GEOMETRY AND RANK OF FIBERED HYPERBOLIC 3-MANIFOLDS

1. Introduction

Recall that the rank of a finitely generated group is the minimal number of elements needed to generate it. M. White [Whi02] has proven that the injectivity radius of M is bounded above by some function of $\text{rank}(\pi_1(M))$. Building on a technique that he introduced, we show that if M is a hyperbolic 3-manifold fibering over the circle with fiber Σ_g and $\text{rank}(\pi_1(M)) \neq 2g+1$, then the diameter of M is bounded above by some function of its injectivity radius. Equivalently, after fixing g and ϵ there are at most finitely many such M for which $\text{rank}(\pi_1(M)) \neq 2g+1$.

Let Σ_g be the closed orientable surface of genus g and $\phi : \Sigma_g \to \Sigma_g$ a homeomorphism. We can construct a 3-manifold M_ϕ, called the *mapping torus* of ϕ, as the quotient space

$$M_\phi = \Sigma_g \times [0,1]/\sim,$$

where $(x,0) \sim (\phi(x),1)$. Thurston [Thu98] has proven that if the map $\phi : \Sigma_g \to \Sigma_g$ is pseudo-anosov then M_ϕ can be given a hyperbolic metric.

The fundamental group of M_ϕ is given by an HNN-extension

$$1 \to \pi_1(\Sigma_g) \to \pi_1(M_\phi) \to \mathbb{Z} \to 1.$$

Since $\text{rank}(\pi_1(\Sigma_g)) = 2g$ it follows that $\text{rank}(\pi_1(M_\phi)) \leq 2g+1$. It is not hard to construct examples where this inequality is strict, but it seems likely that if the gluing map is complicated enough then equality should hold. As an illustration of this, J. Souto proved in [Sou05] that given a pseudo-anosov map $\phi : \Sigma_g \to \Sigma_g$, we have for sufficiently large powers ϕ^n of ϕ that $\text{rank}(\pi_1(M_{\phi^n})) = 2g+1$. Our main result is the following extension of Souto's theorem.

Theorem 1.1. *Given $\epsilon > 0$ and a closed orientable surface Σ_g, there are at most finitely many ϵ-thick hyperbolic 3-manifolds M fibering over S^1 with fiber Σ_g for which rank$(\pi_1(M)) \neq 2g+1$.*

Recall that the *injectivity radius* of a hyperbolic manifold M, written $\text{inj}(M)$, is defined to be half the length of a shortest homotopically essential loop in M, and M is called *ϵ-thick* if $\text{inj}(M) \geq \epsilon$.

The *Heegaard genus* of a closed 3-manifold M is the smallest $g = g(M)$ such that M can be obtained by gluing two genus g handlebodies along their boundaries. Then $\text{rank}(\pi_1(M)) \leq g(M)$ and in fact there are currently no known hyperbolic 3-manifolds for which $\mathrm{rank}(\pi_1(M)) \neq g(M)$. In [BS05], Bachman and Schleimer proved that if the translation distance of ϕ in the curve complex $C(\Sigma_g)$ is large, then the Heegaard genus of M_ϕ is $2g+1$. It is likely that the conclusion of Theorem 1.1 is true under similar assumptions, but it is not yet clear to us how to prove this.

Before beginning the bulk of this paper, let us sketch the idea behind the proof of Theorem 1.1. Let M be a hyperbolic 3-manifold fibering over the circle with fiber Σ_g. Following a technique of White [Whi02], we find a graph X and a π_1-surjective mapping $f : X \to M$ whose image has as small length as is possible. We show that if M has large diameter it is most efficient for X to use small edges to fill out the fundamental group of the fiber and a long edge to navigate M's waist. Intuitively, this is a reflection of the fact that if M has a lower injectivity radius bound then it can be given an explicit fiber bundle structure in which the fibers have bounded diameter, but our proof uses a different approach. The subgraph of X consisting of all small edges then has rank equal to $\mathrm{rank}(\pi_1(\Sigma_g)) = 2g$, implying that $\pi_1(X)$, and therefore $\pi_1(M)$, has rank $2g+1$.

The paper is organized as follows. We begin in Section 2 by recalling some standard facts from the theory of Kleinian groups. In Section 3, we use a lemma of Souto and a compactness argument to provide geometry bounds for certain covers of doubly degenerate hyperbolic manifolds homeomorphic to $\Sigma_g \times \mathbb{R}$. The minimal length graphs above are formally introduced in Section 4 and Section 5 contains a proof of Theorem 1.1. We finish with an appendix that fleshes out a result due to Souto, [Sou06], that gives a convenient decomposition for minimal length π_1-surjective graphs in closed hyperbolic 3-manifolds.

Acknowledgements: I thank Justin Malestein and Benson Farb for their helpful comments and Juan Souto for many conversations, advice and insight.

2. Preliminaries

Let M be a hyperbolic 3-manifold with finitely generated fundamental group. For the sake of simplicity, we will assume that M has no cusps. An important result of P. Scott in 3-manifold topology states that M admits a *compact core,* that is a compact submanifold N whose inclusion into M is a homotopy equivalence, [Sc73]. The connected components of $M \setminus N$ are called the *ends* of M. Marden asked in the 1970s whether M is always homeomorphic to the interior of its compact core; this was recently proven to be true by Agol [Agol04] and Calegari-Gabai [CG06]. Consequently, if E is an end of M then E is homeomorphic to $\partial E \times [0, \infty)$, where ∂E is the boundary component of N facing E.

