1. Introduction

The classical Ward correspondence establishes a bijective correspondence between a certain class of holomorphic vector bundles on $\mathbb{P}(\mathbb{C})^3$ with the class of complex instantons on Minkowski space. In this note, we prove a partial quantum analogue of this result. More precisely, we show that to this same class of vector bundles on commutative $\mathbb{P}^3$, we can associate an instanton on quantum Minkowski space (i.e., a quantum instanton). The inverse procedure remains to be investigated, and it is hoped that the much larger family of semi-stable torsion free sheaves on $\mathbb{P}(\mathbb{C})^3$ will also give rise to quantum instantons.

Ward’s construction utilizes the Penrose double fibration twistor diagram,

$$
\begin{array}{c}
\mathbb{M} \\
\mu \\
F \\
\nu \\
\mathbb{P}
\end{array}
\begin{array}{c}
\mu \\
\nu \\
\mathbb{F} \\
\mathbb{M}
\end{array}
$$

where $\mathbb{F}$ denotes the flag variety parametrizing $\{ \text{lines} \subset \text{planes} \subset \mathbb{C}^4 \}$, $\mathbb{M}$ denotes the Grassmannian of $\{ \text{planes} \subset \mathbb{C}^4 \}$, and the maps $\mu$ and $\nu$ are the natural projections. Frenkel and Jardim have obtained a remarkable quantization of this diagram in which both the spaces $\mathbb{M}$ and $\mathbb{F}$ are replaced by noncommutative spaces $\mathbb{M}_q$ and $\mathbb{F}_q$ respectively, but the space $\mathbb{P}$ remains undeformed,

$$
\begin{array}{c}
\mathbb{M}_q \\
\mu \\
\mathbb{F}_q \\
\nu \\
\mathbb{P}
\end{array}
\begin{array}{c}
\mu \\
\nu \\
\mathbb{F}_q \\
\mathbb{M}_q
\end{array}
$$

Thus, while the actual space time $\mathbb{M}$ has been quantized, the space of light rays $\mathbb{P}$ (i.e., twistor space) remains unchanged, perfectly in line with Penrose’s philosophy that the space of light rays is more fundamental that points in space-time!

Deformation of commutative geometries have various interesting starting points, including A. Connes’s functional analytic approach and M. Artin’s algebroid-geometric approach. The deformation of space-time considered by I. Frenkel and M. Jardim is based on another remarkable deformation theory, namely the theory of quantum groups. While the theory of compact quantum groups has been well-studied from the point of view of Connes’s theory, the space-time which we
consider is based on the theory of non-compact quantum groups, the functional analytic treatment
of which is still nascent. Thus, in this paper, we shall regard our noncommutative spaces from the
algebro-geometric point of view, hoping to return to the same subject from the seemingly richer
functional analytic point of view at a later time.

From the point of view of noncommutative algebraic geometry, a noncommutative space $X$

is really a category, modeled on the category of coherent sheaves on a variety. Following A.
Grothendieck, it was well understood that many of the standard constructions that one performs
in the study of an algebraic variety can be understood purely in terms of the category of sheaves
on this same variety. This philosophy, while rendered somewhat obscure by theory of Topoi,
might certainly trace among its many concrete origins Serre’s explicit description of the category
of coherent sheaves on a projective variety in terms on modules over the graded coordinate ring
of the variety. The standard approach to noncommutative projective algebraic geometry is then to
begin with a noncommutative graded ring (perhaps satisfying some special growth conditions) and
define a noncommutative space as (a quotient of the) category of graded modules over this variety.
Indeed, we shall adopt this point of view and using it illustrate how one can transform classical
data on $\mathbb{P}$ to noncommutative data on $M_q$ using the quantum double fibration diagram above.

Instantons are defined by certain certain differential equations on Minkowski space, and Ward’s
insight was to use the Penrose’s twistor methods to translate this differential constraint into a purely
holomorphic constraint on the related twistor space. So, prior to discussing instantons on quantum
Minkowski space, we shall need to understand how to formulate and assemble differential information
on such spaces. Fortunately, the theory of quantum groups and especially the notion of an $R$-matrix provide an adequate replacement for the ordinary differential calculus. The definition
of instantons as vector bundles with anti-self-dual curvature is then straightforward. In addition
to instantons, Penrose has shown that solutions to many of the classical equations of mathematical
physics (e.g. Maxwell, Dirac, Einstein) may be reinterpreted as holomorphic information on
twistor space. We believe that the quantum analogues of these equations shall also admit such a
reinterpretation using the techniques we introduce in this note, and we hope to come back to this
question in the future.

The plan of this paper is as follows: in section 2, we set down some basic definitions of
noncommutative geometry; following this, we introduce in section 3 algebras which will play
the role of a quantum flag variety and quantum Minkowski space and fit them into the quantum
double fibration described above; in section 4, we study the quantum differential calculus; and
in section 5, we prove a quantum analogue of the Ward correspondence; in section 6, we show
how the data of classical complex instanton bundle satisfied the hypothesis of our quantum Ward
correspondence and hence encodes a quantum instanton; and finally, we conclude in section 7 with
some remarks about planned future directions.

2. Recollections from Noncommutative Algebraic Geometry

2.1. Notations. Let $k$ be a noncommutative Noetherian $\mathbb{C}$-algebra with unit, and let $A = \oplus_{i \geq 0} A_i$ be
a graded Noetherian $k$-algebra. We shall define $\text{mod}(A)$ to be the category of finitely generated
right $A$-modules, $\text{gr}(A)$ to be the category of finitely generated graded right $A$-modules, and $\text{tors}(A)$
to be subcategory of $\text{gr}(A)$ consisting of finite dimensional graded $A$-modules. Finally, we denote
by $\text{qgr}(A) = \text{gr}(A)/\text{tors}(A)$ to be the quotient category, whose objects are objects of $\text{gr}(A)$ and
morphisms are given, for $M, N \in \text{qgr}(A)$ corresponding to $M, N \in \text{gr}(A)$, by

$$\text{Hom}_{\text{qgr}(A)}(\bar{M}, \bar{N}) = \lim_{\mathcal{M}'} \text{Hom}_{\text{gr}(A)}(M', N)$$
for \( M/M' \in \text{tors}(A) \). The natural projection shall be denoted \( \pi : \gr(A) \to \qgr(A) \), and for any \( M \in \gr(A) \) or \( \qgr(A) \) we shall denote by \( M_n \) the set of homogeneous elements of degree \( n \). For every \( n \in \mathbb{Z} \), there exists a shift functor \( [n] : \gr(A) \to \gr(A) \) which sends \( M \mapsto M[n] \), where \( M[n]_i = M_{n+i} \). This functor then descends to a functor, which we also denote, \( [n] : \qgr(A) \to \qgr(A) \).

