemma 4.3. *Let U be a complementary component of a positive extended leaf \hat{K} with $\partial_u U = (k, k')$. If $(\hat{K}_i)_{i=1}^\infty$ is a sequence of extended positive leaves in U , and $\varinjlim \hat{K}_i$ intersects \hat{K} , then $\overline{\varinjlim} \partial_u \hat{K}_i \subset \{k, k'\}$.*

Proof. This is an immediate consequence of Lemma 4.1. \square

If \hat{K} is an extended leaf, we can find extended leaves that lie arbitrarily close to \hat{K} in each complementary component:

Lemma 4.4. *Let U be a complementary component of an extended positive leaf \hat{K} . If $C \subset U$ is compact, then there is a positive leaf $L \subset U$ whose closure \bar{L} separates \hat{K} from C .*

Proof. We will deal with the positive case for concreteness. Let us begin by finding an *extended* leaf that separates \hat{K} from C .

Choose an arc $A \subset U$ that separates \hat{K} from C , and let B be the union of all positive extended leaves that intersect A . This is obviously connected, and it is compact by property (3) of an upper semicontinuous decomposition. Let V be the component of $\hat{P} \setminus (\hat{K} \cup B)$ that lies between \hat{K} and B . Then $B' = \bar{V} \cap B$ is a connected set that separates A from \hat{K} . We will show that B' is contained in a single extended leaf \hat{L} .

Note that $\partial_u V = (k, b) \cup (b', k')$, where $k, k' \in \partial_u \hat{K}$ and $b, b' \in \partial_u B'$.

Given a point $p \in B'$, choose a sequence of points $p_i \in V$ that converge to p , and let $\hat{L}_i = \hat{D}^+(p_i)$ be the corresponding positive extended leaves. Then $\overline{\varinjlim} \hat{L}_i \subset \hat{D}^+(p)$. The ends of \hat{L}_i converge to a subset of $\{b, b'\}$, so $\hat{D}^+(p) = \hat{D}^+(b)$ or $\hat{D}^+(b')$.

This works for every $p \in B'$, so

$$B' \subset \widehat{\mathcal{D}}^+(b) \cup \widehat{\mathcal{D}}^+(b').$$

In fact, $\widehat{\mathcal{D}}^+(b) = \widehat{\mathcal{D}}^+(b')$, since otherwise this would be a separation of B' . Therefore, $\widehat{L} := \widehat{\mathcal{D}}^+(b) = \widehat{\mathcal{D}}^+(b')$ separates K from C .

It remains to find a subleaf of \widehat{L} whose closure separates \widehat{K} from C . We use the following fact (see, e.g., [9] Lemma 2.7): if A is a subset of the plane that separates points x and y , then some component of A separates x and y .

Let x and y be points in P that lie in the components of $\widehat{P} \setminus \widehat{L}$ that contain \widehat{K} and C respectively. Since $\widehat{L} \cap P$ separates x and y , some component $L \subset \widehat{L} \cap P$ does too. This is a subleaf of \widehat{L} , and \overline{L} separates \widehat{K} from C . \square

4.4. Master leaves. Given a point $z \in S_\infty^2$, the *master leaf* corresponding to z is

$$Z := \{z \rightarrow\} \cup \{\rightarrow z\} \subset \widehat{P}.$$

In other words, Z is the union of all extended leaves, both positive and negative, that map to z .

Lemma 4.5. *Each master leaf is connected.*

Proof. Fix a master leaf Z corresponding to a point $z \in S_\infty^2$. Let $B_1 \supset B_2 \supset \dots$ be a nested sequence of closed balls around z in $\overline{\mathbb{H}^3}$ with $\bigcap_i B_i = z$. For each i , let $C_i = \pi_P(B_i \cap \mathbb{H}^3)$.

The \overline{C}_i are compact and connected, so $\overline{C} := \bigcap_i \overline{C}_i$ is compact and connected. We will show that $Z = \overline{C}$.

Let $p \in Z \cap P$. This corresponds to a flowline with one end at z , which clearly intersects every B_i . Then $p \in C_i$ for each i , and hence $p \in \overline{C}$.

Let $p \in Z \cap S_u^1$. For each i , let U_i be a neighborhood of z in S_∞^2 that is contained in B_i . Then $\langle \rightarrow U_i \rangle$ is an open neighborhood of p , and $\{\rightarrow U_i\} = \langle \rightarrow U_i \rangle \cap P$, so $p \in \overline{\{\rightarrow U_i\}}$. Furthermore, $\{\rightarrow U_i\} \subset C_i$ for each i by the preceding argument, so $p \in \overline{C}$.

