ON DENSENESS OF HOROSPHERES IN HIGHER RANK HOMOGENEOUS SPACES

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Abstract. Let G be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. Let N denote a maximal horospherical subgroup of G, and $P = MAN$ the minimal parabolic subgroup which is the normalizer of N. Let $\mathcal{E}$ denote the unique P-minimal subset of $\Gamma\backslash G$ and let $\mathcal{E}_0$ be a P°-minimal subset. We consider a notion of a horospherical limit point in the Furstenberg boundary G/P and show that the following are equivalent for any $[g] \in \mathcal{E}_0$,

(1) $gP \in G/P$ is a horospherical limit point;
(2) $[g]NM$ is dense in $\mathcal{E}$;
(3) $[g]N$ is dense in $\mathcal{E}_0$.

The equivalence of (1) and (2) is due to Dal'bo in the rank one case. We also show that unlike convex cocompact groups of rank one Lie groups, the NM-minimality of $\mathcal{E}$ does not hold in a general Anosov homogeneous space.

1. Introduction

Let G be a connected semisimple real algebraic group. Let (X, d) denote the associated Riemannian symmetric space. Let $P = MAN$ be a minimal parabolic subgroup of G with fixed Langlands decomposition, where A is a maximal real split torus of G, M the maximal compact subgroup of P commuting with A and N the unipotent radical of P. Note that N is a maximal horospherical subgroup of G, which is unique up to conjugations.

Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \log A$ so that $\log N$ consists of positive root subspaces, and we set $A^+ = \exp \mathfrak{a}^+$. This means that N is a contracting horospherical subgroup in the sense that for any a in the interior of A^+,

$$N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}.$$

Let Γ be a *Zariski dense* discrete subgroup of G. In this paper, we are interested in the topological behavior of the action of the horospherical subgroup N on $\Gamma\backslash G$ via the right translations. When $\Gamma < G$ is a cocompact lattice, every N-orbit is dense in $\Gamma\backslash G$, i.e., the N-action on $\Gamma\backslash G$ is minimal. This is due to Hedlund [11] for $G = \mathrm{PSL}_2(\mathbb{R})$ and to Veech [19] in general. Dani gave a full classification of possible orbit closures of N-action for any lattice $\Gamma < G$ [6].

Oh is partially supported by NSF grant DMS-1900101.

For a general discrete subgroup $\Gamma < G$, the quotient space $\Gamma\backslash G$ does not necessarily admit a dense N-orbit, even a dense NM-orbit, for instance in the case where Γ does not have a full limit set. Let $\mathcal{F}$ denote the Furstenberg boundary G/P. We denote by $\Lambda = \Lambda_\Gamma$ the limit set of Γ,

$$\Lambda = \{\lim_{i\to\infty} \gamma_i(o) \in \mathcal{F} : \gamma_i \in \Gamma\}$$

where $o \in X$ and the convergence is understood as in Definition 2.2. This definition is independent of the choice of $o \in X$. The limit set Λ is known to be the unique Γ-minimal subset of $\mathcal{F}$ (see [1, 9, 14]). Thus the set

$$\mathcal{E} = \{[g] \in \Gamma\backslash G : gP \in \Lambda\}$$

is the unique P-minimal subset of $\Gamma\backslash G$. For a given point $[g] \in \mathcal{E}$, the topological behavior of the horospherical orbit $[g]N$ (or of $[g]NM$) is closely related to the ways in which the orbit $\Gamma(o)$ approaches gP along its limit cone. The limit cone $\mathcal{L} = \mathcal{L}_\Gamma$ of Γ is defined as the smallest closed cone of $\mathfrak{a}^+$ containing the Jordan projection $\lambda(\Gamma)$. It is a convex cone with non-empty interior: $\operatorname{int}\mathcal{L} \neq \emptyset$ [1]. If $\operatorname{rank} G = 1$, then $\mathcal{L} = \mathfrak{a}^+$. In higher ranks, the limit cone of Γ depends more subtly on Γ.

Horospherical limit points. Recall that in the rank one case, a horoball in X based at $\xi \in \mathcal{F}$ is a subset of the form $gN(\exp \mathfrak{a}^+)(o)$ where $g \in G$ is such that $\xi = gP$ [5]. Our generalization to higher rank of the notion of a horospherical limit point involves the limit cone of Γ. By a Γ-tight horoball based at $\xi \in \mathcal{F}$, we mean a subset of the form $\mathcal{H}_\xi = gN(\exp \mathcal{C})(o)$ where $g \in G$ is such that $\xi = gP$ and $\mathcal{C}$ is a closed cone contained in $\operatorname{int}\mathcal{L} \cup \{0\}$. For $T > 0$, we write

$$\mathcal{H}_\xi(T) = gN(\exp(\mathcal{C} - \mathcal{C}_T))o$$

where $\mathcal{C}_T = \{u \in \mathcal{C} : \|u\| < T\}$ for a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}$.

Definition 1.1. We call a limit point $\xi \in \Lambda$ a horospherical limit point of Γ if one of the following equivalent conditions holds:

- there exists a Γ-tight horoball $\mathcal{H}_\xi$ based at ξ such that for any $T > 1$, $\mathcal{H}_\xi(T)$ contains some point of $\Gamma(o)$;
- there exist a closed cone $\mathcal{C} \subset \operatorname{int}\mathcal{L} \cup \{0\}$ and a sequence $\gamma_j \in \Gamma$ satisfying that $\beta_\xi(o, \gamma_j o) \in \mathcal{C}$ for all $j \geq 1$ and $\beta_\xi(o, \gamma_j o) \to \infty$ as $j \to \infty$, where β denotes the $\mathfrak{a}$-valued Busemann map (Definition 2.3).

See Lemma 3.3 for the equivalence of the above two conditions. We denote by

$$\Lambda_h \subset \Lambda$$

the set of all horospherical limit points of Γ. The attracting fixed point y_γ of a loxodromic element $\gamma \in \Gamma$ whose Jordan projection $\lambda(\gamma)$ belongs to $\operatorname{int}\mathcal{L}$ is always a horospherical limit point (Lemma 3.5). Moreover, for any $u \in \operatorname{int}\mathcal{L}$, any u-directional radial limit point ξ (i.e, $\xi = gP$ for some $g \in G$ such that $\limsup_{t\to\infty} \Gamma g \exp(tu) \neq \emptyset$) is also a horospherical limit point (Lemma 5.3).