2.2. **Serre’s Theorem.** We recall here the fundamental theorem of Serre which can really be seen as the starting point of noncommutative projective algebraic geometry. In this section, all objects are commutative as we are working in the framework of classical algebraic geometry.

Let \( X \) be an affine algebraic variety, and let \( \mathcal{E} \) be a vector bundle on \( X \). Then there exists a map,

\[
\pi : \mathbb{P}(\mathcal{E}) \to X
\]

where \( \mathbb{P}(\mathcal{E}) = \text{Proj}(\oplus_{d \geq 0} \text{Sym}^d(\mathcal{E})) \). Furthermore, we have on \( \mathbb{P}(\mathcal{E}) \) a natural ample line bundle \( \mathcal{O}(1) \). We shall denote by \( \mathcal{O}(n) = \mathcal{O}(1)^{\otimes n} \) and if \( \mathcal{F} \) is any sheaf on \( \mathbb{P}(\mathcal{E}) \), we write \( \mathcal{F}(n) \) for \( \mathcal{F} \otimes \mathcal{O}(n) \).

**Theorem 2.3** (Serre). With the notation as above, denote by \( \text{Coh}(\mathbb{P}(\mathcal{E})) \) the category of coherent sheaves on \( \mathbb{P}(\mathcal{E}) \). Let \( R = \oplus_{d \geq 0} \Gamma(\mathbb{P}(\mathcal{E}), \mathcal{O}(d)) \) be the graded coordinate algebra. Then the functor,

\[
\Gamma_* : \text{Coh}(\mathbb{P}(\mathcal{E})) \to \text{qgr}(R)
\]

defined by \( \Gamma_*(\mathcal{F}) = \oplus_{d \geq 0} \Gamma(\mathbb{P}(\mathcal{E}), \mathcal{F}(d)) \) is an equivalence of categories.

**Remark 2.4.** Suppose \( X = \text{Spec}(A) \), and let \( \mathcal{E} = \mathcal{O}_X \). Then \( \mathbb{P}(\mathcal{E}) = X \) and the so \( \text{Coh}(\mathbb{P}(\mathcal{E})) = \text{Coh}(X) \mod (A) \) and so the above theorem actually contains affine algebraic geometry as a special case.

2.5. **Noncommutative Spaces.** It was realized by Artin (Manin, and probably others...) that a profitable setting for noncommutative algebraic geometry is furnished by turning Serre’s theorem into a definition of a noncommutative projective variety.

**Definition 2.6.** Let \( A = \oplus_{j \geq 0} A_j \) be a graded \( k \)-algebra, where as before \( k \) denotes a noncommutative Noetherian \( \mathbb{C} \)-algebra with unit. Then we shall define the noncommutative space associated to \( A \), denoted either \( \text{Coh}(\text{Proj}A) \) or \( \text{Proj}(A) \), as the category \( \text{qgr}(A) \). In the case that \( A = k \) is affine, we define a noncommutative affine space, still denoted by \( \text{Coh}(A) \) or \( \text{Coh}(\text{Spec}A) \), as the category \( \text{mod} \ A \).

The classical constructions of pullback and pushforward can be naturally imported into the noncommutative framework. Let \( X = \text{Proj}(A), Y = \text{Proj}(B) \) be noncommutative spaces together with a map \( Y \xrightarrow{f} X \) defined by a map of \( k \)-algebras \( f : A \to B \).

**Definition 2.7.** Let \( \mathcal{F} \in \text{Coh}(X) \) corresponds to \( F \in \gr(A) \) via \( \pi(F) = \mathcal{F} \). Then we define the pullback \( f^*(\mathcal{F}) \in \text{Coh}(Y) \) via

\[
f^*(\mathcal{F}) := \pi(F \otimes_A B).
\]

This construction has an adjoint \( f_* : \text{Coh}(Y) \to \text{Coh}(X) \) defined as follows: let \( \mathcal{E} \in \text{Coh}(Y) \), and choose a representative \( E \in \gr(B) \) for \( \mathcal{E} \). Then we may define

\[
f_*(\mathcal{E}) := \pi(\text{Hom}_{\text{qgr}(B)}(B, E)).
\]

The right \( A \)-module structure on \( f_*(\mathcal{E}) \) is given by

\[
(\varphi \cdot a)(b) = \varphi(a \cdot b), \quad \text{where} \quad \varphi \in \text{Hom}_{\text{qgr}(B)}(B,E), b \in B, a \in A
\]
Remark 2.8. If \( L : E \to F \) is only a \( \mathbb{C} \)-linear map of \( A \)-modules, we may still apply \( f_* \) to obtain a \( \mathbb{C} \)-linear map
\[
 f_*(L) : f_*(E) \to f_*(F).
\]
This shall be important for us, as we shall often be interested in pushing forward differentials.

The usual projection formula continues to hold with the appropriate modification of hypotheses. We state it here only in the context in which we shall need it.

**Lemma 2.9 (Projection Formula).** Let \( \mathcal{E} \in \text{Coh}(Y) \). Then we have,
\[
 f_*(f^* \mathcal{E} \otimes_B B[n]) = \mathcal{E} \otimes_A f_*(B[n])
\]
where \( B[n] \) is the \( B \)-bimodule obtained by shifting \( B \), and \( f_*(B[n]) \) inherits the natural \( A \)-bimodule structure.

3. **Quantum Twistor Geometry**

3.1. **Classical Double Fibration.** Let \( \mathbb{T} \) denote a 4-dimensional complex vector space, sometimes called the vector space of twistors.

3.2. **Flag Varieties.** Fixing \( \mathbb{T} \) as above, we define the following algebraic varieties,
\[
 \mathcal{F} := \text{Fl}(1,2; \mathbb{T})
\]
the flag variety parametrizing \((l, P)\) where \( l \subset \mathbb{T} \) is a line, \( P \subset \mathbb{T} \) is plane, and \( l \subset P \);
\[
 \mathbb{M} := \text{Fl}(2, \mathbb{T}) = \text{Gr}(2, \mathbb{T})
\]
the Grassmannian of planes in \( \mathbb{T} \); and
\[
 \mathbb{P} := \text{Fl}(1, \mathbb{T}) = \mathbb{P}^3(\mathbb{C})
\]
the projective space of lines in \( \mathbb{T} \).