Define the *convex core* of M to be the smallest convex submanifold $\mathrm{CC}(M) \subset M$ whose inclusion is a homotopy equivalence. An end E of M is called *convex-cocompact* if $E \cap \mathrm{CC}(M)$ is compact, and *degenerate* otherwise. A convex-cocompact end is geometrically a warped product, where the metric on level surfaces of $E \cong \partial E \times [0, \infty)$ grows exponentially with the distance to the boundary of the convex core. The geometry of degenerate ends is more subtle; before discussing it we need to review some facts about negatively curved surfaces in hyperbolic 3-manifolds.

Definition 2.1. Let M be a hyperbolic 3-manifold. A *simplicial hyperbolic surface* in M is a map $f : S \to M$, where

- S is a closed surface equipped with a triangulation T
- f maps each face of T to a totally geodesic triangle in M
- for each vertex $v \in T$ the angles between the f-images of the edges adjacent to v sum to at least 2π.

If $f : S \to M$ is a simplicial hyperbolic surface then we get a path-metric on S by requiring that f preserves path lengths. The metric is smooth and hyperbolic away from the vertices of T, at which there are possible excesses of angle. By the Gauss-Bonnet Theorem, we have $\mathrm{vol}(S) \leq 2\pi|\chi(S)|$. Since the diameter of S is bounded above by its volume and injectivity radius, we obtain:

Lemma 2.1 (Bounded Diameter Lemma, Thurston [Thu78])**.** *Assume $f : \Sigma_g \to M$ is an ϵ-thick simplicial hyperbolic surface. Then* $\mathrm{diam}(S) \leq \frac{4}{\epsilon}(2g-2)$.

Mahler's Compactness Theorem [BP92] states that the moduli space of ϵ-thick (smooth) hyperbolic surfaces is compact. Therefore, any invariant of hyperbolic surfaces that varies continuously over moduli space will have an upper bound on ϵ-thick surfaces. The following Proposition allows us to transfer all of these bounds to ϵ-thick simplicial hyperbolic surfaces; note that this provides a new proof of Lemma 2.1, albeit without the explicit constant.

Proposition 2.2 (Smooth Dominates Simplicial)**.** *Let S be a closed surface and d a metric on S coming from a simplicial hyperbolic surface. Then there exists a smooth hyperbolic metric d_{hyp} on S such that for all $x, y \in S$*

$$\frac{1}{C} d(x, y) \leq d_{hyp}(x, y),$$

where $C > 0$ depends only on the topological type of S. Note that if d is ϵ-thick then d_{hyp} is $\frac{\epsilon}{C}$-thick.

Proof of Proposition 2.2. Working in polar coordinates in a small neighborhood around each singular point of d, we can explicitly deform d to obtain a smooth metric d' that is bilipschitz to d and has Gaussian curvature $K \leq -1$.

The argument is very similar to the proof of the 2π-Theorem of Gromov and Thurston [BH96], so we will omit it here. Now, if done properly the bilipschitz constant between d and d' will depend only on the angles d has around the points in its singular locus. Since the Gauss-Bonnet Theorem gives an upper bound for the sum of these angles, d and d' are in fact C-bilipschitz for some C depending only on the topological type of S.

To finish the proof, let d_{hyp} be the hyperbolic metric in the conformal class of d'. The Ahlfors-Schwartz Lemma [Ahl73] states that distances measured in d' are less than or equal to distances in d_{hyp}; this proves the desired inequality. □

Using Proposition 2.2 and a based version of Mahler's Compactness Theorem we can also prove the following.

Corollary 2.3 (Short Markings)**.** *Set $\Gamma = \pi_1(\Sigma_g)$ and fix a generating set $X \subset \Gamma$. Then given $\epsilon, g > 0$ there is a constant L such that whenever $f : \Sigma_g \to M$ is an ϵ-thick simplicial hyperbolic surface and $p \in \Sigma_g$, there is an isomorphism $\Phi : \Gamma \to \pi_1(\Sigma_g, p)$ such that the image of each element of X can be represented by a loop based at p of length less than L.*

2.1. **The Geometry of Degenerate Ends.** Assume that E is a degenerate end of a hyperbolic 3-manifold M. Canary proved in [Can96] that there is a neighborhood of E in which every point lies in the image of a simplicial hyperbolic surface whose inclusion into E is a homotopy equivalence – this result is usually known as Canary's Filling Theorem. If E is incompressible and M is ϵ-thick, then the Bounded Diameter Lemma applies; we therefore have an exhaustion of E by surfaces with diameter bounded above by some function of ϵ and the genus of ∂E. In fact, although we will not need this, one can use a geometric limit argument to show that there exists a homeomorphism $E \cong \partial E \times [0, \infty)$ so that a similar diameter bound applies to the level surfaces $\partial E \times \{t\}$.

Example 2.2. Let M_ϕ be the mapping torus of a pseudo-anosov map $\phi : \Sigma_g \to \Sigma_g$. As mentioned in the introduction, $\pi_1(M_\phi)$ decomposes as

$$1 \to \pi_1(\Sigma_g) \to \pi_1(M_\phi) \to \mathbb{Z} \to 1.$$

Let N be the cyclic cover of M_ϕ coming from the subgroup $\pi_1(\Sigma_g)$. Then N is homeomorphic to $\Sigma_g \times \mathbb{R}$, and since it regularly covers a closed manifold we have $\mathrm{CC}(N) = N$, implying that N is totally degenerate. Note that unwrapping a fiber bundle structure on M_ϕ gives a product structure for N with fibers of bounded diameter.