Remarks 1.2.

(1) There exists a notion of horospherical limit points in the geometric boundary associated to a symmetric space, see [10]. When $\operatorname{rank} G \geq 2$, this notion and the one considered here are different.
(2) Unlike the rank one case, a sequence $\gamma_i(o) \in \mathcal{H}_\xi(T_i)$, with $T_i \to \infty$, does not necessarily *converge* to ξ for a Γ-tight horoball $\mathcal{H}_\xi$ based at ξ. It is hence plausible that a general discrete group Γ would support a horospherical limit point outside of its limit set.

Denseness of horospheres. The following theorem generalizes Dal'bo's theorem [5] to discrete subgroups in higher rank semisimple Lie groups:

Theorem 1.3. *Let $\Gamma < G$ be a Zariski dense discrete subgroup. For any $[g] \in \mathcal{E}$, the following are equivalent:*

(1) $gP \in \Lambda_h$;
(2) $[g]NM$ is dense in $\mathcal{E}$.

Remarks 1.4. Conze and Guivarc'h considered the notion of a horospherical limit point for Zariski dense discrete subgroups Γ of $\mathrm{SL}_d(\mathbb{R})$ using the description of $\mathrm{SL}_d(\mathbb{R})/P$ as the full flag variety and the standard linear action of Γ on $\mathbb{R}^d$ [4]. By duality, this notion coincides with ours and hence the special case of Theorem 1.3 for $G = \mathrm{SL}_d(\mathbb{R})$ also follows from [4, Theorem 4.2][1].

In order to extend Theorem 1.3 to N-orbits, we fix a P°-minimal subset $\mathcal{E}_0$ of $\Gamma\backslash G$ where P° denotes the identity component of P. Clearly, $\mathcal{E}_0 \subset \mathcal{E}$. Since $P = P^\circ M$, any P°-minimal subset is a translate of $\mathcal{E}_0$ by an element of the finite group $M^\circ\backslash M$, where M° is the identity component of M. Denote by $\mathfrak{D}_\Gamma = \{\mathcal{E}_0, ..., \mathcal{E}_p\}$ the finite collection of all P°-minimal sets in $\mathcal{E}$. In order to understand N-orbit closures it is hence sufficient to restrict to $\mathcal{E}_0$.

The following is a refinement of Theorem 1.3:

Theorem 1.5. *Let $\Gamma < G$ be a Zariski dense discrete subgroup. For any $[g] \in \mathcal{E}_0$, the following are equivalent:*

(1) $gP \in \Lambda_h$;
(2) $[g]N$ is dense in $\mathcal{E}_0$.

Remark 1.6. We may consider horospherical limit points outside the context of Λ. In this case our proofs of Theorems 1.3 and 1.5 show that if $gP \in \mathcal{F}$ is a horospherical limit point, then the closures of $[g]MN$ and $[g]N$ contain $\mathcal{E}$ and $\mathcal{E}_i$, for some $\mathcal{E}_i \in \mathfrak{D}_\Gamma$, respectively.

For $G = \mathrm{SO}^\circ(n, 1)$, $n \geq 2$, Theorem 1.5 was proved in [16]. When G has rank one and $\Gamma < G$ is convex cocompact, every limit point is horospherical and Winter's mixing theorem [20] implies the N-minimality of $\mathcal{E}_0$.

[1] However the claim in [4, Theorem 6.3] is incorrect.

Directional horospherical limit points. We also consider the following seemingly much stronger notion:

Definition 1.7. For $u \in \mathfrak{a}^+$, a point $\xi \in \mathcal{F}$ is called u-horospherical if there exists a sequence $\gamma_j \in \Gamma$ such that $\sup_j \|\beta_\xi(o, \gamma_j o) - \mathbb{R}_+ u\| < \infty$ and $\beta_\xi(o, \gamma_j o) \to \infty$ as $j \to \infty$.

Denote by $\Lambda_h(u)$ the set of u-horospherical limit points. Surprisingly, it turns out that every horospherical limit point is u-horospherical *for all* $u \in \operatorname{int} \mathcal{L}$:

Theorem 1.8. *For all* $u \in \operatorname{int} \mathcal{L}$*, we have*

$$\Lambda_h = \Lambda_h(u).$$

Existence of non-dense horospheres. A finitely generated subgroup $\Gamma < G$ is called an Anosov subgroup (with respect to P) if there exists $C > 0$ such that for all $\gamma \in \Gamma$, $\alpha(\mu(\gamma)) \geq C|\gamma| - C$ for all simple roots α of $(\mathfrak{g}, \mathfrak{a}^+)$, where $\mu(\gamma) \in \mathfrak{a}^+$ denotes the Cartan projection of γ and $|\gamma|$ is the word length of γ with respect to a fixed finite generating set of Γ.

For Zariski dense Anosov subgroups of G, almost all NM-orbits are dense in $\mathcal{E}$ and almost all N-orbits are dense in $\mathcal{E}_0$ with respect to any Patterson-Sullivan measure on Λ ([14], [15]). In particular, the set of all horospherical limit points has full Patterson-Sullivan measures.

On the other hand, as Anosov subgroups are regarded as higher rank generalizations of convex cocompact subgroups, it is a natural question whether the minimality of the NM-action persists in the higher rank setting. It turns out that it is not the case. Our example is based on Thurston's theorem [18, Theorem 10.7] together with the following observation on the implication of the existence of a Jordan projection of an element of Γ lying in the boundary $\partial\mathcal{L}$ of the limit cone.