Let $p \in \widehat{P} \setminus Z$. For i sufficiently large, we can find neighborhoods V_\pm of $\hat{e}^\pm(p)$ in S_∞^2 such no flowline in $\{V_- \rightarrow V_+\}$ intersects B_i , and hence $\{V_- \rightarrow V_+\}$ is disjoint from C_i . Here, we're using the fact that each flowline is contained in a neighborhood of uniform size of the corresponding geodesic.

If $p \in P$ then it is contained in $\{V_- \rightarrow V_+\}$, so $p \notin \overline{C}$. If $p \in S_u^1$, then it is contained in the interior of $\overline{\{V_- \rightarrow V_+\}} \cap S_u^1$. Only the frontier of this set may be contained in \overline{C}_i , so $p \notin \overline{C}$. \square

A master leaf is called *trivial* if it is contained in S_u^1 . As with extended leaves, trivial master leaves are points.

4.5. The universal sphere. Take two copies of the compactified flowspace, and identify their boundaries to form a sphere.

That is, let

$$i^\pm : \widehat{P} \rightarrow \widehat{P}^\pm$$

be homeomorphisms, and let

$$S_u^2 = \widehat{P}^+ \sqcup \widehat{P}^- / i^+(a) \sim i^-(a) \text{ for each } a \in S_u^1.$$

We will think of \widehat{P}^+ and \widehat{P}^- as the northern and southern hemispheres of the universal sphere S_u^2 . These meet at the equator, which is identified with S_u^1 .

We can think of the endpoint maps as maps

$$\hat{e}^\pm : \widehat{P}^\pm \rightarrow S_\infty^2$$

defined on the hemispheres of the universal sphere. These agree on the equator, so they define a map

$$\tilde{e} : S_u^2 \rightarrow S_\infty^2.$$

The preimage $\tilde{e}^{-1}(z)$ of a point $z \in S_\infty^2$ is the union of all positive and negative extended leaves that map to z , where we think of positive leaves as lying in the northern hemisphere and negative leaves as lying in the southern hemisphere. The collection

$$\tilde{\mathcal{D}} = \{\tilde{e}^{-1}(z) \mid z \in S_\infty^2\}$$

of such point preimages forms an upper semicontinuous decomposition of S_u^2 .

Notice that the projection

$$\pi : S_u^2 \rightarrow \widehat{P}$$

that identifies the two hemispheres takes $\tilde{Z} = \tilde{e}^{-1}(z)$ to a corresponding master leaf Z . We will call $\tilde{Z} \in \tilde{\mathcal{D}}$ a *lifted master leaf*.

Lemma 4.6. *Lifted master leaves are compact, connected, and nonseparating.*

Proof. Let $\tilde{Z} = \tilde{e}^{-1}(z)$. This is compact since it is a closed subset of the compact space.

If \tilde{Z} were disconnected, then we could write it as a disjoint union

$$\tilde{Z} = A \sqcup B$$

of compact sets A and B . Then $\pi(A)$ and $\pi(B)$ are disjoint. Indeed, $\pi(A)$ does not intersect $\pi(B)$ in S_u^1 since π restricts to a homeomorphism on the equator. Also, $\pi(A)$ does not intersect $\pi(B)$ in P , since otherwise the corresponding master leaf Z would have positive and negative subleaves that intersect in P . Therefore,

$$Z = \pi(A) \sqcup \pi(B)$$

would be a separation of Z . But Z is connected, so \tilde{Z} must be connected.

It remains to show that \tilde{Z} is nonseparating. Notice that S_∞^2 is identified with the quotient

$$S_u^2 / \tilde{\mathcal{D}}$$

obtained by collapsing each lifted master leaf to a point. If \tilde{Z} were separating then it would map to a cut point in S_∞^2 , which is impossible. \square

As an application, we have the following important lemma.

Lemma 4.7 (No bigons). *Let \hat{K} and \hat{L} be extended leaves of any type (i.e. both positive, both negative, or one positive and one negative). Then $\partial_u \hat{K}$ intersects $\partial_u \hat{L}$ in at most one point.*

Proof. We will use following fact from classical analysis situs (see Theorem II.5.28a in [22]): If A and B are compact connected subsets of S^2 , and $A \cap B$ is disconnected, then $A \cup B$ separates S^2 .

Think of \hat{K} and \hat{L} in the appropriate hemispheres of S_u^2 . If they intersect in S_u^1 then they are contained in a lifted master leaf \tilde{Z} . If they intersect at more than one point in S_u^1 , then \tilde{Z} is separating, contradicting the preceding lemma. \square

Remark 4.8. The universal sphere can be used to build a compactification of \tilde{M} called the *flow ideal compactification* (cf. [5] and [6]).