A remarkable theorem of Thurston [Thu78] says that essentially the only example in which an incompressible degenerate end infinitely covers is the cyclic cover of a mapping torus. Canary used his Filling Theorem to extend Thurston's result to the compressible case, [Can96].

Theorem 2.4 (The Covering Theorem, [Thu78], [Can96])**.** *Assume that N and M are hyperbolic 3-manifolds without cusps and $\pi : N \to M$ is a Riemannian covering map. Then if E is a degenerate end of N, either π is finite to one on E or M is compact and π factors as*

$$N \xrightarrow{\textit{finite to one}} N' \xrightarrow{\textit{cyclic}} M' \xrightarrow{\textit{finite to one}} M \ ,$$

where $N' \cong \Sigma_g \times \mathbb{R}$ is the cyclic cover of a finite cover M' of M that fibers over the circle. In the latter case, the fibers of N' lift to surfaces in N homotopic to ∂E.

2.2. **Algebraic and Geometric Convergence.** Let Γ be a finitely generated group and consider a sequence of discrete and faithful representations $\rho_i : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$. If $\rho_i \to \rho$ pointwise as maps $\Gamma \to \mathrm{PSL}(2,\mathbb{C})$ we usually say that (ρ_i) is *algebraically convergent,* with ρ as its algebraic limit. Alternatively, we can consider the sequence of subgroups $\Gamma_i = \rho_i(\Gamma) \subset \mathrm{PSL}(2,\mathbb{C})$; if these converge to a subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ in the Hausdorff topology on closed subsets of $\mathrm{PSL}(2,\mathbb{C})$ then we say that $\Gamma_i \to \Gamma$ *geometrically.* The case where the two notions of convergence agree is useful enough to warrant additional terminology. Specifically, if $\rho_i \to \rho$ algebraically and $\rho_i(\Gamma) \to \rho(\Gamma)$ geometrically than one says that $\rho_i \to \rho$ *strongly.* The following criterion for strong convergence comes from work of Evans [Eva04], Anderson and Canary [AC96] and the resolution of Marden's conjecture by Agol [Agol04] and Calegari-Gabai [CG06].

Theorem 2.5 (No Parabolics Implies Strong Convergence)**.** *Assume that Γ is a finitely generated group which is not virtually abelian and $\rho_i : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ is a sequence of discrete and faithful representations converging algebraically to $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$. If $\rho(\gamma)$ is hyperbolic for all $\gamma \in \Gamma$, then the convergence is strong.*

There is a nice way to interpret the geometric convergence of a sequence of subgroups $\Gamma_i \to \Gamma_\infty \subset \mathrm{PSL}(2,\mathbb{C})$ in terms of the quotient manifolds $M_i = \mathbb{H}^3/\Gamma_i$. If we fix a basepoint and baseframe (p, f) for $\mathbb{H}^3$, for each i we can take the projection (p_i, f_i) as a basepoint and baseframe for M_i. Then $\Gamma_i \to \Gamma_\infty$ geometrically if there exist sequences of positive numbers $\epsilon_i \to 0$ and $R_i \to \infty$, and $(1+\epsilon_i)$-bilipschitz maps $\phi_i : \mathrm{B}(p_i, R_i) \to M_\infty$ sending (p_i, f_i) to (p_∞, f_∞). For future reference, we will call the maps ϕ_i a sequence of almost isometric maps coming from geometric convergence. Note that using this as our definition, we can now speak about a geometrically convergent sequence of framed hyperbolic 3-manifolds, or even based hyperbolic 3-manifolds if we forget about the presence of a baseframe.

3. Short Graphs in Doubly Degenerate $\Sigma_g \times \mathbb{R}$

Assume that M is a hyperbolic 3-manifold without cusps that is homeomorphic to $\Sigma_g \times \mathbb{R}$. Using Waldhausen's Cobordism Theorem [Wal68], it is not hard to see that there is an explicit homeomorphism $M \cong \Sigma_g \times \mathbb{R}$ such that $\mathrm{CC}(M)$ sits inside M as either

- $\Sigma_g \times [0,1]$, in which case M is convex cocompact
- $\Sigma_g \times [0,\infty)$, in which case M is called singly degenerate
- $\Sigma_g \times \mathbb{R}$, and then M is called doubly degenerate.

We mentioned in the introduction that Theorem 1.1 is an extension of an earlier theorem of Souto [Sou05]. A key step in Souto's proof was the following application of the Covering Theorem (see Section 2) to covers of doubly degenerate hyperbolic manifolds homeomorphic to $\Sigma_g \times \mathbb{R}$.

Lemma 3.1 ([Sou05])**.** *Let M be a doubly degenerate hyperbolic 3-manifold homeomorphic to $\Sigma_g \times \mathbb{R}$ and let $\Gamma \subset \pi_1(M)$ be a proper subgroup of rank at most $2g$. Then Γ is free, infinite index and convex-cocompact.*

Proof. First, note that Γ is isomorphic to the fundamental group of a surface S covering Σ_g. If S is closed that it has genus greater than g; this is impossible by the assumption that $\mathrm{rank}(\Gamma) \leq 2g$. So S is non-compact, implying that Γ is free and infinite index in $\pi_1(M)$.