Proposition 1.9. *Let* $\Gamma < G$ *be a Zariski dense discrete subgroup. For any loxodromic element* $\gamma \in \Gamma$*, we have*

$$\lambda(\gamma) \in \operatorname{int} \mathcal{L} \quad \text{if and only if} \quad \{y_\gamma, y_{\gamma^{-1}}\} \subset \Lambda_h$$

where y_γ *and* $y_{\gamma^{-1}}$ *denote the attracting fixed points of* γ *and* γ^{-1} *respectively.*

In particular, if $\lambda(\Gamma) \cap \partial\mathcal{L} \neq \emptyset$*, then* $\Lambda \neq \Lambda_h$ *and hence there exists a non-dense* NM*-orbit in* $\mathcal{E}$*.*

Thurston's work [18] provides many examples of Anosov subgroups satisfying that $\lambda(\Gamma) \cap \partial\mathcal{L} \neq \emptyset$. To describe them, let Σ be a a torsion-free cocompact lattice of $\mathrm{PSL}_2(\mathbb{R})$ and let $\pi : \Sigma \to \mathrm{PSL}_2(\mathbb{R})$ be a discrete faithful representation. Let $0 < d_-(\pi) \leq d_+(\pi) < \infty$ be the minimal and maximal geodesic stretching constants:

$$d_+(\pi) = \sup_{\sigma \in \Sigma - \{e\}} \frac{\ell(\pi(\sigma))}{\ell(\sigma)} \quad \text{and} \quad d_-(\pi) = \inf_{\sigma \in \Sigma - \{e\}} \frac{\ell(\pi(\sigma))}{\ell(\sigma)} \tag{1.1}$$

where $\ell(\sigma)$ denotes the length of the closed geodesic in the hyperbolic manifold $\Sigma \backslash \mathbb{H}^2$ corresponding to σ and $\ell(\pi(\sigma))$ is defined similarly.

Consider the following self-joining subgroup

$$\Gamma_\pi := (\mathrm{id} \times \pi)(\Sigma) = \{(\sigma, \pi(\sigma)) : \sigma \in \Sigma\} < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}).$$

It is easy to see that Γ is an Anosov subgroup of $G = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. Moreover when π is not a conjugate by a Möbius tranformation, Γ_π is Zariski dense in G (cf. [12, Lemma 4.1]). Identifying $\mathfrak{a} = \mathbb{R}^2$, the Jordan projection $\lambda(\gamma_\pi)$ of $\gamma_\pi = (\sigma, \pi(\sigma)) \in \Gamma_\pi$ is given by $(\ell(\sigma), \ell(\pi(\sigma))) \in \mathbb{R}^2$. Hence the limit cone $\mathcal{L}$ of Γ_π is given by

$$\mathcal{L} := \{(v_1, v_2) \in \mathbb{R}^2_{\geq 0} : d_-(\pi)v_1 \leq v_2 \leq d_+(\pi)v_1\}.$$

Thurston [18, Theorem 10.7] showed that $d_+(\pi)$ is realized by a simple closed geodesic of $\Sigma\backslash\mathbb{H}^2$ in *most of cases*, which hence provides infinitely many examples of Γ_π which satisfy $\lambda(\Gamma_\pi) \cap \partial\mathcal{L} \neq \emptyset$. Therefore Proposition 1.9 implies (in this case, we have $NM = N$):

Corollary 1.10. *There are infinitely many non-conjuagte Zariski dense Anosov subgroups $\Gamma_\pi < \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ with non-dense NM-orbits in $\mathcal{E}$.*

We close the introduction by the following question (cf. [13],[17]):

Question 1.11. For a simple real algebraic group G with $\operatorname{rank} G \geq 2$, is every discrete subgroup $\Gamma < G$ with $\Lambda = \Lambda_h = \mathcal{F}$ necessarily a cocompact lattice in G?

Acknowledgments. We would like to thank Dick Canary and Pratyush Sarkar for helpful conversations regarding Corollary 1.10. The first named author would also like to thank Subhadip Dey and Ido Grayevsky for helpful and enjoyable discussions. We thank the anonymous referee for pointing out to us the paper [4].

2. Preliminaries

Let G be a connected, semisimple real algebraic group. We fix, once and for all, a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$,

Choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A := \exp \mathfrak{a}$ and $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M, and we set N to be the maximal contracting horospherical subgroup: for $a \in \operatorname{int} A^+$,

$$N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}.$$

We set $P = MAN$, which is the unique minimal parabolic subgroup of G, up to conjugation.

For $u \in \mathfrak{a}$, we write $a_u = \exp u \in A$. We denote by $\|\cdot\|$ the norm on $\mathfrak{g}$ induced by the Killing form. Consider the Riemannian symmetric space $X := G/K$ with the metric induced from the norm $\|\cdot\|$ on $\mathfrak{g}$ and $o = K \in X$.

Let $\mathcal{F} = G/P$ denote the Furstenberg boundary. Since K acts transitively on $\mathcal{F}$ and $K \cap P = M$, we may identify $\mathcal{F} = K/M$. We denote by $\mathcal{F}^{(2)}$ the unique open G-orbit in $\mathcal{F} \times \mathcal{F}$.

Denote by $w_0 \in K$ the unique element in the Weyl group such that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$; it is the longest Weyl element. We then have $\check{P} := w_0 P w_0^{-1}$ is an opposite parabolic subgroup of G, with $\check{N}$ its unipotent radical. The map $\mathrm{i} = -\mathrm{Ad}_{w_0} : \mathfrak{a}^+ \to \mathfrak{a}^+$ is called the opposition involution.

For $g \in G$, we consider the following visual maps

$$g^+ := gP \in \mathcal{F} \quad \text{and} \quad g^- := gw_0P \in \mathcal{F}.$$

Then $\mathcal{F}^{(2)} = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}$.

Any element $g \in G$ can be uniquely decomposed as the commuting product g_h, g_e, g_u, where g_h, g_e, and g_u are hyperbolic, elliptic and unipotent elements respectively. The Jordan projection of g is defined as the element $\lambda(g) \in \mathfrak{a}^+$ satisfying $g_h = \varphi \exp \lambda(g) \varphi^{-1}$ for some $\varphi \in G$.