Take $\hat{P} \times [-1, 1]$, and collapse each vertical interval in $S_u^1 \times [-1, 1]$ to a point. This is a compactification of $\tilde{M} \simeq \mathbb{H}^3$ whose boundary is S_u^2 . It can be pictured as a “lens” foliated by the vertical segments connecting $i^-(p)$ to $i^+(p)$ for each $p \in P$. Collapsing the decomposition \tilde{D} we recover the usual compactification $\bar{\mathbb{H}}^3$ of \mathbb{H}^3 , together with the foliation by flowlines.

4.6. Linking. Two disjoint subsets $A, B \subset S_u^1$ are *n-linked* if B intersects exactly n complementary components of A ; equivalently, if A intersects exactly n complementary components of B .

The extended leaves through a point $p \in P$ are said to be *n-linked* if their ends are n -linked in S_u^1 . They are ($\geq n$)-*linked* if they are m -linked for $m \geq n$, and *linked* if they are (≥ 2)-linked.

By the following lemma, the extended leaves through each point are n -linked for some finite n .

Lemma 4.9. *Let \hat{K} and \hat{L} be the positive and negative extended leaves through a point $p \in P$. Then $\partial_u \hat{L}$ intersects only finitely many complementary components of $\partial_u \hat{K}$.*

Proof. If $\partial_u \hat{L}$ intersects infinitely many complementary components of $\partial_u \hat{K}$, then we can find ends $k_i \in \partial_u \hat{K}$ and $l_i \in \partial_u \hat{L}$ so that $\lim_{i \rightarrow \infty} k_i = \lim_{i \rightarrow \infty} l_i$. The ends of extended leaves are closed, so this means that \hat{K} intersects \hat{L} in S_u^1 , a contradiction. \square

The idea of linking and n -linking works as well for leaves. If the leaves through a point are n -linked then the extended leaves through that point are ($\geq n$)-linked.

The following lemma is an immediate application of the Pigeonhole Principle.

Lemma 4.10 (Linking Pigeonhole Principle). *Suppose that \hat{K} and \hat{L} are ($\geq n$)-linked, and let A be a subset of S_u^1 that contains the ends of \hat{K} . If A has less than n components then it also contains an end of \hat{L} .*

5. CLOSED ORBITS

We now turn to the problem of finding closed orbits.

5.1. Closed orbits and master leaves.

Lemma 5.1. *If $g \in \pi_1(M)$ fixes a nontrivial master leaf, then it represents a closed orbit.*

Proof. Let α and ρ be the attracting and repelling fixed points of g in S_∞^2 , and let A and R be the corresponding master leaves. These are the only master leaves fixed by g .

Suppose that R is nontrivial. Then $R \cap P$ is nonempty. A point $p \in R \cap P$ corresponds to a flowline with one endpoint at ρ . Applying g takes the other

endpoint closer to α , so the forward orbit

$$g(p), g^2(p), g^3(p), \dots$$

remains in a bounded subset of P . The Brouwer Plane Translation Theorem² then implies that g has a fixed point in P , and hence represents a closed orbit.

If A is nontrivial, then replace g by g^{-1} . This interchanges A and R , and we can use the same argument. \square

5.2. Closed orbits and the universal circle. A group Γ of orientation-preserving homeomorphisms of S^1 is said to be *pA-like* if for each $g \in \Gamma$, some positive power g^n has an even number of fixed points, alternately attracting and repelling.

We will see that the action of $\pi_1(M)$ on S_u^1 is pA-like.

Lemma 5.2. *Let g be an element of $\pi_1(M)$. If g fixes some point in S_u^1 , then it has an even number of fixed points, alternately attracting and repelling.*

Proof. Let α and ρ be the attracting and repelling fixed points of g in S_∞^2 . Let F be the set of fixed points of g in S_u^1 . We will show that $F = \langle \alpha \rangle \cup \langle \rho \rangle$, where each point in $\langle \alpha \rangle$ is attracting, and each point in $\langle \rho \rangle$ is repelling.

If $x \in S_u^1$ is not in $\langle \alpha \rangle$ or $\langle \rho \rangle$, then $\lim_{i \rightarrow \infty} e_u(g^i(x)) = \alpha$, so $\lim_{i \rightarrow \infty} g^i(x) \in \langle \alpha \rangle$. Similarly, $\lim_{i \rightarrow -\infty} g^i(x) \in \langle \rho \rangle$.