Now let M_Γ be the cover of M corresponding to Γ. The fundamental group of M_Γ is free, so the resolution of Marden's Conjecture by Agol [Agol04] and Calegari-Gabai [CG06] and a topological argument imply that M_Γ is homeomorphic to the interior of a handlebody. Thus M_Γ has a single end, E. If E is degenerate then the Covering Theorem implies that the cover $M_\Gamma \to M$ is finite-to-one on E, and thus on all of M_Γ. Since Γ is infinite index this cannot be, so E is convex-cocompact. □

To prove Theorem 1.1, we need an improved version of Lemma 3.1 that gives a diameter bound for the convex core of $\mathbb{H}^3/\Gamma$ in terms of $\mathrm{inj}(M)$ and the length of a set of loops in M generating Γ. Our proof will be a compactness argument: we define a topology on the set of wedges of k bounded length loops in ϵ-thick doubly degenerate hyperbolic 3-manifolds homeomorphic to $\Sigma_g \times \mathbb{R}$, show that the resulting space is compact and then use continuity to show that there is an upper bound for the corresponding convex core diameters.

Definition 3.1. Define $\mathcal{G} = \mathcal{G}(\epsilon, L, k)$ to be the space of pairs (M, f), where

(1) M is a doubly degenerate ϵ-thick hyperbolic 3-manifold homeomorphic to $\Sigma_g \times \mathbb{R}$
(2) $f : \wedge_k \mathbb{S}^1 \to M$ is a lipschitz map from the wedge of k circles, endowed with some fixed metric, that has image of total length less than L.

We say that $(M_i, f_i) \to (M_\infty, f_\infty)$ if

(1) $(M_i, \star_i)$ converges geometrically to $(M_\infty, \star_\infty)$, where $\star_i$ is the wedge point of $f_i(\wedge_k \mathbb{S}^1)$
(2) there is a sequence ϕ_i of almost isometric maps coming from the geometric convergence in (1) such that $\phi_i \circ f_i : \wedge_k \mathbb{S}^1 \to M$ converges pointwise to $f_\infty : \wedge_k \mathbb{S}^1 \to M_\infty$.

Proposition 3.2. *$\mathcal{G}$ is compact.*

Proof. Let (M_i, f_i) be a sequence in $\mathcal{G}$ and assume that $\star_i \in M_i$ is the wedge point of $f_i(\wedge_k \mathbb{S}^1)$. For each i, Canary's Filling Theorem gives a simplicial hyperbolic surface in M_i with image passing through $\star_i$; using the short markings of these surfaces provided by Corollary 2.3 we can construct representations $\rho_i : \pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbb{C})$ with $\mathbb{H}^3/\rho_i(\Sigma_g) \cong M_i$ so that a fixed base point $\star \in \mathbb{H}^3$ projects to each $\star_i$ and up to passing to a subsequence, ρ_i converges algebraically to some $\rho_\infty : \pi_1(\Sigma_g) \to \mathrm{PSL}(2, \mathbb{C})$. Since our lower bound on injectivity radius persists through algebraic limits, $\rho_\infty(\pi_1(\Sigma_g))$ contains no parabolics. Theorem 2.5 then implies that $\rho_i \to \rho_\infty$ strongly.

Set $M_\infty = \mathbb{H}^3/\rho_\infty(\Sigma_g)$ and let $\star_\infty \in M$ be the projection of $\star$. The fundamental group of M_∞ is isomorphic to $\pi_1(\Sigma_g)$, so Bonahon's Tameness Theorem ([Bon86]) implies that $M_\infty \cong \Sigma_g \times \mathbb{R}$. Moreover, it follows from strong convergence and (Lemma 2.4, [McM96]) that M_∞ is doubly degenerate. Since the sequence of based hyperbolic 3-manifolds $(M_i, \star_i)$ converges geometrically to $(M_\infty, \star_\infty)$, we can construct a map $f_\infty : \wedge_k \mathbb{S}^1 \to M_\infty$ by applying Arzela-Ascoli's Theorem to the sequence of maps $\phi_i \circ f_i : \wedge_k \mathbb{S}^1 \to M_\infty$ we get by composing f_i with a sequence of almost isometric maps coming from geometric convergence. Clearly (M_i, f_i) converges to (M_∞, f_∞) in $\mathcal{G}$. □

Corollary 3.3. *Let M be a doubly degenerate ϵ-thick hyperbolic 3-manifold homeomorphic to $\Sigma_g \times \mathbb{R}$ and let $p \in M$ be a basepoint. Assume that $\Gamma \subset \pi_1(M, p)$ is a proper subgroup that can be generated by $2g$ loops based at p of length less than L. Then Γ is convex cocompact and the diameter of the convex core of $\mathbb{H}^3/\Gamma$ is bounded above by some constant depending only on L, ϵ and g.*

Proof. Observe that Γ determines an element $(M, f) \in \mathcal{G} = \mathcal{G}(\epsilon, L, 2g)$, with the extra property that f is not π_1-surjective. The subset of $\mathcal{G}$ consisting of pairs (M, f) for which f is not π_1-surjective is closed in $\mathcal{G}$, and therefore compact by Proposition 3.2. Lemma 3.1 implies that for all such (M, f) the cover $M_{\pi_1 f}$ of M corresponding to the π_1-image of f is convex cocompact. It is not hard to see that if $(M_i, f_i) \to (M_\infty, f_\infty) \in \mathcal{G}$ then $(M_i)_{\pi_1 f_i} \to (M_\infty)_{\pi_1 f_\infty}$ algebraically after picking appropriate markings. The diameter of the convex core of a hyperbolic 3-manifold is continuous with respect to algebraic convergence, so the diameter of the convex core of $(M_i)_{\pi_1 f_i}$ varies continuously over $\mathcal{G}$. This proves the claim. □

4. Carrier Graphs

In the following, assume M is a closed hyperbolic 3-manifold.