3.3. **Double Fibration.** These three spaces are related by the following double fibration diagram, where the maps \( \mu \) and \( \nu \) are projections defined by \( \mu(l, P) = l \) and \( \mu(l, P) = P \), for \((l \subset P) \in \mathcal{F}\),
\[
 \begin{array}{ccc}
 \mathcal{F} & \xrightarrow{\mu} & \mathbb{M} \\
 \downarrow{\nu} & & \downarrow{\nu} \\
 \mathbb{P} & & \mathbb{M}
 \end{array}
\]

3.4. **Local Coordinates.** If \( a, b \in \mathbb{T} \) are two linearly independent vectors, then we denote the two dimensional plane spanned by \( a \) and \( b \) by \([a, b] \subset \mathbb{T}\), and we use the same notation to refer to the corresponding element of \( \mathbb{M} \).

Let \( \phi_I : \mathbb{H}_\mathbb{C} \to \mathbb{M} \) be the coordinate map,
\[
 \begin{pmatrix}
 z_{11} & z_{12} \\
 z_{21} & z_{22}
 \end{pmatrix} \mapsto \begin{pmatrix}
 z_{11} \\
 z_{12} \\
 z_{21} \\
 1
 \end{pmatrix}, \quad \begin{pmatrix}
 z_{22} \\
 0 \\
 0 \\
 1
 \end{pmatrix}.
\]
It is easy to see that \( \phi_I \) is an isomorphism, and we denote by \( \mathbb{M}^I = \phi_I(\mathbb{H}_\mathbb{C}) \). Writing \( \mathbb{F}^I = \nu^{-1}(\mathbb{M}^I) \) and \( \mathbb{P}^I = \mu(\mathbb{F}^I) \), we have the following local twistor diagram.
Lemma 3.5. In the notation above, $\mathbb{F}^l \cong \mathbb{M}^l \times \mathbb{P}^l(\mathbb{C})$. If we take coordinates $(z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}, [x_0 : x_1])$ for $\mathbb{F}^l$, then the double fibration diagram may be expressed in local coordinates as follow,

\[
\begin{pmatrix} z_{11}x_0 + z_{12}x_1 : z_{21}x_0 + z_{22}x_1 : x_0 : x_1 \end{pmatrix} \in \mathbb{F}^l
\]

\[
\begin{pmatrix} z \end{pmatrix} \in \mathbb{M}^l
\]

Proof. By definition $\mathbb{F}^l$ consists of all pairs $(|l|, |p|)$ where $|l|$ is a one dimensional subspace of the plane spanned by vectors $\vec{a} = \langle z_{11}, z_{21}, 1, 0 \rangle$ and $\vec{b} = \langle z_{12}, z_{22}, 0, 1 \rangle$. Consider the map,

\[
(z, [x_0 : x_1]) \mapsto ([x_0a + x_1b], [\vec{a}, \vec{b}]).
\]

It is easy to check that this is the required isomorphism from $\mathbb{M} \times \mathbb{P}^l(\mathbb{C}) \to \mathbb{F}^l$. The expression for $\mu$ and $\nu$ is now obvious. \hfill \square

3.6. Quantum Double Fibration. In this section, we introduce the deformation of the diagram $\mathbb{?}$ above which keeps $\mathbb{P}^l$ commutative, but replaces $\mathbb{M}^l$ and $\mathbb{F}^l$ with noncommutative spaces.

3.7. Quantum Flag Varieties. — We first note the following simple consequence of Lemma [?].

Lemma 3.8. Let us define the graded commutative algebra

\[
\mathcal{P}^l = \mathbb{C}[z_{11}x_1 + z_{12}x_2, z_{21}x_1 + z_{22}x_2, x_1, x_2] \subset \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}, x_1, x_2]
\]

where $z_{ij}$ is given degree 0 and $x_i$ have degree 1. Then we have

\[
\mathbb{P}^l = \text{Proj}(\mathcal{P}^l).
\]

Definition 3.9. Let $\mathcal{M}^I_q$ be the unital associative algebra over $\mathbb{C}(q)$ ($q$ is an indeterminate) generated by $z_{11}, z_{12}, z_{21}, z_{22}$ together with the following six relations,

\[
[z_{11}, z_{12}] = 0, \quad [z_{21}, z_{22}] = 0, \quad [z_{11}, z_{22}] = [z_{12}, z_{21}]
\]

\[
z_{11}z_{21} = q^{-2}z_{21}z_{12} \quad z_{12}z_{21} = q^{-2}z_{21}z_{12}, \quad z_{12}z_{22} = q^{-2}z_{22}z_{12}
\]

We define the quantum affine Minkowski space to be $\mathbb{M}^l_q = \text{Proj}(\mathcal{M}^I_q) = \text{mod} \mathcal{M}^I_q$

Finally, we shall define the quantum analogue of the local flag variety $\mathbb{F}^l$.

Definition 3.10. Let $\mathcal{F}^l_q$ be the unital $\mathcal{M}_q^I$-algebra generated by $x_1, x_2$ subject to the following relations,

\[
x_1z_{11} = q^{-2}z_{11}x_1, \quad x_1z_{12} = q^{-2}z_{12}x_1, \quad x_1z_{21} = z_{21}x_1 + (q^{-2} - 1)z_{11}x_2
\]

\[
[x_1, x_2] = 0, \quad x_1z_{22} = z_{22}x_1 + (q^{-2} - 1)z_{12}x_2, \quad [x_2, x_{ij}] = 0
\]

We define the quantum local flag variety to be $\mathcal{F}^l_q = \text{Proj}(\mathcal{F}^l_q)$

Remark 3.11. As a $\mathbb{C}(q)$-vector space, we have $\mathcal{F}^l_q = \mathcal{M}^I_q[x_1, x_2]$. 

5
3.12. **Quantum Double Fibration.** — Our analogue of the classical double fibration will be the following map of algebras,

\[
\begin{array}{ccc}
\mathcal{P}_q & \xrightarrow{\mu} & \mathcal{M}_q \\
\xrightarrow{\nu} & & \\
\mathcal{P}_q & \xrightarrow{\nu} & \mathcal{M}_q
\end{array}
\]

where the maps \( \mu \) and \( \nu \) are both the obvious inclusions.