In particular, if $x \in S_u^1 \setminus \langle \alpha \rangle \cup \langle \rho \rangle$, then $g(x) \neq x$. Therefore, $F \subset \langle \alpha \rangle \cup \langle \rho \rangle$.

Let I be a complementary interval of F . Then g acts as a translation on I , fixing its endpoints. One of these endpoints, a_I , is attracting, while the other, r_I , is repelling (with respect to points in I). Then $a_I \in \langle \alpha \rangle$ and $r_I \in \langle \rho \rangle$. Indeed, take $x \in I \setminus \langle \alpha \rangle \cup \langle \rho \rangle$. Then $a_I = \lim_{i \rightarrow \infty} g^i(x) \in \langle \alpha \rangle$, and $r_I = \lim_{i \rightarrow -\infty} g^i(x) \in \langle \rho \rangle$.

Let $a \in \langle \alpha \rangle$. If $g(a) \neq a$, then a is contained in some complementary interval J of F . Then $\lim_{i \rightarrow -\infty} g^i(a) = r_J$. But $\langle \alpha \rangle$ is g -invariant, so this means that $r_J \in \langle \alpha \rangle$, a contradiction. Hence $\langle \alpha \rangle \subset F$. Similarly, $\langle \rho \rangle \subset F$, so $F = \langle \alpha \rangle \cup \langle \rho \rangle$.

Each $a \in \langle \alpha \rangle$ is the boundary of two complementary intervals of F . It is attracting with respect to each of these intervals, and hence attracting overall. Similarly, each point in $\langle \rho \rangle$ is repelling. Clearly, the attracting fixed points must alternate with the repelling fixed points. \square

Lemma 5.3. *Let g be an element of $\pi_1(M)$. Then some positive power g^n has fixed points in S_u^1*

Proof. Suppose that $F = \emptyset$.

Let $J = (a, r)$ be a complementary interval of $\langle \alpha \rangle \cup \langle \rho \rangle$ bounded by $a \in \langle \alpha \rangle$ and $r \in \langle \rho \rangle$. If $g^i(J) = g^j(J)$ for some $i > j > 0$ then $g^{i-j}(J) = J$, so $g^{i-j}(a) = a$ and we are done.

Otherwise, the intervals $J, g^1(J), g^2(J), \dots$ are disjoint, hence their diameters must go to zero. After taking a subsequence, $g^i(J)$ converges to a point, and we have $\lim_{i \rightarrow \infty} g^i(a) = \lim_{i \rightarrow \infty} g^i(b)$. This contradicts the fact that $\langle \alpha \rangle \cap \langle \rho \rangle = \emptyset$. \square

Lemma 5.4. *Suppose that $g \in \pi_1(M)$ does not represent a closed orbit. Then it acts on S_u^1 with exactly two fixed points in an attracting-repelling pair.*

²If f is a homeomorphism of the plane with a bounded forward orbit, then f has a fixed point. See, e.g., [10]

Proof. Note that $F \neq \emptyset$. Indeed, if $F = \emptyset$ then g must fix a point in P by Brouwer's fixed point theorem, and hence represent a closed orbit.

Let $\langle \alpha \rangle$ and $\langle \rho \rangle$ be the attracting and repelling fixed point sets as in the preceding lemmas. Notice that $\langle \alpha \rangle = \partial_u A$ and $\langle \rho \rangle = \partial_u R$, where A and R are the master leaves corresponding to α and ρ .

If $\langle \alpha \rangle$ were disconnected then A would be nontrivial, and if $\langle \rho \rangle$ were disconnected then R would be nontrivial. Either way, g would represent a closed orbit by Lemma 5.1. \square

5.3. The closing lemma. We will now prove our first major result.

Homotopy Closing Lemma. *Let $(g_i)_{i=1}^{\infty}$ be an ω -sequence for a recurrent point $p \in P$. If the extended leaves through p are linked, then each g_i represents a closed orbit for i sufficiently large.*

We begin with the (≥ 3) -linked case.

Proposition 5.5 (Homotopy closing for (≥ 3) -links). *Let (g_i) be an ω -sequence for a recurrent point $p \in P$. If the extended leaves through p are (≥ 3) -linked, then each g_i represents a closed orbit for i sufficiently large.*

Proof. Let $\hat{K} = \widehat{\mathcal{D}}^+(p)$ and $\hat{L} = \widehat{\mathcal{D}}^-(p)$ be the extended leaves through p . For each i , let $\hat{K}_i = g_i(\hat{K})$ and $\hat{L}_i = g_i(\hat{L})$.