Definition 4.1. A *carrier graph* for M is a graph X and a map $f : X \to M$ which induces a surjection on fundamental groups.

Standing Assumption: In this paper we are interested in generating sets of minimal size, which correspond to carrier graphs with $\mathrm{rank}(\pi_1(X)) = \mathrm{rank}(\pi_1(M))$. From now on all carrier graphs will be assumed to have this property.

If a carrier graph $f : X \to M$ is rectifiable, we can pull back path lengths in M to obtain an pseudo-metric on X. Collapsing to a point each zero-length segment in X yields a new carrier graph endowed with an actual metric. Define the *length* of a metrized carrier graph to be the sum of the lengths of its edges, and a *minimal length carrier graph* to be a carrier graph which has smallest length (over all metrized carrier graphs of minimal rank). An argument using Arzela-Ascoli's Theorem, [Whi02], shows that minimal length carrier graphs exist in any closed hyperbolic 3-manifold.

The following Proposition shows that minimal length carrier graphs are geometrically well behaved.

Proposition 4.1 (White, [Whi02])**.** *Assume $f : X \to M$ is a minimal length carrier graph in a closed hyperbolic 3-manifold M. Then X is trivalent with $2(\mathrm{rank}(\pi_1(M)) - 1)$ vertices and $3(\mathrm{rank}(\pi_1(M)) - 1)$ edges, each edge in X maps to a geodesic segment in M, the angle between any two adjacent edges is $\frac{2\pi}{3}$, and the image of any simple closed path in X is an essential loop in M.*

We conclude this section with a technical result that is instrumental in our proof of Theorem 1.1. A slightly more general result was proven by Souto in [Sou06], but the proof given there is somewhat incomplete. We include a full proof of the more general result in Appendix A.

Proposition 4.2 (Chains of Bounded Length)**.** *Let M be a closed hyperbolic 3-manifold with $f : X \to M$ a minimal length carrier graph. Then we have a sequence of (possibly disconnected) subgraphs*

$$\emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X$$

such that the length of any edge in $Y_{i+1} \setminus Y_i$ is bounded above by some constant depending only on $\mathrm{inj}(M)$, $\mathrm{rank}(\pi_1(M))$, $\mathrm{length}(Y_i)$ and the diameters of the convex cores of the covers of M corresponding to $f_*(\pi_1(Y_i^j))$, where $Y_i^1, \ldots, Y_i^n$ are the connected components of Y_i.

5. Proof of Theorem 1.1

Fix $\epsilon, g > 0$ and assume that $M \to \mathbb{S}^1$ is an ϵ-thick hyperbolic 3-manifold fibering over the circle with fiber Σ_g. The goal of this section is to prove that there are only finitely many cases where $\mathrm{rank}(\pi_1(M)) \neq 2g+1$. We begin, however, with a quick computation concerning M's girth.

Definition 5.1. The *waist length* of M, denoted $\mathrm{waist}(M)$, is the smallest length of a loop in M that projects to a generator of $\pi_1(S^1)$.

Proposition 5.1 (Fibered 3-Manifolds Have High BMI)**.** *Let M be an ϵ-thick hyperbolic 3-manifold fibering over the circle with fiber Σ_g. Then*

$$2\,\mathrm{diam}(M) - \frac{16}{\epsilon}(2-2g) \leq \mathrm{waist}(M) \leq 2\,\mathrm{diam}(M).$$

Proof. Assume that γ is a loop realizing the waist length of M. Canary's Filling Theorem [Can96] implies that every point in the cyclic cover of M corresponding to the fundamental group of the fiber lies in the image of a simplicial hyperbolic surfaces for which the inclusion map is a homotopy equivalence. Projecting down, this provides an exhaustion of M by simplicial hyperbolic surfaces in the homotopy class of the fiber. By homological considerations, any such surface must intersect γ. The Bounded Diameter Lemma (Lemma 2.1) then implies that $\mathrm{diam}(M) \leq \frac{1}{2}\,\mathrm{waist}(M) + \frac{8}{\epsilon}(2-2g)$. This establishes the first inequality.

For the second, recall that the fundamental group of M is generated by the set of all loops in M of length less than $2\,\mathrm{diam}(M)$. Any generating set for $\pi_1(M)$ must contain a loop that encircles M's waist, so the waist length of M is at most twice its diameter. □

There are only finitely many hyperbolic 3-manifolds with diameter less than a given constant. Together with Proposition 5.1, this gives a similar finiteness result for thick hyperbolic 3-manifolds fibering over the circle with a fixed fiber and bounded waist length.

We are now ready to prove the main result of this note.

