VIRASORO OPERATORS AND A COMPUTATION OF AN INTERSECTION NUMBER ON HILBERT SCHEMES OF POINTS ON A SURFACE

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1. INTRODUCTION

Let $X$ be a smooth projective complex surface, $X^{[n]}$ denote the Hilbert scheme of points on $X$, and $\Omega_X = \bigoplus_{n=0}^{\infty} A^*(X^{[n]}, \mathbb{Q})$ the direct sum of all the Chow groups. Nakajima and Grojnowski have (independently) constructed in a geometric action of an infinite dimensional Lie algebra on $\Omega_X$. Furthermore, Nakajima was able to identify this algebra with the Heisenberg algebra up to a constant which he showed was equal to a certain intersection number on $X^{[n]}$. In this note, we determine this constant. Two other computations of this constant also exist in the literature: using the link between symmetric products of an embedded curve, symmetric functions, and certain vertex operators which was first noted in [Gr], Nakajima gave a recursive computation of this intersection number; in another direction, [ES] analyzed the geometry of certain incidence schemes (see also section 2.2?) parametrizing pairs of subschemes $\xi \subset \eta$, and in this way were able to relate intersections on $X^{[n]}$ to those on $X^{[n-1]}$. Our approach, which is very similar to that of [ES], can be seen as an attempt to rephrase their computation in the language of vertex operators (actually we just use the two Virasoro operators $L_{\pm 1}$ here).

In section 2, we review some basic facts about the geometry of Hilbert schemes. In section 3, we review the work of Nakajima, Grojnowski, and Lehn on the connections between Hilbert Schemes, Heisenberg algebras, and Virasoro algebras. Finally, in section 4, we complete the computation of the intersection number.

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2. BASIC GEOMETRY OF HILBERT SCHEMES

Let $X^{[n]}$ be the Hilbert scheme of $n$-points on the surface $X$. We usually write $\xi$ for a length $n$ zero-dimensional subscheme of $X$ (i.e., a closed point of $X^{[n]}$) and $I_\xi$ for its ideal sheaf. We have the fundamental,

**Theorem 2.1.** Let $n \geq 1$ and $X$ be a smooth, irreducible, projective surface. Then $X^{[n]}$ is smooth, irreducible, and projective of dimension $2n$.

Let us recall that the punctual Hilbert scheme $M_n(P) \subset X^{[n]}$ which parametrize subschemes $\xi \subset X$ which are supported at a fixed point $P \in X$. Similarly, let $M_n$ parametrize subschemes supported at some point in $X$ (i.e., $M_n = \bigcup_{P \in X} M_n(P)$). We have the following punctual analog of the above theorem.

**Theorem 2.2.** Let $n \geq 1$. Then $M_n(P)$ is an irreducible variety of dimension $n - 1$ and $M_n$ is an irreducible variety of dimension $n + 1$.

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For $n \geq m > 0$ we denote the incidence scheme

$$X^{[n,m]} := \{ (\xi, \eta) \in X^{[n]} \times X^{[m]} | \xi \subset \eta \},$$
equipped with natural projections $X^{[n]} \xrightarrow{\pi_n} X^{[n,m]} \xrightarrow{\pi_m} X^{[m]}$. In the case when $m = n + 1$ we also have the map $X^{[n,n+1]} \to X$ which sends $(\xi, \eta) \mapsto \text{Supp}(I_\eta / I_\xi)$. Also, when $m = n + 1$, we have the following

**Theorem 2.3.** $X^{[n,n+1]}$ is irreducible, and smooth of dimension $2n + 2$. Furthermore, if we define $Z_n := \{ (\xi, x) \in X^{[n]} \times X | x \in \text{Supp}(\xi) \}$, then we have

$$X^{[n,n+1]} \cong Bl_{Z_n}X^{[n]} \times X,$$

where we identify the exceptional divisor $E = E_n = \{ (\xi, \eta) \in X^{[n,n+1]} | \text{Supp}(\xi) = \text{Supp}(\eta) \}$.

As before, we have the following punctual analogues of the above,

**Theorem 2.4.** Let $M_{n,n+1} = \pi_{n+1}^{-1}(M_{n+1})_{\text{red}}$ is irreducible of dimension $n + 2$. Similarly, $M_{n,n+1}(P) = \pi_{n+1}^{-1}(M_{n+1}(P))_{\text{red}}$ of dimension $n$.

There is the usual intersection form on $X^{[n]}$,

$$< \alpha, \beta > := \int_{X^{[n]}} \alpha \beta$$

for $\alpha, \beta \in A^*(X^{[n]})$.

It will be convenient for us to renormalize this and define,

$$(\alpha, \beta) := (-1)^n < \alpha, \beta >$$

for $\alpha, \beta \in A^*(X^{[n]})$.

Note that this is the negative of the intersection form on the surface itself.

### 3. Heisenberg and Virasoro Correspondences

**Heisenberg Correspondence.** For $n \in \mathbb{N}$, following Nakajima, we define the incidence varieties

$$Q^{[l,l+n]} := \{ (\xi, x, \eta) \in X^{[l]} \times X \times X^{[l+n]} | \xi \subset \eta, |\eta| - |\xi| = nx \},$$

with projections $\pi_l, \rho$, and $\pi_{l+n}$ to $X^{[l]}, X$, and $X^{[l+n]}$ respectively and where $|\eta| - |\xi| = nx$ means that $\text{Supp}(I_\eta / I_\xi) = x$. Using these incidence varieties, we define operators $p_{-n} : A^*(X) \to \text{End}(\delta_X)$ via

$$p_{-n}(\alpha)(y) = \pi_{l+n}([Q^{[l,l+n]}] \cdot \rho^* \alpha \cdot \pi_l y)$$

where $y \in A^*(X^{[l]})$. For $n \in \mathbb{N}$, define $p_n(\alpha) := p_{-n}(\alpha)^*$ as the adjoint with respect to the form $(\cdot, \cdot)$ described above.