Suppose that there are infinitely many g_i that do not represent closed orbits, and take a subsequence consisting only of these. By Lemma 5.1, $\hat{K}_i \neq \hat{K}$ and $\hat{L}_i \neq \hat{L}$ for each i .

The \hat{K}_i can visit at most finitely many complementary components of \hat{K} . Indeed, let U_i be the complementary component that contains \hat{K}_i for each i . Then $\partial_u U_i$ contains the ends of \hat{K}_i , hence it must also contain an end of \hat{L}_i . If infinitely many of the U_i were distinct, the diameters of the corresponding complementary intervals would go to zero. Then \hat{K}_i and \hat{L}_i would both accumulate on the same point in S_u^1 , and $\partial_u \hat{K} \cap \partial_u \hat{L} \neq \emptyset$, a contradiction.

Therefore, infinitely many \hat{K}_i lie in a single complementary interval $U = U_i$. After taking a subsequence, $\partial_u \hat{K}_i$ accumulates on at most two points, the boundary points of $\partial_u U$. Since \hat{L}_i is (≥ 3) -linked with \hat{K}_i , the Linking Pigeonhole Principle says that \hat{L}_i also accumulates on one of these points. Again, this means that \hat{K} and \hat{L} intersect at this point, a contradiction. \square

In fact, any (≥ 3) -linked pair of extended leaves \hat{K}, \hat{L} that meet in P are fixed by some $g \in \pi_1(M)$. Simply take a point $p \in \hat{K} \cap \hat{L}$, and an ω -limit point $q \in \omega(p)$. The elements g_i of an ω -sequence eventually take \hat{K} and \hat{L} to the extended leaves through q , so $g_i^{-1}g_j$ fixes \hat{K} and \hat{L} for large $i \neq j$.

Remark 5.6. The reader may notice that our proof works with the extended leaves through p , quickly forgetting about p itself. In fact, p may not correspond to a closed orbit.

For example, start with the suspension flow of a pseudo-Anosov diffeomorphism, which is both quasigeodesic and pseudo-Anosov. Blow up some singular orbit γ to a solid torus foliated by parallel closed orbits. One may perturb the flow on this solid torus, breaking some of the closed orbits while keeping the flow quasigeodesic.

Using coarse contraction-expansion we can extend this to the 2-linked case.

Proposition 5.7 (Homotopy closing for 2-links). *Let (g_i) be an ω -sequence for a recurrent point $p \in P$. If the extended leaves through p are 2-linked, then each g_i represents a closed orbit for i sufficiently large.*

Proof. Let $\hat{K} = \hat{\mathcal{D}}^+(p)$ and $\hat{L} = \hat{\mathcal{D}}^-(p)$ be the extended leaves through p . For each i , let $\hat{K}_i = g_i(\hat{K})$, and $\hat{L}_i = g_i(\hat{L})$.

Suppose that infinitely many g_i do not represent closed orbits, and take a subsequence consisting only of these. In particular, this means that $\hat{K}_i \neq \hat{K}$ and $\hat{L}_i \neq \hat{L}$ for each i . By Lemma 5.4, each g_i acts on S_u^1 with exactly two fixed points, a_i and r_i , in an attracting-repelling pair.

As in the preceding lemma, we can assume that each \hat{K}_i is contained in a single complementary component U of \hat{K} . The corresponding complementary interval is of the form $\partial_u U = (k, k')$, where $k, k' \in \partial_u \hat{K}$ (Figure 5). Taking a further subsequence, the points $k_i := g_i(k)$ and $k'_i := g_i(k')$ converge to either k or k' . Furthermore, $\lim k_i \neq \lim k'_i$, since otherwise $\lim k_i \in \overline{\lim} \hat{L}_i$.