Theorem 1.1. *Given $\epsilon, g > 0$ there are at most finitely many ϵ-thick hyperbolic 3-manifolds M fibering over $\mathbb{S}^1$ with fiber Σ_g for which $\mathrm{rank}(\pi_1(M)) \neq 2g+1$.*

Proof. Assume that M is an ϵ-thick hyperbolic 3-manifold fibering over the circle with fiber Σ_g and $\mathrm{rank}(\pi_1(M)) \leq 2g$. We will show that the waist length of M is bounded by some constant depending only on ϵ and g.

Let $f : X \to M$ be a minimal length carrier graph. By Proposition 4.2, there is a constant L and a chain of (possibly disconnected) subgraphs

$$\emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X$$

with $\mathrm{length}(Y_{i+1})$ bounded above by some constant depending only on ϵ, g, $\mathrm{length}(Y_i)$ and the diameters of the convex cores of the covers of M corresponding to the fundamental groups of the connected components of Y_i.

Assume for the moment that no connected component of Y_i runs all the way around M's waist, so that Y_i lifts homeomorphically to the cyclic cover $M_{\pi_1(\Sigma_g)}$ of M. Since $\text{rank}(\pi_1(X)) \leq 2g$, the components of Y_i have even smaller rank and thus cannot generate the fundamental group of $M_{\pi_1(\Sigma_g)}$. Therefore Corollary 3.3 applies to bound the diameters of the associated convex cores in terms of $\text{length}(Y_i)$, ϵ and g. It follows that $\text{length}(Y_{i+1})$ is also bounded above by $\text{length}(Y_i)$, ϵ and g.

Applying this argument iteratively, we obtain a length bound for the first Y_i that has a component which navigates the waistline of M. The length bound depends on the number of steps in the iteration process, but since there are at most $3(\text{rank}(\pi_1(M)) - 1)$ edges in X the number of subgraphs in our chain is also limited. Therefore we have that the waist length of M is bounded by a function of ϵ and g. □

Under slight modifications the proof of Theorem 1.1 shows that for mapping tori with large waist length there is only one Nielsen equivalence class of minimal size generating sets for $\pi_1(M)$. The interested reader may compare our proof with [Sou06] for more details.

Appendix A. Chains of Bounded Length

We prove here the generalization of Proposition 4.2 promised in Section 4. The idea of the proof given below was originally sketched by Souto in [Sou06] - the purpose of this Appendix is to fill in some missing details.

Assume that $M = \mathbb{H}^3/\Gamma$ is a closed hyperbolic 3-manifold and $f : X \to M$ is a minimal length carrier graph. Choose an edge $e \subset X$ and a subgraph $Y \subset X$. Our first goal will be to provide a useful definition of the length of e relative to the subgraph Y; this should vanish when $e \subset Y$ and should agree with the hyperbolic length of $f(e)$ when neither of the vertices of e lies inside Y. If we consider X to be embedded as a subset of M, then relative length will be similar to the length e has outside of the hyperbolic convex hulls of the components of Y that e touches, but we need to do our measurements in the universal cover and throw out sections of e that lie inside some of the thin parts of M.

To clarify this, fix a universal covering $\pi_X : \tilde{X} \to X$ and a lift $\tilde{f} : \tilde{X} \to \mathbb{H}^3$ of f. Assume that a vertex v of e lies in a connected component $Z_v \subset Y$ and choose lifts $\hat{e}, \hat{Z}_v \subset \tilde{X}$ of e and Z that touch above v. Let $\Gamma_{\tilde{f}(\hat{Z}_v)}$ be the subgroup of $\Gamma \cong \pi_1(M)$ that leaves $\tilde{f}(\hat{Z}_v)$ invariant.

Definition A.1 (Thick Convex Hulls)**.** The *thick convex hull of* $\tilde{f}(\hat{Z}_v)$*,* written $\text{TCH}(\tilde{f}(\hat{Z}_v))$, is the smallest convex set K containing $\tilde{f}(\hat{Z}_v)$ such that for every $\gamma \in \Gamma_{\tilde{f}(\hat{Z}_v)}$ and $x \in \mathbb{H}^3 \setminus K$, we have $\text{d}(\gamma(x), x) \geq 1$.

Definition A.2 (Edge Length Relative to a Subgraph)**.** Define the *length of e relative to Y*, denoted $\mathrm{length}_Y(e)$, to be the length of the part of $\tilde{f}(\hat{e})$ that lies outside of $\mathrm{TCH}(\tilde{f}(\hat{Z}_v))$ for each vertex v of e contained in Y.

It is easy to see that the relative length of e is well-defined, independent of the lifts chosen above. The definition is a bit less complicated if we assume that X is embedded as a subset of M. For then we can lift e directly to $\mathbb{H}^3$ along with any connected components of Y that e touches, and then measure the length of e's lift outside of the thick convex hulls of the lifted subgraphs. In the proofs below, we will assume X to be embedded in order to remove a level of notational hinderance. The arguments will be exactly the same in the general case.

Although an edge can have very long absolute length while having short length relative to a subgraph Y, we can bound this difference if we have some control over the geometry of the covers of M corresponding to the fundamental groups of the components of Y.