We also set $p_0(\alpha) = 0$.

The utility of the above definition lies in the following,

**Theorem 3.1.** The operators $p_n(\alpha) \in \text{End}(\delta_X) (n \in \mathbb{Z}, \alpha \in A^*(X))$ generate an algebra with commutation relations

$$[p_n(\alpha), p_m(\beta)] = c_m \delta_{m-n}(\alpha, \beta) \cdot \text{Id}_{\delta_X}$$

where the $c_m$ are universal constants independent of the surface $X$ and $\alpha$ and $\beta$.

Further, Nakajima conjectured $c_m = m$. This allows us to identify, $\delta_X$ with the irreducible representation of a Heisenberg algebra which is the Fock space modeled on $A^*(X) = \mathbb{C}$. From now on, denote $1 = 1 \in A^*(pt)$. In the operator language, $M_n(P) = p_{-n}([P]) \cdot 1$ and $M_n = p_{-n}([X]) \cdot 1$.

**Virasoro Correspondence.** The irreducible representation of the Heisenberg algebra described above carries the action of a Virasoro algebra, which Lehn [Le] has described geometrically. For our purposes, we will just need two of these Virasoro operators, $\mathcal{L}_{\pm 1}$ which have particulary transparent geometric definitions.

Let $E = E_n = \{ (\xi, \eta) \in X^{[n,n+1]} | \text{Supp}(\xi) = \text{Supp}(\eta) \} \subset X^{[n,n+1]}$ denote the exceptional divisor. Then, define

1. $\mathcal{L}_{+1} : y \mapsto \pi_{n+1}(\pi_n^*(y) \cdot (E))$ for $y \in A^*(X^{[n+1]})$
2. $\mathcal{L}_{-1} : y \mapsto \pi_{n+1}(\pi_n^*(y) \cdot (E))$ for $y \in A^*(X^{[n]})$

Note that these operators are adjoint with respect to $(\cdot, \cdot)$.
The basic geometric fact which we need for our calculation is the following. The proof is is explicit in [Le] (actually, in Lehn’s paper, he proves (ii) and then shows how (i) follows formally assuming Nakajima’s constant $c_m = m$). Also, the argument below is (essentially) contained in [ES]. We sketch the argument here for convenience.

**Proposition 3.2.** Let $P \in X$ denote a point in the surface $X$. Then,

(i) $Z(p_-(P)) \cdot I = -np_{-n+1}(P) \cdot I$

(ii) $Z(p_-(X)) \cdot I = np_{-n-1}(X) \cdot I$.

**Proof.** For simplicity, we assume that $\alpha = \{P\}$. The argument works similarly for $\alpha = \{P\}$.

(i) A dimension argument and the fact that $M_{n,n+1}$ is irreducible implies that $\pi_{n+1}^*\{M_{n+1}\} = l[M_{n,n+1}]$ for some integer $l$. We can determine $l$ by the projection formula: since the degree of $\chi^{n,n+1} \to \chi^n$ is $n + 1$, we get that

$$\pi_{n+1}(X^{n,n+1}) = \pi_{n+1}(X^{n,n+1}) \cdot \pi_{n+1}(X^{n,n+1}) = \pi_{n+1}(X^{n,n+1}) \cdot [M_{n,n+1}] = (n + 1)[M_{n,n+1}].$$

Thus, $E \cdot \pi_{n+1}(M_{n+1}) = -(n + 1)[M_{n,n+1}]$. Since $\pi_n$ is proper and $M_{n,n+1} \to M_n$ has degree 1, we conclude part (i).

(ii) First note that as above we can conclude $E \cdot \pi_n\{M_n\} = l[M_{n,n+1}]$ for some integer $l$. We can conclude $l = n$ via the projection formula: since $E \to X^n$ has degree $n$ and $M_{n+1,n+1} \to M_n$ has degree 1, we write

$$\pi_n(E \cdot \pi_n^*\{M_n\}) = \pi_n(E) \cdot [M_n] = n[M_n].$$

Finally, we conclude the proof of (ii) by noting that $M_{n,n+1} \to M_{n+1}$ has degree 1.

□

4. Computation of an Intersection Number

Let $n \in \mathbb{N}$. Let us first note that $c_n = -(M_n(P), M_n) = -(p_{-n}(P), 1, p_{-n}(X) \cdot 1)$. Indeed, by Nakajima’s theorem $[p_n(P), p_{-n}(X)] \cdot 1 = -c_n$ since $p_n(\alpha) \cdot 1 = 0$. Now, by adjointness of $p_{-n}$,

$$-c_n = (p_n(P), p_{-n}(X) \cdot 1, 1) = (p_{-n}(X) \cdot 1, p_{-n}(P) \cdot 1).$$

Now, using and the adjointness of the form $(\cdot, \cdot)$ with respect to $Z_{\pm 1}$ and Proposition 3.2, we have

$$(n + 1)c_n = (M_n(P), Z_{n+1}M_{n+1}) = (Z_{-1}M_n(P), M_{n+1}) = -nc_{n+1}.$$ 

Since $c_1 = 1$, we may recursively determine $c_n = n$.

**References**