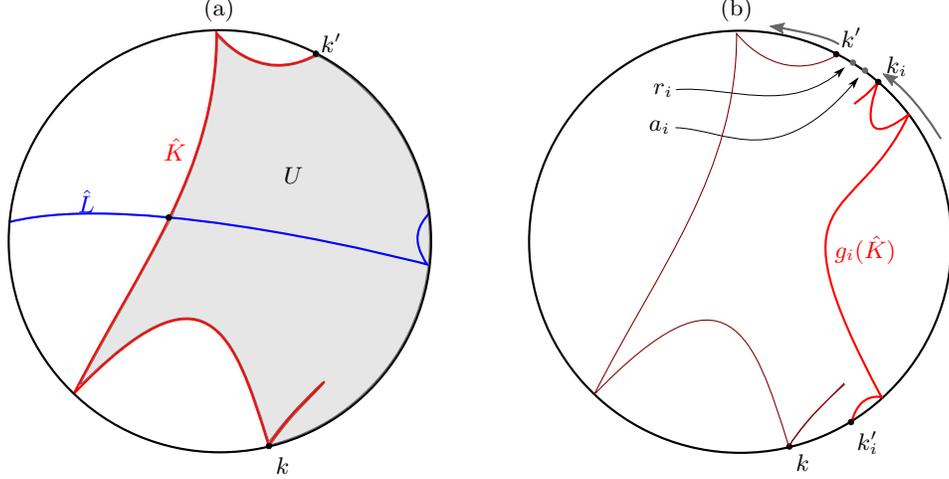


FIGURE 5. Step 1.

Step 1:

Suppose that $\lim k_i = k'$ and $\lim k'_i = k$.

For i sufficiently large, g_i takes the interval (k', k) to a disjoint interval (k'_i, k_i) . Then g_i has no fixed points in either of these two intervals, so a_i and r_i are contained in either (k, k'_i) or (k_i, k') . In fact, a_i and r_i must lie in the same one of these intervals as illustrated in Figure 5(b). Then a_i and r_i both accumulate on \hat{K} , so

$$\lim_{i \rightarrow \infty} \hat{e}(a_i) = \hat{e}^+(\hat{K}) = \lim_{i \rightarrow \infty} \hat{e}(r_i),$$

which contradicts Lemma 3.1.

Step 2:

Suppose that $\lim k_i = k$ and $\lim k'_i = k'$. We will find negative leaves N and N' that cross \hat{K} near k and k' respectively, and are pulled inward by the g_i . This will produce two repelling fixed points, contradicting our assumptions.

Special case: Suppose that \hat{K} contains a single subleaf K , so that $\hat{K} = \overline{K}$. Let \hat{N}_k and $\hat{N}_{k'}$ be the negative extended leaves through k and k' . Then $K \subset$

$V \cap V'$, where V is a complementary component of \hat{N}_k , and V' is a complementary component of $\hat{N}_{k'}$.

By Lemma 4.7, $k' \notin \hat{N}_k$, so $k' \subset V$. Similarly, $k \subset V'$.

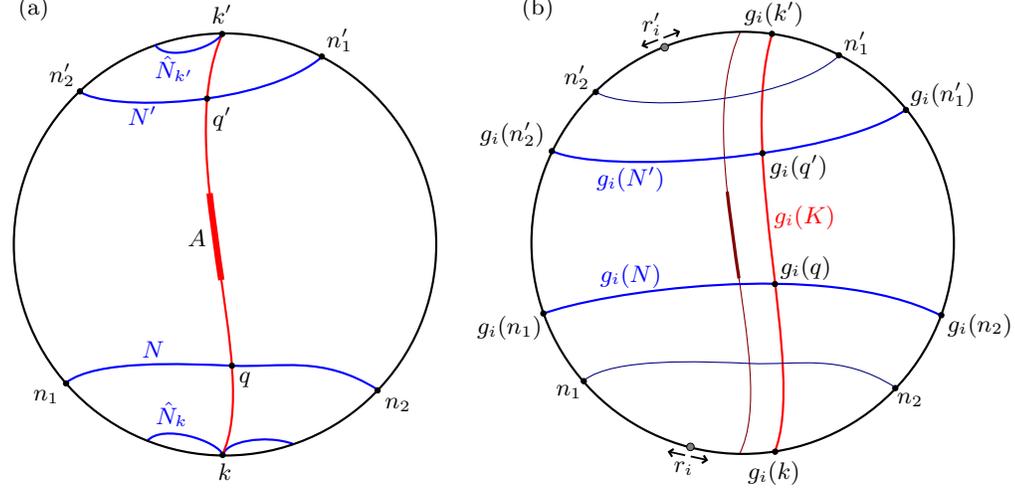


FIGURE 6. Homotopy closing for 2-links: special case.

Let

$$A = K \cap \{e^-(p) \rightarrow e^+(p)\}.$$

By coarse contraction, $\overline{\lim} g_i(q) \subset A$ for each $q \in K$.

Choose negative leaves N and N' that separates A from \hat{N}_k and $\hat{N}_{k'}$ respectively, and label their ends as in Figure 6(a).