An element $g \in G$ is called loxodromic if $\lambda(g) \in \operatorname{int} \mathfrak{a}^+$; in this case, g_u is necessarily trivial. For a loxodromic element $g \in G$, the point $\varphi^+ \in \mathcal{F}$ is called the attracting fixed point of g, which we denote by y_g. For any loxodromic element $g \in G$ and $\xi \in \mathcal{F}$ with $(\xi, y_{g^{-1}}) \in \mathcal{F}^{(2)}$, we have $\lim_{k\to\infty} g^k \xi = y_g$ and the convergence is uniform on compact subsets.

Note that for any loxodromic element $g \in G$,

$$\lambda(g^{-1}) = \mathrm{i}\, \lambda(g).$$

Let $\Gamma < G$ be a Zariski dense discrete subgroup of G. The limit cone $\mathcal{L} = \mathcal{L}_\Gamma$ of Γ is the smallest closed cone of $\mathfrak{a}^+$ containing $\lambda(\Gamma)$. It is a convex cone with non-empty interior [1].

We will use the following simple lemma.

Lemma 2.1. *For any $v \in \lambda(\Gamma)$ and $\zeta \in \mathcal{F}$, there exists a loxodromic element $\gamma \in \Gamma$ with $\lambda(\gamma) = v$ and a neighborhood U of ζ in $\mathcal{F}$ such that $\{y_\gamma\} \times U$ is a relatively compact subset of $\mathcal{F}^{(2)}$ and as $k \to \infty$,*

$$\gamma^{-k}\zeta \to y_{\gamma^{-1}} \quad \textit{uniformly on } U.$$

Proof. Let $\zeta \in \mathcal{F}$. Choose $\gamma_1 \in \Gamma$ such that $\lambda(\gamma_1) = v$. Since the set of all loxodromic elements of Γ is Zariski dense in G [2] and $\mathcal{F}^{(2)}$ is Zariski open in $\mathcal{F} \times \mathcal{F}$, there exists $\gamma_2 \in \Gamma$ such that $(\zeta, \gamma_2 y_{\gamma_1}) \in \mathcal{F}^{(2)}$. Let $\gamma = \gamma_2\gamma_1\gamma_2^{-1}$, so that $y_\gamma = \gamma_2 y_{\gamma_1}$. It now suffices to take any neighborhood U of ζ such that $U \times \{\gamma_2 y_{\gamma_1}\}$ is a relatively compact subset of $\mathcal{F}^{(2)}$. ∎

Convergence of a sequence in X to $\mathcal{F}$. By the Cartan decomposition $G = KA^+K$, for $g \in G$, we may write

$$g = \kappa_1(g) \exp(\mu(g)) \kappa_2(g) \in KA^+K$$

where $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of g, is uniquely determined, and $\kappa_1(g), \kappa_2(g) \in K$. If $\mu(g) \in \operatorname{int} \mathfrak{a}^+$, then $[\kappa_1(g)] \in K/M = \mathcal{F}$ is uniquely determined.

Let Π be the set of simple roots for $(\mathfrak{g}, \mathfrak{a})$. For a sequence $g_i \to G$, we say $g_i \to \infty$ regularly if $\alpha(\mu(g_i)) \to \infty$ for all $\alpha \in \Pi$. Note that if $g_i \to \infty$ regularly, then for all sufficiently large i, $\mu(g_i) \in \operatorname{int} \mathfrak{a}^+$ and hence $[\kappa_1(g_i)]$ is well-defined.

Definition 2.2. A sequence $p_i \in X$ is said to converge to $\xi \in \mathcal{F}$ if there exists $g_i \to \infty$ regularly in G with $p_i = g_i(o)$ and $\lim_{i\to\infty}[\kappa_1(g_i)] = \xi$.

P°-minimal subsets. We denote by $\Lambda \subset \mathcal{F}$ the limit set of Γ, which is defined as

$$\Lambda = \{\lim \gamma_i(o) : \gamma_i \in \Gamma\}. \tag{2.1}$$

For a non-Zariski dense subgroup, Λ may be an empty set. For $\Gamma < G$ Zariski dense, this is the unique Γ-minimal subset of $\mathcal{F}$ ([1], [14]).

It follows that the following set $\mathcal{E}$ is the unique P-minimal subset of $\Gamma\backslash G$:

$$\mathcal{E} = \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\}.$$

Let P° denote the identity component of P. Then $\mathcal{E}$ is a disjoint union of at most $[P : P^\circ]$-number of P°-minimal subsets. We fix one P°-minimal subset $\mathcal{E}_0$ once and for all. Note that any P°-minimal subset is then of the form $\mathcal{E}_0 m$ for some $m \in M$. We set

$$\Omega := \{[g] \in \Gamma\backslash G : g^+, g^- \in \Lambda\} \quad \text{and} \quad \Omega_0 := \Omega \cap \mathcal{E}_0. \tag{2.2}$$

Busemann map. The Iwasawa cocycle $\sigma : G \times \mathcal{F} \to \mathfrak{a}$ is defined as follows: for $(g, \xi) \in G \times \mathcal{F}$ with $\xi = [k]$ for $k \in K$, $\exp \sigma(g, \xi)$ is the A-component of gk in the KAN decomposition, that is,

$$gk \in K \exp(\sigma(g, \xi))N.$$

The $\mathfrak{a}$-valued Busemann function $\beta : \mathcal{F} \times X \times X \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(ho, go) := \sigma(h^{-1}, \xi) - \sigma(g^{-1}, \xi).$$

We note that for any $g \in G$, $\xi \in \mathcal{F}$, and $x, y, z \in X$,

$$\beta_\xi(x, y) = \beta_{g\xi}(gx, gy), \quad \text{and} \quad \beta_\xi(x, y) = \beta_\xi(x, z) + \beta_\xi(z, y). \tag{2.3}$$

In particular, $\beta_\xi(o, go) \in \mathfrak{a}$ is defined by

$$g^{-1}k_\xi \in K \exp(-\beta_\xi(o, go))N, \tag{2.4}$$

and hence $\beta_P(o, a_u o) = u$ for any $u \in \mathfrak{a}$. For $h, g \in G$, we set $\beta_\xi(h, g) := \beta_\xi(ho, go)$.