3.13. **Alternative Presentation.** — Though we have not mentioned explicitly the quantum groups behind the definitions above, the reader can find such definitions in [?], or also see the appendix.

4. **Algebras of Exterior Differential Forms on Quantum Spaces**

Let \( X \) be a smooth commutative algebraic variety. Then we would like to provide an analogue of the de Rahm complex on \( \mathcal{M}_q \) for each of our quantum spaces \( \mathcal{P}_q \) and \( \mathcal{M}_q \) in this section. We begin with \( \mathcal{M}_q \) as the differential calculus on \( \mathcal{P}_q \) built up from than on \( \mathcal{M}_q \).

**Definition 4.2.** We define the de Rahm complex on \( \mathcal{M}_q \) as follows,

1. \( \Omega^0_{\mathcal{M}_q} = \mathcal{M}_q \)
2. \( \Omega^1_{\mathcal{M}_q} \) is the \( \mathcal{M}_q \)-bimodule generated by \( dz_{ij} \) subject to relations summarized in the following table, (note: the multiplication should be read from row to column, so for example, \( dz_{11}z_{11} = q^{-2}z_{11}dz_{11} \))

<table>
<thead>
<tr>
<th>dz_{11}</th>
<th>q^{-2}z_{11}dz_{11}</th>
<th>qz_{12}dz_{12}</th>
<th>q^{-2}z_{12}dz_{12} + (q^{-2} - 1)z_{11}dz_{21} + (q^{-2} - 1)z_{21}dz_{12} + (q^{-2} - 1)z_{21}dz_{12} + \cdots</th>
</tr>
</thead>
<tbody>
<tr>
<td>dz_{12}</td>
<td>q^{-2}z_{11}dz_{12}</td>
<td>q^{-2}z_{12}dz_{12}</td>
<td>q^{-2}z_{12}dz_{12} + (q^{-2} - 1)z_{11}dz_{21} + (q^{-2} - 1)z_{21}dz_{12} + (q^{-2} - 1)z_{21}dz_{12} + \cdots</td>
</tr>
<tr>
<td>dz_{21}</td>
<td>qz_{12}dz_{21}</td>
<td>qz_{12}dz_{21} - (q^2 - 1)z_{21}dz_{22}</td>
<td>qz_{21}dz_{21}</td>
</tr>
<tr>
<td>dz_{22}</td>
<td>qz_{12}dz_{22}</td>
<td>qz_{12}dz_{22}</td>
<td>qz_{21}dz_{22}</td>
</tr>
</tbody>
</table>

3. \( \Omega^2_{\mathcal{M}_q} \) is the \( \mathcal{M}_q \)-bimodule generated by symbols \( dz_{ij} \wedge dz_{kl} \) subject following relations summarized in the following table,

<table>
<thead>
<tr>
<th>dz_{11}</th>
<th>dz_{12}</th>
<th>dz_{21}</th>
<th>dz_{22}</th>
</tr>
</thead>
<tbody>
<tr>
<td>dz_{11}</td>
<td>0</td>
<td>-q^{-2}dz_{11}dz_{11}</td>
<td>-dz_{21}dz_{11}</td>
</tr>
<tr>
<td>dz_{12}</td>
<td>0</td>
<td>0</td>
<td>-q^{-2}(dz_{21}dz_{12} + (q^{-2} - 1)dz_{11}dz_{22})</td>
</tr>
<tr>
<td>dz_{21}</td>
<td>*</td>
<td>*</td>
<td>0</td>
</tr>
<tr>
<td>dz_{22}</td>
<td>*</td>
<td>*</td>
<td>0</td>
</tr>
</tbody>
</table>

The differentials \( d, d^1 \) are then defined by the properties,

\[
d(z_{ij}) = dz_{ij} \quad d(ab) = dab + adb, \quad \text{for } a, b \in \mathcal{M}_q \\
d^1(fdz_{ij}) = dfdz_{ij}, \quad d^1(dz_{ij}f) = -dz_{ij}, df \quad \text{for } f \in \mathcal{M}_q
\]
Remark 4.3. It is easy to see that considered either as a left-module or as a right-module, the bimodules $\Omega_{\mathcal{M}_q^l}$ are free.

Furthermore, we define partial derivatives $\partial_{ij}$ using the formula,

$$df = \sum_{i,j} dz_{ij} \partial_{ij} f \quad \text{for} \quad f \in \mathcal{M}_q^l.$$ 

An alternative presentation of this differential calculus in the more compact R-matrix formulation (from which it was originally derived) is given in the appendix.

Lemma 4.4. We have a decomposition of $(\mathcal{M}_q^l$)-modules, $\Omega_{\mathcal{M}_q^l}^2 = \Omega_{\mathcal{M}_q^l}^{2,+} \oplus \Omega_{\mathcal{M}_q^l}^{2,-}$, where

$$\Omega_{\mathcal{M}_q^l}^{2,+} = \mathcal{M}_q^l (dz_{11} \wedge dz_{12}, dz_{21} \wedge dz_{22}, dz_{11} \wedge dz_{22} - dz_{12} \wedge dz_{21})$$
and

$$\Omega_{\mathcal{M}_q^l}^{2,-} = \mathcal{M}_q^l (dz_{11} \wedge dz_{21}, dz_{12} \wedge dz_{22}, dz_{11} \wedge dz_{22} + q^2 dz_{12} \wedge dz_{21}).$$

We refer to these as the self-dual and anti-self dual two forms on quantum Minkowski space.

Note: this is in contrast to the formulas give in [?]. Which is correct?

4.5. Differential Forms on $\mathcal{F}_q^l$. — We shall only need to define the following part of the de Rahm complex for our purposes,

$$d : \mathcal{F}_q^l \to \Omega_{\mathcal{F}_q^l}^1.$$ 

Definition 4.6. We define $\widetilde{\Omega}_{\mathcal{F}_q^l}$ to be the graded $\mathcal{F}_q^l$-module generated by symbols $dz_{ij}, dx_1, dx_2$ subject to the following relations. [insert relations] We then define $\Omega_{\mathcal{F}_q^l}$ to be the submodule of $\widetilde{\Omega}_{\mathcal{F}_q^l}$ generated by $dz_{ij}$ and $x_1 dx_2 - x_2 dx_1$.