Lemma A.1. *Assume that M is a closed hyperbolic 3-manifold, $f : X \to M$ is a minimal length carrier graph, Y is a subgraph of X and e is an edge of $X \setminus Y$. Then* $\mathrm{length}(e)$ *is bounded by a constant depending only on* $\mathrm{length}_Y(e)$*,* $\mathrm{length}(Y)$*,* $\mathrm{inj}(M)$*,* $\mathrm{rank}(\pi_1(M))$ *and the diameters of the convex cores of the covers of M corresponding to the components of Y that e touches.*

Proof. As mentioned above, we forget about f and assume that X is embedded as a subset of M. Suppose that e shares a vertex with a connected component $Z \subset Y$, and let $\tilde{e}, \tilde{Z} \subset \mathbb{H}^3$ be lifts that touch above that vertex. It suffices to show that the Hausdorff distance from $\tilde{Z}$ to $\mathrm{TCH}(\tilde{Z})$ is bounded by the quantities mentioned in the statement of the Lemma. For since X is minimal length, $\tilde{e} \cap \mathrm{TCH}(\tilde{Z})$ must minimize the distance from $\tilde{e} \cap \partial\,\mathrm{TCH}(\tilde{Z})$ to $\tilde{Z}$; thus a bound on the Hausdorff distance between $\tilde{Z}$ and $\mathrm{TCH}(\tilde{Z})$ limits the length that $\tilde{e}$ can have inside of $\mathrm{TCH}(\tilde{Z})$.

We first claim that the hyperbolic distance from $\tilde{Z}$ to $\mathrm{CH}(\Lambda(\Gamma_{\tilde{Z}}))$ is bounded above by a constant depending only on $\mathrm{inj}(M)$ and $\mathrm{rank}(\pi_1(M))$. Choose an infinite piecewise geodesic path $\gamma \subset \tilde{Z}$ that projects to a simple closed curve in Y and let $g \in \Gamma_{\tilde{Z}}$ be the corresponding deck transformation. Taking a maximal sequence of consecutive edges of γ that project to distinct edges in M yields a subpath γ' whose g-translates cover γ. Note that the orthogonal projection of γ' to $\mathrm{axis}(g)$ has length equal to the translation distance of g, which is at least $\mathrm{inj}(M)$. By Lemma 4.1, X has $3(\mathrm{rank}(\pi_1(M)) - 1)$ edges; the number of edges in γ' can certainly be no greater than this. Thus there is an edge of γ whose orthogonal projection to $\mathrm{axis}(g)$ has length at least $\frac{\mathrm{inj}(M)}{3(\mathrm{rank}(\pi_1(M))-1)}$. It follows from elementary hyperbolic geometry that there is a point on this edge whose distance from axis(g) is bounded above by a constant depending on that length; this proves the claim.

Now $\tilde{Z}$ and $\mathrm{CH}(\Lambda(\Gamma_{\tilde{Z}}))$ are both invariant under the action of $\Gamma_{\tilde{Z}}$ with quotients of bounded diameter, so our limit on the hyperbolic distance between them translates into a bound on their Hausdorff distance. But if $\tilde{Z}$ is Hausdorff-close to a convex set then it must also be Hausdorff-close to its convex hull, $\mathrm{CH}(\tilde{Z})$. Since the Hausdorff distance from $\mathrm{CH}(\tilde{Z})$ to $\mathrm{TCH}(\tilde{Z})$ is controlled by $\mathrm{inj}(M)$, we have a bound on the Hausdorff distance between $\tilde{Z}$ and $\mathrm{TCH}(\tilde{Z})$. □

For a subgraph $Z \subset X$, we define the *length of Z relative to Y* to be

$$\mathrm{length}_Y(Z) = \sum_{\text{edges } e \subset Z} \mathrm{length}_Y(e).$$

Using our definition of relative length, we can streamline the formulation of Proposition 4.2. The statement given earlier follows from this one after applying Lemma A.1.

Proposition A.2 (Chains of Bounded Length). *There is a universal constant L with the property that if M is a closed hyperbolic 3-manifold and $f : X \to M$ is a minimal length carrier graph then we have a sequence of (possibly disconnected) subgraphs*

$$\emptyset = Y_0 \subset Y_1 \subset \ldots \subset Y_k = X$$

such that $\mathrm{length}_{Y_i}(Y_{i+1}) < L$ *for all i.*

Proof. It is a standard fact in hyperbolic geometry that there exist a universal constant $C > 0$ with the following property:

(1) any path in $\mathbb{H}^3$ made of geodesic segments of length at least C connected with angles at least $\frac{\pi}{3}$ is a quasi-geodesic.

There is also a constant $D > C$ such that

(2) if $N \subset \mathbb{H}^3$ contains the axis of a hyperbolic isometry γ and $\mathrm{d}(x, \gamma(x)) \geq 1$ for all $x \in \mathbb{H}^3 \setminus N$, then $\mathrm{d}(x, \gamma(x)) \geq C$ for all $x \in \mathbb{H}^3 \setminus \mathcal{N}_D(N)$,
(3) any geodesic ray emanating from a convex subset $K \subset \mathbb{H}^3$ that leaves $\mathcal{N}_D(K)$ meets $\partial\mathcal{N}_D(K)$ in an angle of at least $\frac{\pi}{3}$,

and finally a constant $B > 0$ for which

(4) any geodesic exiting a convex subset $K \subset \mathbb{H}^3$ will exit $\mathcal{N}_D(K)$ after an additional length less than B.

We will show that if Y is any subgraph of X then there is an edge in $X \setminus Y$ of length at most $L = C + 2B$ relative to Y; applying this iteratively will give the chain of subgraphs in the statement of the Proposition.