Shadows. For $q \in X$ and $r > 0$, we set $B(q,r) = \{x \in X : d(x,q) \le r\}$. For $p = g(o) \in X$, the shadow of the ball $B(q,r)$ viewed from p is defined as

$$O_r(p,q) := \{(gk)^+ \in \mathcal{F} : k \in K,\ gk \operatorname{int} A^+ o \cap B(q,r) \neq \emptyset\}.$$

Similarly, for $\xi \in \mathcal{F}$, the shadow of the ball $B(q,r)$ as viewed from ξ is

$$O_r(\xi,q) := \{h^+ \in \mathcal{F} : h \in G \text{ satisfies } h^- = \xi,\ ho \in B(q,r)\}.$$

Lemma 2.3. [14, Lemma 5.6 and 5.7]

(1) There exists $\kappa > 0$ such that for any $g \in G$ and $r > 0$,

$$\sup_{\xi \in O_r(g(o),o)} \|\beta_\xi(g(o),o) - \mu(g^{-1})\| \le \kappa r.$$

(2) If a sequence $p_i \in X$ converges to $\xi \in \mathcal{F}$, then for any $0 < \varepsilon < r$, we have

$$O_{r-\varepsilon}(p_i,o) \subset O_r(\xi,o) \subset O_{r+\varepsilon}(p_i,o)$$

for all sufficiently large i.

3. Horospherical limit points

Let $\Gamma < G$ be a Zariski dense discrete subgroup. A Γ-tight horoball based at $\xi \in \mathcal{F}$ is a subset of the form $\mathcal{H}_\xi = gN(\exp \mathcal{C})(o)$ where $g \in G$ is such that $\xi = gP$ and $\mathcal{C}$ is a closed cone contained in $\operatorname{int} \mathcal{L} \cup \{0\}$. For $T > 0$, we write $\mathcal{H}_\xi(T) = gN(\exp(\mathcal{C} - \mathcal{C}_T))o$. We recall the definition from the introduction:

Definition 3.1. We say that $\xi \in \mathcal{F}$ is a horospherical limit point of Γ if there exists a Γ-tight horoball $\mathcal{H}_\xi$ based at ξ such that $\mathcal{H}_\xi(T) \cap \Gamma(o) \neq \emptyset$ for all $T > 1$.

In this section we provide a mostly self-contained proof of the following theorem:

Theorem 3.2. *Let $[g] \in \mathcal{E}$. The following are equivalent:*

(1) $g^+ = gP \in \Lambda$ is a horospherical limit point;

(2) $[g]NM$ is dense in $\mathcal{E}$.

The main external ingredient in our proof is the density of the group generated by the Jordan projection $\lambda(\Gamma)$, due to Benoist [2], that is,

$$\mathfrak{a} = \overline{\langle \lambda(\Gamma) \rangle}$$

for every Zariski dense discrete subgroup $\Gamma < G$. In fact, for every cone $\mathcal{C} \subset \mathcal{L}$ with non-empty interior, there exists a Zariski dense subgroup $\Gamma' < \Gamma$ with $\mathcal{L}_{\Gamma'} \subset \mathcal{C}$ (see [1]); therefore we have

$$\mathfrak{a} = \overline{\langle \lambda(\Gamma) \cap \operatorname{int} \mathcal{L} \rangle}.$$

It is convenient to use a characterization of horospherical limit points in terms of the Busemann function.

Lemma 3.3. *For $\xi \in \Lambda$, we have $\xi \in \Lambda_h$ if and only if there exists a closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ and a sequence $\gamma_j \in \Gamma$ satisfying*

$$\beta_\xi(o, \gamma_j o) \to \infty \quad \textit{and} \quad \beta_\xi(o, \gamma_j o) \in \mathcal{C} \textit{ for all large } j \geq 1. \tag{3.1}$$

Proof. Let $\xi = gP \in \Lambda_h$ be as defined in Definition 3.1. Then there exists $\gamma_j = gpn_j a_{u_j} k_j \in \Gamma$ for some $p \in P$, $n_j \in N$, $k_j \in K$ and $u_j \to \infty$ in some closed cone $\mathcal{C}$ contained in $\operatorname{int} \mathcal{L} \cup \{0\}$. Fix some closed cone $\mathcal{C}' \subset \operatorname{int} \mathcal{L} \cup \{0\}$ whose interior contains $\mathcal{C}$. Note that

$$\begin{aligned} \beta_\xi(o, \gamma_j o) &= \beta_{gP}(e, g) + \beta_{gP}(g, gpn_j a_{u_j}) \\ &= \beta_P(g^{-1}, e) + \beta_P(e, p) + \beta_P(e, n_j) + \beta_P(e, a_{u_j}) \\ &= \beta_P(g^{-1}, p) + u_j. \end{aligned}$$

Therefore the sequence $\beta_\xi(o, \gamma_j) - u_j$ is uniformly bounded. Since $u_j \in \mathcal{C}$, $\beta_\xi(o, \gamma_j o) \in \mathcal{C}'$ for all large j. Therefore (3.1) holds. For the other direction, let γ_j and $\mathcal{C}$ satisfy (3.1) for $\xi = gP$ for $g \in G$. Since $G = gNAK$, we may write $\gamma_j = gn_j a_{u_j} k_j$ for some $n_j \in N, u_j \in \mathfrak{a}$ and $k_j \in K$. By a similar computation as above, the sequence $\beta_\xi(o, \gamma_j o) - u_j$ is uniformly bounded. It follows that $u_j \in \mathcal{C}'$ for all large j and $u_j \to \infty$. Therefore for any $T > 1$, there exists $j > 1$ such that $\gamma_j(o) \in gN \exp(\mathcal{C}' - \mathcal{C}'_T)(o)$. This proves $\xi \in \Lambda_h$. □