As before, we may define partial derivatives $\partial_{ij} = \frac{\partial}{\partial z_{ij}}$ and $\partial_k = \frac{\partial}{\partial x_k}$ through relation

$$df = \sum_{ij} dz_{ij} \partial_{ij}(f) + \sum_k dx_k \partial_k(f),$$
for $f = f(z_{ij}, x_k) \in \mathcal{F}_q^l$. Note that it is clear from the form of the above relations that if $f \in (\mathcal{F}_q^l)_0 = \mathcal{M}_q^l$, then the two definitions of $\partial_{ij} f$ coincide.

5. QUANTUM WARD CORRESPONDENCE

The classical Ward correspondence establishes a bijective correspondence between holomorphic $GL(n, \mathbb{C})$-instantons on $\mathbb{M}^l$ and locally free sheaves on $\mathbb{P}^l$ with a certain triviality (called, $\mathbb{M}^l$-triviality) condition. Our aim in this section will be to establish one half of this correspondence, but first we review the theory of instantons on $\mathbb{M}^l_q$.

5.1. Vector Bundles on Noncommutative Spaces. Recall that the classical Serre-Swan theorem establishes an equivalence of categories between the category of vector bundles over an affine space Spec$(A)$ and the category of projective $A$-modules. Thus, we make the following.

Definition 5.2. A vector bundle over the noncommutative affine space $\mathbb{M}^l_q$ is a projective right $\mathcal{M}_q^l$-module.

The analogue of locally free sheaf on a projective variety is provided by the following homological criterion, easily seen to reduce to the usual notion on a commutative space.
Definition 5.3. Let \( X = \text{Proj}(A) \). Then a coherent sheaf \( \mathcal{F} = \pi(F) \in \text{Coh}(X) \) is said to be \textit{locally free} if it satisfies the following homological condition,

\[
\text{Ext}^p(A, F) = 0 \quad \text{for} \quad p \geq 1 .
\]

5.4. Instantons on \( \mathbb{M}_q^l \). Having introduced both the notion of vector bundle and differential calculus, we may now proceed to define instantons. We define a \textit{connection} \( \nabla \) on a projective right \( \mathbb{M}_q^l \)-module \( E \) to be a \( C \)-linear map

\[
\nabla : E \to E \otimes_{\mathbb{M}_q^l} \Omega_{\mathbb{M}_q^l}^1
\]

which satisfies the following condition,

\[
\nabla(\sigma f) = \nabla(\sigma)f + \sigma df.
\]

We may uniquely extend \( \nabla \) to a map, also denoted \( \nabla \),

\[
\nabla : E \otimes_{\mathbb{M}_q^l} \Omega_{\mathbb{M}_q^l}^1 \to E \otimes_{\mathbb{M}_q^l} \Omega_{\mathbb{M}_q^l}^2
\]

which satisfies

\[
\nabla(\sigma \otimes \omega) = \nabla(\sigma)\omega + \sigma \nabla(\omega) \quad \text{for} \quad \sigma \in E, \omega \in \Omega_{\mathbb{M}_q^l}^1 .
\]

We define the \textit{curvature} of \((E, \nabla)\) to be the \( \mathbb{M}_q^l \)-linear map

\[
\nabla^2 = \nabla \circ \nabla : E \otimes_{\mathbb{M}_q^l} \Omega_{\mathbb{M}_q^l}^0 \to E \otimes_{\mathbb{M}_q^l} \Omega_{\mathbb{M}_q^l}^2
\]

The splitting of \( \Omega_{\mathbb{M}_q^l}^2 \) described above allows us to define an \textit{instanton} as a vector bundle \( E \) on \( \mathbb{M}_q^l \) equipped with a connection \( \nabla \) whose curvature satisfies the \textit{anti-self-duality condition},

\[
\nabla^2 = \nabla \circ \nabla : E \otimes_{\mathbb{M}_q^l} \Omega_{\mathbb{M}_q^l}^0 \to E \otimes_{\mathbb{M}_q^l} \Omega_{\mathbb{M}_q^l}^1 .
\]

5.5. Grassmann Connections. — A particularly important class of connections is provided by the following construction. Let \( P : (\mathbb{M}_q^l)^n \to (\mathbb{M}_q^l)^n \) be an \( \mathbb{M}_q^l \)-linear map such that \( P^2 = P \). Then \( E = \text{Im}(P) \) is a projective module, which can be equipped with the \textit{Grassman connection} defined as a composition,

\[
\nabla_P : E \hookrightarrow (\mathbb{M}_q^l)^n \xrightarrow{d} (\mathbb{M}_q^l)^n \otimes_{\mathbb{M}_q^l} \Omega_{\mathbb{M}_q^l}^1 \xrightarrow{P \otimes \text{id}} E \otimes_{\mathbb{M}_q^l} \Omega_{\mathbb{M}_q^l}^1
\]

The curvature of \( \nabla_P \) is easily verified to be given by the formulas,

\[
\nabla^2_P = PdPdP = dP(1 - P)dP = dPdPP.
\]

5.6. \( \mathbb{M}_q^l \)-Triviality. In all the follows, the following condition on a locally free sheaf \( \mathcal{E} \) on \( \mathbb{P}^l \) will be crucial.

Definition 5.7. A sheaf \( \mathcal{E} \) will be called \( \mathbb{M}_q^l \)-\textit{trivial} if

1. \( \mu^* \mathcal{E} \) is locally free;
2. the natural map \( \nu^* \nu_* \mu^* \mathcal{E} \to \mu^* \mathcal{E} \) is an isomorphism.

In section ?, we shall show how to construct a class of \( \mathbb{M}_q^l \)-trivial sheaves.
5.8. **Main Result.** We are now finally ready to state our main result,

**Theorem 5.9** (Quantum Ward Correspondence). Let $\mathcal{E}$ be a $\mathbb{M}^l_q$-trivial sheaf on $\mathbb{P}^l$. Then $\mathcal{E} = v_\star \mu^* \mathcal{E}$ is a projective module on $\mathbb{M}^l_q$ which admits a connection $\nabla$ whose curvature is anti-self-dual.