Choose points $q \in N \cap K$ and $q' \in N' \cap K$. When i be large, $g_i(q)$ and $g_i(q')$ are close to A , and the ends of $g_i(N)$ and $g_i(N')$ are arranged as in Figure 6(b). In particular, $g_i(n_1, n_2) \supseteq (n_1, n_2)$ and $g_i(n'_1, n'_2) \supseteq (n'_1, n'_2)$, so g_i must have at least two repelling fixed points, $r_i \in (n_1, n_2)$ and $r'_i \in (n'_1, n'_2)$. This is a contradiction.

General case: We return to the general case, where \hat{K} may contain more than one subleaf.

Again, let \hat{N}_k be the negative extended leaf through k . By Lemma 4.7, $k' \notin \hat{N}_k$, so k' is contained in some complementary component V of \hat{N}_k . The generic picture is illustrated in Figure 7(a).

Notice that \hat{K} may have some “bad subleaves” that are not contained in V . If

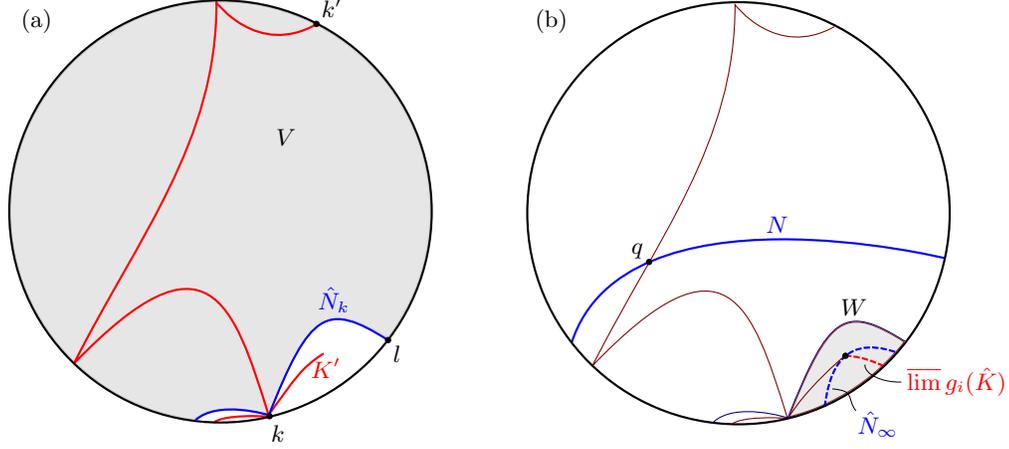
$$A = \hat{K} \cap \{e^-(p) \rightarrow e^+(q)\}$$

intersects one of these bad subleaves, then $A \not\subset V$, and it doesn't make sense to ask for a leaf that separates A from \hat{N}_k . Instead, we will use

$$A^* = A \cap V.$$

Take a negative leaf $N \subset V$ that links \hat{K} , and let $q \in N \cap \hat{K}$. We will show that $\overline{\lim} g_i(q) \subset A^*$.

Indeed, suppose that $g_i(q)$ accumulates on a point q_∞ in a bad subleaf K' . This subleaf must intersect \bar{U} , since $\hat{K}_i \subset U$ for all i . Using Lemma 4.7 we see that $\partial_u \bar{K}' = k$. The complementary component W of \hat{N}_k that contains K' corresponds to a complementary interval of the form $\partial_u W = (k, l) \subset (k, k')$.


 FIGURE 7. Bad subleaves of \hat{K} .

Take a subsequence so that $\lim g_i(q) = q_\infty$. Then $\overline{\lim} g_i(N)$ is contained in the negative extended leaf \hat{N}_∞ through q_∞ , which lies in W . But N and \hat{K} are linked, so \hat{K}_i must accumulate on some point in (l, k) . This is impossible, since $\overline{\lim} g_i(\hat{K}) \subset \hat{K}$, which has no ends in (l, k) . See Figure 7(b). Thus $\overline{\lim} g_i(q) \subset A^*$.

Finally, let N be a negative leaf that separates A^* from \hat{N}_k , and choose a point $q \in N \cap \hat{K}$. Take i to be large, so that k_i is close to k , k'_i is close to k' , and $g_i(q)$ is close to A^* . See Figure 8. As in the special case, g_i must have a repelling fixed point near k .

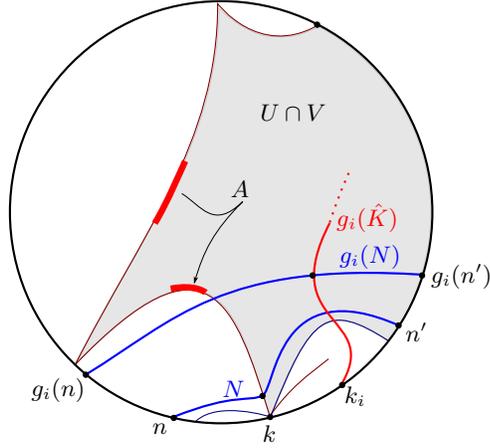


FIGURE 8. The general case.