We note that condition (3.1) is independent of the choice of basepoint o. Indeed, for any $g \in G$ and $\xi \in \mathcal{F}$ and for all $\gamma \in \Gamma$ we have

$$\beta_\xi(o, \gamma o) = \beta_\xi(o, go) + \beta_\xi(go, \gamma go) + \beta_\xi(\gamma go, \gamma o),$$

and hence

$$\begin{aligned} \|\beta_\xi(o, \gamma o) - \beta_\xi(go, \gamma go)\| &= \|\beta_\xi(o, go) + \beta_\xi(\gamma go, \gamma o)\| \\ &= \|\beta_\xi(o, go) - \beta_{\gamma^{-1}\xi}(o, go)\| \\ &\leq 2 \cdot \max_{\eta \in \mathcal{F}} \|\beta_\eta(o, go)\|. \end{aligned}$$

Since this bound is independent of $\gamma \in \Gamma$, condition (3.1) implies that for any $p = go \in X$,

$$\beta_\xi(p, \gamma_j p) \to \infty \quad \text{and} \quad \beta_\xi(p, \gamma_j p) \in \mathcal{C} \text{ for all large } j. \tag{3.2}$$

Let us now consider the following seemingly stronger condition for a limit point being horospherical:

Definition 3.4. For $u \in \mathfrak{a}^+$, a point $\xi \in \mathcal{F}$ is called a u-horospherical limit point if for some $p \in X$ (and hence for any $p \in X$), there exists a constant $R > 0$ and a sequence $\gamma_j \in \Gamma$ satisfying

$$\beta_\xi(p, \gamma_j p) \to \infty \quad \text{and} \quad \|\beta_\xi(p, \gamma_j p) - \mathbb{R}_+ u\| < R \quad \text{for all } j.$$

We denote the set of u-horospherical limit points by $\Lambda_h(u)$.

By G-invariance of the Busemann map, the set of horospherical (resp. u-horospherical) limit points is Γ-invariant. Therefore for $x = [g] \in \Gamma\backslash G$, we may say $x^+ := \Gamma g P$ horospherical (resp. u-horospherical) if g^+ is.

For $u \in \mathfrak{a}$, we call $x \in \Gamma\backslash G$ a u-periodic point if $xa_u = xm_0$ for some $m_0 \in M$; note that $xa_{\mathbb{R}u}M_0$ is then compact. Note that for $u \in \operatorname{int}\mathfrak{a}^+$, the existence of a u-periodic point is equivalent to the condition that $u \in \lambda(\Gamma)$.

Lemma 3.5. *Let $u \in \mathfrak{a}^+$. If $x \in \Gamma\backslash G$ is u-periodic, then $x^+ \in \mathcal{F}$ is a u-horospherical limit point.*

Proof. Since x is u-periodic, there exist $g \in G$ with $x = [g]$ and $\gamma \in \Gamma$ such that $\gamma = ga_u m g^{-1}$ for some $m \in M$, and $y_\gamma = g^+ \in \Lambda$. Moreover, for any $k \geq 1$

$$\beta_{gP}(go, \gamma^k go) = \beta_P(o, a_u^k o) = ku.$$

This implies gP is u-horospherical. □

Proposition 3.6. *Let $x \in \Gamma\backslash G$. If x^+ is u-horospherical for some $u \in \lambda(\Gamma)$ then the closure $\overline{xN}$ contains a u-periodic point.*

Proof. Choose $g \in G$ so that $x = [g]$. We may assume without loss of generality that $g = k \in K$, since $kanN = kNa$, and a translate of a u-periodic point by an element of A is again a u-periodic point. Since $u \in \lambda(\Gamma)$, there exists a u-periodic point, say, $x_0 \in \Gamma\backslash G$. It suffices to show that

$$\overline{[k]N} \cap x_0 AM \neq \emptyset \tag{3.3}$$

as every point in x_0AM is u-periodic.

Since k^+ is u-horospherical and using (2.4), there exists $R > 0$ and sequences $\gamma_j \in \Gamma$, $u_j \to \infty$ in $\mathfrak{a}^+$ and $k_j \in K$ and $n_j \in N$ satisfying $\gamma_j^{-1}k = k_j a_{-u_j} n_j$ or

$$k_j = \gamma_j^{-1} k n_j^{-1} a_{u_j}, \tag{3.4}$$

with $\|\mathbb{R}_+u - u_j\| < R$ for all j. Let $\ell_j \to \infty$ be a sequence of integers satisfying

$$\|\ell_j u - u_j\| < R + \|u\| \quad \text{for all } j \geq 1. \tag{3.5}$$

By passing to a subsequence, we may assume without loss of generality that $\gamma_j^{-1}kP$ converges to some $\xi_0 \in \mathcal{F}$. Since $\check{N}P$ is Zariski open and Γ is Zariski dense, we may choose $g_0 \in G$ such that $x_0 = [g_0]$ and $g_0^{-1}\xi_0 \in \check{N}P$. Let $h_0 \in \check{N}$ be such that $\xi_0 = g_0h_0P$. Since $g_0\check{N}P$ is open and $\gamma_j^{-1}kP \to g_0h_0P$, we may assume that for all j, there exists $h_j \in \check{N}$ satisfying $g_0h_jP = \gamma_j^{-1}kP = k_jP$ with $h_j \to h_0$. Let $p_j = a_{v_j}m_j\tilde{n}_j \in P = AMN$ be such that $g_0h_jp_j = k_j$; since $h_j \to h_0$ and the product map $\check{N} \times P \to \check{N}P$ is a diffeomorphism, the sequence p_j, as well as $v_j \in \mathfrak{a}$, are bounded.