Since $\mu^* \mathcal{E}$ is locally free by assumption and the map $v$ is a projection, it is clear that $\mathcal{E}$ is a projective module. In order to obtain an anti-self-dual connection on $\mathcal{E}$ we proceed by "pushing forward the relative connection from the relative de Rahm complex." Rather than develop the full machinery of relative differentials in the noncommutative case, we shall prefer to proceed by a more ad-hoc method. Thus, we shall proceed in the following three steps:

5.10. **Step 1: Pushing forward the relative de Rahm complex.** —

**Lemma 5.11.** Let us define the maps $\beta$ and $\gamma$ in terms of the partial derivatives introduced above,

$$
\beta : f \mapsto \begin{pmatrix}
-x_1 \partial_{12} + x_2 \partial_{11} \\
-x_1 \partial_{22} + x_2 \partial_{21}
\end{pmatrix}f
$$

and

$$
\gamma : \begin{pmatrix}
g_1 \\
g_2
\end{pmatrix} \mapsto (x_1 \partial_{22} - x_2 \partial_{21})g_1 + (x_2 \partial_{11} - x_1 \partial_{12})g_2
$$

Then, there exists isomorphisms of right $\mathcal{M}^l_q$-modules $\delta_1, \delta_2, \delta_3$ such that we have a commutative diagram,

$$
\begin{array}{cccc}
\mathcal{F}^l_q & \rightarrow & \mathcal{F}^l_q(1) \oplus \mathcal{F}^l_q(1) & \rightarrow & \mathcal{F}^l_q(2) \\
\downarrow v_\star & & \downarrow v_\star & & \downarrow v_\star \\
\mathcal{M}^l_q & \rightarrow & (\mathcal{M}^l_q)^4 & \rightarrow & (\mathcal{M}^l_q)^3 \\
\downarrow \delta_1 & & \downarrow \delta_2 & & \downarrow \delta_3 \\
\mathcal{M}^l_q & \rightarrow & \Omega^1_{\mathbb{M}^l_q} & \rightarrow & \Omega^2_{\mathbb{M}^l_q, +}
\end{array}
$$

where $\pi_+ : \Omega^2_{\mathbb{M}^l_q} \rightarrow \Omega^2_{\mathbb{M}^l_q, +}$ is the projection onto the self-dual forms.

**Proof.** First, we note that in $v_\star(\mathcal{F}^l_q(1)) = \mathcal{F}^l_q(1) = \langle x_1, x_2 \rangle \cdot \mathcal{M}^l_q$ and $v_\star(\mathcal{F}^l_q(2)) = \langle x_1^2, x_2 x_1, x_2^2 \rangle \cdot \mathcal{M}^l_q$ by Lemma 5.8. Thus,

$$
v_\star(\mathcal{F}^l_q(1) \oplus \mathcal{F}^l_q(1)) = \langle x_1, x_2 \rangle \cdot \mathcal{M}^l_q \oplus \langle x_1, x_2 \rangle \cdot \mathcal{M}^l_q,
$$

and we shall refer to elements of this module as a column of vectors $\begin{pmatrix} x_1 r + x_2 s \\ x_1 t + x_2 u \end{pmatrix}$ where $r, s, t, u \in \mathcal{M}^l_q$. In these coordinates, we have the following description of $v_\star \beta :$ Let $f \in \mathcal{M}^l_q$, then

$$
\beta(f) = \begin{pmatrix}
-x_1 \partial_{12} f + x_2 \partial_{11} f \\
-x_1 \partial_{22} f + x_2 \partial_{21} f
\end{pmatrix}.
$$

Thus, if we define the maps, $\delta_1 = \text{Id}$ and $\delta_2$ via,

$$
\delta_2 \begin{pmatrix} x_1 r + x_2 s \\ x_1 t + x_2 u \end{pmatrix} \mapsto dz_{11} s - dz_{12} r + dz_{21} u - dz_{22} t
$$

then we can easily verify that the left half of the above diagram satisfies the conclusion of the claim.

As to the right half, we first obtain the following explicit expression for $\pi_+ \circ d$. In order to state the desired expression, let us recall that a basis of self-dual and anti-self-dual forms are given by
the following expressions,
\[
\begin{align*}
    a &= dz_{11} \land dz_{12} \\
    b &= dz_{21} \land dz_{22} \\
    c &= dz_{11} \land dz_{21} \\
    d &= dz_{12} \land dz_{22} \\
    \omega_+ &= dz_{11} \land dz_{22} - dz_{12} \land dz_{21} \\
    \omega_- &= dz_{11} \land dz_{22} + dz_{12} \land dz_{21}
\end{align*}
\]
Further, in these coordinates, we have
\[
dz_{11} \land dz_{22} = \frac{\omega_+ + \omega_-}{2} \quad \text{and} \quad dz_{12} \land dz_{21} = \frac{\omega_+ - \omega_-}{2}.
\]
Now, let \( \omega \in \Omega^1_q \), which we may write as
\[
\omega = dz_{11}A_{11} + dz_{12}A_{12} + dz_{21}A_{21} + dz_{22}A_{22},
\]
where \( A_{ij} \in \mathcal{M}_q^1 \). Then, since both facts that \( d^2 = 0 \) and \( d \) is a derivation still remains valid in the quantum case, we may write,
\[
d\omega = (-)dz_{11}dA_{11} + dz_{12}dA_{12} + dz_{21}dA_{21} + dz_{22}dA_{22}.
\]
Recalling that \( df = \sum_{i,j=1}^2 dz_{ij} \partial_{ij}f \), we obtain the following formula,
\[
d\omega = a(\partial_{11}A_{12} - \partial_{12}A_{11}) + b(\partial_{21}A_{22} - \partial_{22}A_{21}) + c(\partial_{11}A_{21} - \partial_{21}A_{11}) + d(\partial_{12}A_{22} - \partial_{22}A_{12}) + \frac{\omega_+}{2}(-\partial_{22}A_{11} + \partial_{21}A_{12} - \partial_{12}A_{21} + \partial_{11}A_{22}) + \frac{\omega_-}{2}((-\partial_{22}A_{11} - \partial_{21}A_{12} + \partial_{12}A_{21} + \partial_{11}A_{22})
\]
Hence, we may decompose \( d: \Omega^1_q \to \Omega^2_q \) into \( d = d_+ + d_- \), where \( d = \pi_+ \circ d \) and \( \pi_- \circ d \) are the self-dual and anti-self-dual parts of \( d \), by the following expressions,
\[
d_+ \omega = a(\partial_{11}A_{12} - \partial_{12}A_{11}) + b(\partial_{21}A_{22} - \partial_{22}A_{21}) + \frac{\omega_+}{2}(-\partial_{22}A_{11} + \partial_{21}A_{12} - \partial_{12}A_{21} + \partial_{11}A_{22})
\]
and
\[
d_- \omega = b(\partial_{21}A_{22} - \partial_{22}A_{21}) + c(\partial_{11}A_{21} - \partial_{21}A_{11}) + \frac{\omega_-}{2}((-\partial_{22}A_{11} - \partial_{21}A_{12} + \partial_{12}A_{21} + \partial_{11}A_{22})
\]
Let us now define \( \delta_3 \) by the formulas
\[
x_1^2f + x_1x_2g + x_2^2h \mapsto af + \omega_- g + \omega_+ bh.
\]
It is then clear that all the maps on the right hand side of the diagram above satisfy the desires conclusion. For example, we may verify the commutativity of the lower right hand square as follows,
\[
[\text{insert verification}]
\]
\[\Box\]