So, suppose that Y is a subgraph of X. Observe that since the fundamental group of a closed hyperbolic manifold cannot be free, there is an essential closed loop $\gamma \subset X$ that is nullhomotopic in M. Furthermore, since $\pi_1(M)$ does not split as a free product, [Hem76], we can pick γ so that it has no subpath contained entirely in Y that is also a closed loop nullhomotopic in M. Lifting γ to $\mathbb{H}^3$ then gives a closed loop $\tilde{\gamma} \subset \mathbb{H}^3$ such that each time $\tilde{\gamma}$ touches a component of $\pi_M^{-1}(Y)$ it enters and leaves that component using different edges of $\pi_M^{-1}(X \setminus Y)$.

Consider a maximal segment of $\tilde{\gamma}$ that is contained in a component $\tilde{Z}$ of $\pi_M^{-1}(Y)$ and let e and f be the edges that $\tilde{\gamma}$ traverses before and after the segment in $\tilde{Z}$. If e or f has length less than L relative to Y, then we are done. Otherwise, the two edges have a length of at least L left after exiting $\mathrm{TCH}(\tilde{Z})$, so by (4) both of these edges must exit $\mathcal{N}_D(\mathrm{TCH}(\tilde{Z}))$; let e_0 and f_0 be the points where they meet $\partial\mathcal{N}_D(\mathrm{TCH}(\tilde{Z}))$. If the distance between e_0 and f_0 is less than C, then by (2), e and f project to different edges in X. Substituting $\pi_M(e \cap \mathcal{N}_D(\mathrm{TCH}(\tilde{Z}))) \subset X$ with the projection of the geodesic between e_0 and f_0 then yields a new carrier graph for M, and since the new edge has length less than C while the old has length at least D our new carrier graph has shorter length than X. This contradicts the minimality of X, so $\mathrm{d}(e_0, f_0) \geq C$.

We can now create a new closed path in $\mathbb{H}^3$ from $\tilde{\gamma}$ as follows: each time $\tilde{\gamma}$ traverses a component $\tilde{Z}$ of $\pi_M^{-1}(Y)$, replace the part of $\tilde{\gamma}$ that lies inside $\mathcal{N}_D(\mathrm{TCH}(\tilde{Z}))$ by the geodesic with the same endpoints. Then the new path is composed of geodesic segments of length at least C, and by (3), the segments intersect with angles at least $\frac{\pi}{3}$. Therefore it is a quasi-geodesic. Since it is also closed, this is impossible. □

References

[Agol04] I. Agol. *Tameness of hyperbolic 3-manifolds,* preprint (2004).

[AC96] Anderson, J. & Canary, R. *Cores of hyperbolic* 3*-manifolds and limits of Kleinian groups.* Amer. J. Math. 118 (1996), no. 4, 745–779.

[Ahl73] Ahlfors, L. *Conformal invariants.* McGraw-Hill, New York, 1973.

[CG06] Calegari, D. & Gabai, D. *Shrinkwrapping and the taming of hyperbolic 3-manifolds.* J. Amer. Math. Soc. 19 (2006), no. 2, 385–446.

[BS05] Bachman, D. & Schleimer, S. *Surface bundles versus Heegaard splittings.* Comm. Anal. Geom. 13 (2005), no. 5, 903–928.

[BP92] Benedetti, R. & Petronio, C. *Lectures on Hyperbolic Geometry.*Springer, 1992.

[BH96] Bleiler, S. & Hodgson, C. *Spherical space forms and Dehn filling.* Topology 35 (1996), no. 3, 809–833.

[Bon86] Bonahon, F. *Bouts des varietes hyperboliques de dimension 3.* Ann. of math; 124 (1986), 71-158.

[Can93] Canary, R. *Ends of Hyperbolic* 3*-Manifolds. Jour. Amer. Math. Soc.,* **6,** 1-35.

[Can96] Canary, R. *A covering theorem for hyperbolic 3-manifolds and its applications.*Topology 3 (1996).

[Eva04] Evans, R. *Weakly type-preserving sequences and strong convergence.* Geom. Dedicata 108 (2004), 71–92.

[Hem76] Hempel, J. 3-*manifolds.* Princeton University Press, 1976.

[MT98] Matsuzaki, K. & Taniguchi, M. *Hyperbolic Manifolds and Kleinian Groups.* Oxford University Press, 1998.

[McM96] McMullen, C. *Renormalization and 3-manifolds which fiber over the circle.* Annals of Mathematics Studies, 142. Princeton University Press, Princeton, NJ, 1996.

[Sou05] Souto, J. *The Rank of the Fundamental Group of Hyperbolic* 3*-manifolds fibering over the circle,* preprint.

[Sou06] Souto, J. *Rank and Topology of Hyperbolic* 3*-manfolds,* preprint.

[Thu78] W. Thurston. *The geometry and topology of 3-manifolds.* Lecture notes, Princeton University, 1978.

[Thu98] Thurston, W. *Hyperbolic Structures on* 3*-Manifolds, I: Surface Groups and* 3*-Manifolds Fibering over the Circle.* 1998, preprint.

[Sc73] Scott, P. *Compact submanifolds of* 3*-manifolds.* J. London Math. Soc., 7, 246-250.

[Wal68] Waldhausen, F. *On irreducible* 3*-manifolds which are sufficiently large.* Ann. of Math, **87** 1968 56-88.

[Whi02] White, M. *Injectivity radius and fundamental groups of hyperbolic 3-manifolds.*Comm. Anal. Geom. 10 (2002).