Therefore by (3.4), we get for all j,

$$\begin{aligned} g_0 &= k_j p_j^{-1} h_j^{-1} \\ &= \gamma_j^{-1} k n_j^{-1} a_{u_j} (\tilde{n}_j^{-1} m_j^{-1} a_{-v_j}) h_j^{-1} \\ &= \gamma_j^{-1} k n_j^{-1} (a_{u_j} \tilde{n}_j^{-1} a_{-u_j}) a_{u_j} m_j^{-1} a_{-v_j} h_j^{-1} \\ &= \gamma_j^{-1} k n_j^{-1} (a_{u_j} \tilde{n}_j^{-1} a_{-u_j}) m_j^{-1} (a_{u_j - v_j} h_j^{-1} a_{-u_j + v_j}) a_{u_j - v_j}. \end{aligned}$$

Since $h_j^{-1} \in \check{N}$ and $v_j \in \mathfrak{a}$ are uniformly bounded and since $u_j \to \infty$ within a bounded neighborhood of the ray $\mathbb{R}_+ u \in \operatorname{int} \mathfrak{a}^+$, we have

$$\tilde{h}_j = a_{u_j - v_j} h_j^{-1} a_{-u_j + v_j} \to e \quad \text{in } \check{N}.$$

By setting $n'_j = n_j^{-1}(a_{u_j} \tilde{n}_j^{-1} a_{-u_j}) \in N$, we may now write

$$g_0 = \gamma_j^{-1} k n'_j m_j^{-1} \tilde{h}_j a_{u_j - v_j}.$$

Since x_0 is u-periodic, there exists $\gamma_0 \in \Gamma$ such that $\gamma_0 = g_0 a_u m_0 g_0^{-1}$ for some $m_0 \in M$. Hence for all $j \geq 1$,

$$\gamma_0^{-\ell_j} = g_0 a_{-\ell_j u} m_0^{-\ell_j} g_0^{-1} = (\gamma_j^{-1} k n'_j m_j^{-1} \tilde{h}_j a_{u_j - v_j})(a_{-\ell_j u} m_0^{-\ell_j}) g_0^{-1}.$$

In other words,

$$\gamma_j^{-1} k n'_j = \gamma_0^{-\ell_j} g_0 m_0^{\ell_j} a_{-u_j + \ell_j u + v_j} \tilde{h}_j^{-1} m_j.$$

Since the sequence $-u_j + \ell_j u + v_j \in \mathfrak{a}$ is uniformly bounded by (3.5) and $\tilde{h}_j \to e$ in $\check{N}$, we conclude that the sequence $\Gamma k n'_j$ has an accumulation point in $\Gamma g_0 A M$. This proves (3.3). $\square$

It turns out that a horospherical limit point is also u-horospherical for any $u \in \operatorname{int} \mathcal{L}$:

Proposition 3.7. *For each $u \in \operatorname{int} \mathcal{L}$, we have $\Lambda_h = \Lambda_h(u)$.*

Proof. Let $\xi \in \Lambda_h$. By definition, there is a sequence $\gamma_j \in \Gamma$ satisfying $v_j := \beta_\xi(e, \gamma_j) \to \infty$ with the sequence $\|v_j\|^{-1} v_j$ converging to some point $v_0 \in \operatorname{int} \mathcal{L}$. By passing to a subsequence, we may assume that $\gamma_j^{-1} \xi$ converges to some $\xi_0 \in \mathcal{F}$.

Let $u \in \operatorname{int} \mathcal{L}$. We claim that $\xi \in \Lambda_h(u)$. We first consider the case $u \notin \mathbb{R}_+ v_0$. Let $r := \operatorname{rank} G - 1 \geq 0$. Since $\cup_{\gamma \in \Gamma} \mathbb{R}_+ \lambda(\gamma)$ is dense in $\mathcal{L}$, there exist $w_1, \cdots, w_r \in \lambda(\Gamma)$ such that v_0 belongs to the interior of the convex cone spanned by $u, w_1, \cdots, w_r$, so that

$$v_0 = c_0 u + \sum_{\ell=1}^{r} c_\ell w_\ell$$

for some positive constants $c_0, \cdots, c_\ell$.

Since $\|v_j\|^{-1}v_j \to v_0$, we may assume, by passing to a subsequence, that for each $j \geq 1$, we have

$$\|v_j\|^{-1}v_j = c_{0,j}u + \sum_{\ell=1}^{r} c_{\ell,j}w_\ell \tag{3.6}$$

for some positive $c_{\ell,j}$, $\ell = 0, \cdots, r$. Note that for each $0 \leq \ell \leq r$, $c_{\ell,j} \to c_\ell$ as $j \to \infty$.

By Lemma 2.1, we can find a loxodromic element $g_1 \in \Gamma$ and a neighborhood U_1 of ξ_0 such that $\lambda(g_1^{-1}) = w_1$, $\{y_{g_1}\} \times U_1 \subset \mathcal{F}^{(2)}$ and $g_1^{-k}U_1 \to y_{g_1^{-1}}$ uniformly. Applying Lemma 2.1 once more, we can find $g_2 \in \Gamma$ satisfying $\lambda(g_2^{-1}) = w_2$ and a neighborhood $U_2 \subset \mathcal{F}$ of $y_{g_1^{-1}}$ satisfying $\{y_{g_2}\} \times U_2 \subset \mathcal{F}^{(2)}$ and that $g_2^{-k}U_2 \to y_{g_2^{-1}}$ uniformly.

Continuing inductively, we get elements $g_1, ..., g_r \in \Gamma$ and open sets $U_1, ..., U_r \subset \mathcal{F}$ satisfying that for all $\ell = 1, ..., r$,

(1) $w_\ell = \lambda(g_\ell^{-1})$;
(2) $y_{g_{\ell-1}^{-1}} \in U_\ell$;
(3) $g_\ell^{-k}U_\ell \to y_{g_\ell^{-1}}$ uniformly; and
(4) $\{y_{g_\ell}\} \times U_\ell$ is a relatively compact subset of $\mathcal{F}^{(2)}$.

We set $\xi_\ell := y_{g_\ell^{-1}}$ for each $1 \leq \ell \leq r$; so U_ℓ is a neighborhood of $\xi_{\ell-1}$ for each $1 \leq \ell \leq r$.

Since $\mathcal{Q}_{\eta_0} := \{\eta \in \mathcal{F} : (\eta_0, \eta) \in \mathcal{F}^{(2)}\} = \bigcup_{R>0} O_R(\eta_0, o)$ for any $\eta_0 \in \mathcal{F}$ and $U_\ell \subset \mathcal{Q}_{y_{g_\ell}}$ is a relatively compact subset of $\mathcal{F}^{(2)}$, there exists $R_\ell > 0$ such that $U_\ell \subset O_{R_\ell}(y_{g_\ell}, o)$. Since $g_\ell^k o$ converges to y_{g_ℓ} as $k \to +\infty$, by Lemma 2.3(2),

$$O_{R_\ell}(y_{g_\ell}o, o) \subset O_{R_\ell+1}(g_\ell^k o, o) \tag{3.7}$$

for all sufficiently large $k > 1$.