**5.12. Step 2:** If \( \hat{\mathcal{E}} \) is \( \mathcal{M}_q^1 \)-trivial, then setting \( \mathcal{E}' := \mu^* \hat{\mathcal{E}} \), we have \( \mathcal{E}' = \nu^* \mathcal{E} \), and so by the projection formula, we find that
\[
V_n(\mathcal{E}' \otimes \mathcal{F}_q(n)) = \mathcal{E} \otimes \mathcal{F}_q V_n(\mathcal{F}_q(n))
\]
for \( n = 1, 2 \).

**5.13. Step 3:** Finally we establish the twisted version of the above lemma. As above, let \( \xi' = \nu^* \xi \). Since \( \xi \) is projective, we may find \( \mathcal{Q} \) such that \( \mathcal{E} \oplus \mathcal{Q} = (\mathcal{M}_q^1)^n \). We shall denote by \( P \) the natural projection \( \mathcal{E} \oplus \mathcal{Q} \to \mathcal{E} \). Since \( \mathcal{M}_q^1 \to \mathcal{F}_q^1 \) is flat, we may pull back the above
considerations to $\mathcal{P}_q^I$ : i.e., there exists $\mathcal{Q}'$ such that $\mathcal{E}^I \oplus \mathcal{Q}' = (\mathcal{P}_q^I)^n$ and we denote by $P'$ the natural projection from $\mathcal{E}^I \oplus \mathcal{Q}' = (\mathcal{P}_q^I)^n$ to $\mathcal{E}^I$.

We may then define a map $\beta_{\mathcal{E}^I}$ as the composition

$$
\begin{array}{c}
\mathcal{E}^I \\
\downarrow i \\
\mathcal{E}^I (1)^2 \\
\end{array}
\xrightarrow{\beta_{\mathcal{E}^I}}
\begin{array}{c}
(\mathcal{P}_q^I(1) \oplus \mathcal{P}_q^I(1))^n \\
\downarrow P' \\
\mathcal{E}^I (1)^2 \\
\end{array}
$$

where $i$ denotes the natural inclusion and $P'$ (by abuse of notation) denotes the natural projection obtained from $P'$. Similarly, we may define $\gamma_{\mathcal{E}^I}$ as the composition,

$$
\begin{array}{c}
(\mathcal{F}_q^I(1) \oplus \mathcal{F}_q^I(1))^n \\
\downarrow i \\
\mathcal{E}^I (1)^2 \\
\end{array}
\xrightarrow{\gamma_{\mathcal{E}^I}}
\begin{array}{c}
(\mathcal{F}_q^I(2))^n \\
\downarrow P' \\
\mathcal{E}^I (1)^2 \\
\end{array}
$$

with the same notational conventions.

It is then clear that the maps above fit into a commutative diagram

$$
\begin{array}{c}
\mathcal{E}^I \\
\downarrow i \\
\mathcal{E}^I (1)^2 \\
\end{array}
\xrightarrow{\beta_{\mathcal{E}^I}}
\begin{array}{c}
(\mathcal{F}_q^I(1) \oplus \mathcal{F}_q^I(1))^n \\
\downarrow i \\
(\mathcal{F}_q^I(2))^n \\
\end{array}
\xrightarrow{\gamma_{\mathcal{E}^I}}
\begin{array}{c}
\mathcal{E}^I (1)^2 \\
\downarrow i \\
\mathcal{E}^I (1)^2 \\
\end{array}
$$

We may now pushforward the above diagram to obtain the following result.

**Lemma 5.14.** We have the following commutative diagram,

$$
\begin{array}{c}
\mathcal{E}^I \\
\downarrow \nu_\ast \\
\mathcal{E}^I (1)^2 \\
\end{array}
\xrightarrow{\beta_{\mathcal{E}^I}}
\begin{array}{c}
(\mathcal{F}_q^I(1) \oplus \mathcal{F}_q^I(1))^n \\
\downarrow \nu_\ast \\
(\mathcal{F}_q^I(2))^n \\
\end{array}
\xrightarrow{\gamma_{\mathcal{E}^I}}
\begin{array}{c}
\mathcal{E}^I (1)^2 \\
\downarrow \nu_\ast \\
\mathcal{E}^I (1)^2 \\
\end{array}
$$

where $\alpha$ is a connection whose curvature has vanishing self-dual part.

**Proof.** Let us first note that we have the following commutative diagram,
From a careful study of the above diagram, we can verify that \(\alpha\) is the Grassmann connection associated to the projection \(P\). Furthermore, the extension of the Grassmann connection to a map \(\mathcal{E} \otimes \Omega^1 \to \mathcal{E} \otimes \Omega^2\) is given by \(Pd = P(d_+ + d_-)\), and it equally easy to verify that \(\beta = Pd_+\). But since we have \(\beta \circ \alpha = 0\), the curvature of \(\alpha\) must only have anti-self-dual part.

6. CLASSICAL INSTANTONS ARE QUANTUM INSTANTONS

In the previous section, we have shown how to obtain a quantum instanton starting from the data of an \(\mathbb{M}_q\)-trivial sheaf on \(\mathbb{P}^d\). We now show how to obtain a family of such \(\mathbb{M}_q\)-trivial bundles starting from a certain class of classical objects on \(\mathbb{P}^d\) which we now define.