We find another repelling fixed point near k' in the same manner. We had assumed that g_i has only one repelling fixed point, so this is a contradiction. \square

5.4. Recurrent links.

Recurrent Links Lemma. *If Φ is a quasigeodesic flow on a closed hyperbolic 3-manifold M , then some point $p \in P$ is recurrent and has linked extended leaves.*

Together with the Homotopy Closing Lemma, this completes the proof of the Closed Orbits Theorem.

To start,

Lemma 5.8. *Some positive leaf K has least two ends.*

Proof. Suppose that each positive leaf has exactly one end. Define a map

$$r : \widehat{P} \rightarrow S_u^1$$

by

$$r(p) = \partial_u \widehat{\mathcal{D}}^+(p).$$

This is continuous because $\widehat{\mathcal{D}}^+$ is an upper semicontinuous decomposition. Furthermore, r restricts to the identity on S_u^1 , so we have produced a retraction of a closed disc onto its boundary, which is impossible. \square

Lemma 5.9. *The leaves through some $p \in P$ are linked.*

Proof. Let K be a positive leaf with at least two ends, and let $k, k' \in \partial_u \overline{K}$. Let \widehat{N} be the negative extended leaf through k . Then $k' \notin \widehat{N}$, so k' is contained in a complementary component U of \widehat{N} . Let L be a negative leaf whose closure separates k from k' , and take $p \in K \cap L$. \square

Proof of Recurrent Links Lemma. Let $p \in P$ be a point whose leaves are linked, and let $x \in M$ be a point in the corresponding orbit. The ω -limit set $\omega(x)$ is invariant under Φ , so it contains some minimal set. Each minimal set is the closure of an almost-periodic orbit ([1], Theorem 1.7), which is *a fortiori* recurrent. Therefore, we have some $q \in \omega(p)$ that is ω -recurrent.

An ω -sequence for $q \in \omega(p)$ takes the leaves through p to the extended leaves through q , which are therefore linked. \square

6. QUESTIONS

6.1. Coarse transverse hyperbolicity. Our proof of the Homotopy Closing Lemma holds just as well for pseudo-Anosov flows, even ones that are not quasigeodesic. In fact, it should hold for a larger class of *coarsely hyperbolic* flows, defined by the existence of a pair of decompositions that are coarsely contracted/expanded.

It's easy to construct coarsely hyperbolic flows that are neither quasigeodesic nor pseudo-Anosov. For example, start with a pseudo-Anosov flow that is not quasigeodesic and blow up a closed orbit. These examples are quite trivial since the Homotopy Closing Lemma follows easily from the Anosov Closing Lemma for the original flow. It would be interesting to construct a less trivial class of examples.

A flow on a closed 3-manifold M is called *product covered* if the lifted flow on the universal cover \widetilde{M} is conjugate to the vertical flow on \mathbb{R}^3 .

Question. *Let Φ be a product-covered flow on a closed hyperbolic 3-manifold M . Is Φ coarsely hyperbolic?*

More generally:

Question. *Let Φ be a product-covered flow on a closed hyperbolic 3-manifold M . Does Φ have a closed orbit?*

This leads us to the following question.

Question. *Can the fundamental group of a closed hyperbolic 3-manifold act freely on the plane?*

If the answer is no, then each product-covered flow on a closed hyperbolic 3-manifold must have closed orbits.

6.2. Möbius-like groups. In [8] we proposed a very different method for proving the Closed Orbits Theorem.

Let Γ be a group. An action of Γ on a circle S^1 is called *Möbius-like* if each $g \in \Gamma$ is conjugate to a Möbius transformation. It is called *hyperbolic Möbius-like* if each $g \in \Gamma$ is conjugate to a hyperbolic Möbius transformation. A Möbius-like or hyperbolic Möbius-like action is called *Möbius* or *hyperbolic Möbius*, respectively, if it is conjugate to an action by Möbius transformations.

The fundamental group of a closed hyperbolic 3-manifold can never act as a hyperbolic Möbius group (see [8]). The only known examples of Möbius-like actions that are not Möbius are found in [15]. We propose the following conjecture.

Conjecture 6.1. *The fundamental group of a closed hyperbolic 3-manifold can not act as a hyperbolic Möbius-like group.*

By Lemma 5.4, this conjecture implies the Closed Orbits Theorem.

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