**Definition 6.1.** A locally free sheaf \(\hat{\mathcal{E}}\) on \(\mathbb{P}^d\) is said to be \(\mathbb{M}_q\)-free, or sometimes called a complex instanton bundle, if the map \(\nu^* \mu^* \hat{\mathcal{E}} \to \mu^* \hat{\mathcal{E}}\) is an isomorphism.

**Theorem 6.2.** Let \(\hat{\mathcal{E}}\) be \(\mathbb{M}_q\)-free. Then it is also \(\mathbb{M}_q\)-free.

**Proof.** We would like to show that the map
\[
\Psi : \text{Hom}_{qgr}(\mathcal{F}_q^1)(\mathcal{F}_q^1, \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1) \otimes_{\mathbb{M}_q} \mathcal{F}_q^1 \to \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1
\]
\[
\varphi \otimes f \mapsto \varphi(f)
\]
is an isomorphism in \(qgr(\mathcal{F}_q^1)\), where \(\varphi \in \text{Hom}_{qgr}(\mathcal{F}_q^1)(\mathcal{F}_q^1, \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1), f \in \mathcal{F}_q^1\), and we understand that \(\varphi\) is only defined on \(f \in \mathcal{F}_q^1[N]\) for \(N\) sufficiently large.

Let us first show that the above map is surjective. This shall follow from the following two claims.

**Claim 6.3 (A).** There exists \(N\) sufficiently large and \(\sigma_1, \ldots, \sigma_r \in (\hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1)[N]\) such that \(\sigma_1, \ldots, \sigma_r\) generate \(\otimes_{M \geq N}(\hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1)[M]\).

**Claim 6.4 (B).** There exists \(N\) sufficiently large such that given \(\sigma \in (\hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1)[M]\) for \(M \geq N\), there exists \(\varphi \in \text{Hom}_{qgr}(\mathcal{F}_q^1)(\mathcal{F}_q^1, \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1)\) such that \(\sigma \in \text{Im}(\varphi)\).

We defer the proof of these two claims until later, but shall conclude surjectivity from them as follows: By claim A, choose generators \(\sigma_1, \ldots, \sigma_r\) and choose \(\varphi_1, \ldots, \varphi_r\) as in claim B satisfying \(\sigma_i \in \text{Im}(\varphi)\). Then if follows that \(\sigma_1, \ldots, \sigma_r \subset \text{Im}(\Psi)\) and so surjectivity in \(qgr(\mathcal{F}_q^1)\) follows.

Now, we turn to injectivity of \(\Psi\): Suppose we have
\[
y = \varphi_1 f_1 + \cdots \varphi_n f_n, \quad \text{where} \quad \varphi_i \in \text{Hom}_{qgr}(\mathcal{F}_q^1)(\mathcal{F}_q^1, \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1), f_i \in \mathcal{F}_q^1\]
such that \(\Psi(y) = 0\). Using \(\mathbb{M}_q\)-linearity of the tensor product to rewrite this expression as
\[
y = \sum_i \xi_i \otimes x_1^{m_1} x_2^{m_2},
\]
where \(\xi_i \in \text{Hom}_{qgr}(\mathcal{F}_q^1)(\mathcal{F}_q^1, \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1)\). Now, injectivity follows from the following claims.

**Claim 6.5 (C).** Let \(\varphi \in \text{Hom}_{qgr}(\mathcal{F}_q^1)(\mathcal{F}_q^1, \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1)\). Then, we may factor \(\varphi = (q - 1)^k \psi\) where the reduction of \(\Psi\), i.e., the image of \(\varphi\) under the map
\[
\text{Hom}_{qgr}(\mathcal{F}_q^1)(\mathcal{F}_q^1, \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1) \to \text{Hom}_{qgr}(\mathcal{F}_q^1)(\mathcal{F}_q^1, \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1) \otimes_{\mathcal{F}_q^1} \mathcal{F}_q^1 = \text{Hom}_{qgr}(\mathcal{F}_q^1)(\mathcal{F}_q^1, \hat{\mathcal{E}} \otimes_{\mathbb{P}^d} \mathcal{F}_q^1)
\]
is non-zero.
Claim 6.6 (D). Let $\varphi \in \text{Hom}_{qgr(\mathcal{F} I_q)}(\mathcal{F} I_q, \hat{\mathcal{O}} \otimes_{\mathcal{F} I_q} \mathcal{F} I_q)$, and $f \in \mathcal{F} I_q$ such that $\varphi(f)$ is defined. Denote by $\overline{f}$ the reduction of $f$ under the map specialization map $q \to 1$ (i.e., $\mathcal{F} I_q \otimes_{\mathcal{F} I_q} \mathbb{F}^I \to \mathbb{F}^I$), and by $\overline{\varphi}$ as defined in the claim above. Then $\overline{\varphi}$ is defined on $\overline{f}$ and we have $\overline{\varphi}(\overline{f}) = \overline{\varphi(f)}$.

Assuming the above two claims, we may conclude the proof of injectivity as follows: By claim C, we may rewrite the expression for $y$ as

$$y = (q - 1) \sum_i \eta_i \otimes x_1^{n_i}x_2^{m_i} + \sum_j \eta_j \otimes x_1^{n_j}x_2^{m_j}$$

where $\overline{\eta}_k \neq 0$. Then, if we take the reduction of $\Psi(y)$ under the map $\hat{\mathcal{O}} \otimes_{\mathcal{F} I_q} \mathcal{F} I_q \to \hat{\mathcal{O}} \otimes_{\mathcal{F} I_q} \mathbb{F}^I$, we obtain that

$$\sum_j \eta_j(x_1^{n_j}x_2^{m_j}) = \sum_j \overline{\eta}_j(x_1^{n_j}x_2^{m_j}) = 0$$

by claim D. In order for this not to contradict $\mathbb{M}^I$-triviality, we must have that this sum over $j$ is actually an empty summation, and in fact $y$ was of the form

$$y = (q - 1) \sum_i \eta_i \otimes x_1^{n_i}x_2^{m_i}.$$ 

But, the module $\hat{\mathcal{O}} \otimes_{\mathcal{F} I_q} \mathcal{F} I_q$ is free as a $\mathbb{C}[q]$-module, so $\Psi(y) = (q - 1)\Psi(\sum \eta_i \otimes x_1^{n_i}x_2^{m_i}) = 0$ implies that $\Psi(\sum \eta_i \otimes x_1^{n_i}x_2^{m_i}) = 0$. Proceeding inductively, we may establish the desired claim.

\[\square\]