For each $1 \leq \ell \leq r$ and $j \geq 1$, let $k_{\ell,j}$ be the largest integer smaller than $c_{\ell,j}\|v_j\|$. As $\|v_j\| \to \infty$, and $c_{\ell,j} \to c_\ell$, we have $k_{\ell,j} \to \infty$ as $j \to \infty$. By the uniform contraction $g_\ell^{-k}U_i \to \xi_\ell$, there exists $j_0 > 1$ such that for all $j \geq j_0$,

$$\gamma_j^{-1}\xi \in U_1, \quad g_\ell^{-k_{\ell,j}}U_\ell \subseteq U_{\ell+1}, \quad \text{and} \quad U_\ell \subset O_{R_\ell+1}(g_\ell^{k_{\ell,j}}o, o) \tag{3.8}$$

for all $\ell = 1, ..., r$.

For each $j \geq j_0$, we now set

$$\tilde{\gamma}_j := \gamma_j g_1^{k_{1,j}} g_2^{k_{2,j}} \cdots g_r^{k_{r,j}} \in \Gamma.$$

We claim that $\beta_\xi(e, \tilde{\gamma}_j) \to \infty$ as $j \to \infty$ and that

$$\sup_{j \geq j_0} \|\beta_\xi(e, \tilde{\gamma}_j) - \mathbb{R}_+ u\| < \infty; \tag{3.9}$$

this proves that ξ is u-horospherical.

Fix $j \geq j_0$ and for each $1 \leq \ell \leq r$, let $k_\ell := k_{\ell,j}$, $b_\ell := c_{\ell,j}\|v_j\|$, and set

$$h_\ell = g_1^{k_1} g_2^{k_2} \cdots g_\ell^{k_\ell},$$

and $g_0 = e$. The cocycle property of the Busemann function gives that

$$\beta_\xi(e, \tilde{\gamma}_j) = \beta_\xi(e, \gamma_j) - \sum_{\ell=1}^{r} \beta_{h_{\ell-1}^{-1}\gamma_j^{-1}\xi}(g_\ell^{k_\ell}, e). \tag{3.10}$$

By (3.8), $\gamma_j^{-1}\xi \in U_1$ and for each $1 \leq \ell \leq r$,

$$h_{\ell-1}^{-1}\gamma_j^{-1}\xi \in g_\ell^{-k_\ell} \cdots g_1^{-k_1} U_1 \subset U_{\ell+1} \subset O_{R_\ell+1}(g_\ell^{k_\ell} o, o).$$

Hence by Lemma 2.3(1), there exists $\kappa \geq 1$ such that for each $1 \leq \ell \leq r$

$$\|\beta_{h_{\ell-1}^{-1}\gamma_j^{-1}\xi}(g_\ell^{k_\ell}, e) - \mu(g_\ell^{-k_\ell})\| \leq \kappa(R_\ell + 1).$$

Note that for some $C_\ell > 0$, $\|\mu(g_\ell^{-k}) - k\lambda(g_\ell^{-1})\| \leq C_\ell$ for all $k \geq 1$. Since $\lambda(g_\ell^{-1}) = w_\ell$, we get

$$\|\beta_{h_{\ell-1}^{-1}\gamma_j^{-1}\xi}(g_\ell^{k_\ell}, e) - k_\ell w_\ell\| \leq \kappa(R_\ell + 1) + C_\ell.$$

Therefore by (3.10), we obtain

$$\|\beta_\xi(e, \tilde{\gamma}_j) - (v_j - \sum_{\ell=1}^{r} k_\ell w_\ell)\| \leq \kappa \sum_{\ell=1}^{r} (R_\ell + C_\ell + 1).$$

By (3.6), we have

$$c_{0,j}\|v_j\| u = v_j - \sum_{\ell=1}^{r} b_\ell w_\ell.$$

Since $|b_\ell - k_\ell| \leq 1$ and $c_{0,j} > 0$, we deduce that for all $j \geq j_0$,

$$\begin{aligned}
&\|\beta_\xi(e, \tilde{\gamma}_j) - \mathbb{R}_+ u\| \leq \|\beta_\xi(e, \tilde{\gamma}_j) - c_{0,j}\|v_j\| \cdot u\| \\
&\leq \left\| \beta_\xi(e, \tilde{\gamma}_j) - (v_j - \sum_{\ell=1}^{r} k_\ell w_\ell) \right\| + \sum_{\ell=1}^{r} \|k_\ell w_\ell - b_\ell w_\ell\| \\
&\leq \kappa \sum_{\ell=1}^{r} (R_\ell + C_\ell + \|w_\ell\| + 1).
\end{aligned}$$

This proves (3.9), and consequently ξ is u-horospherical for any $u \notin \mathbb{R}_+ v_0$. To show that ξ is v_0-horospherical, fix any $u \notin \mathbb{R}_+ v_0$ and $\tilde{\gamma}_j \in \Gamma$ be a sequence as in (3.9) associated to u. If we set $\tilde{v}_j = \beta_\xi(e, \tilde{\gamma}_j)$, then $\|\tilde{v}_j\|^{-1}\tilde{v}_j$ converges to a unit vector in $\operatorname{int} \mathcal{L}$ proportional to u. Therefore by repeating the same argument only now switching the roles of v_0 and u, we prove that ξ is v_0-horospherical as well. This completes the proof. □

We may now prove theorem 3.2:

Proof of theorem 3.2. Let $g \in G$ be such that $\xi = g^+ \in \Lambda$ is a horospherical limit point. Set $Y := \overline{[g]NM}$. We claim that $Y = \mathcal{E}$. By Benoist [1], the group generated by $\lambda(\Gamma) \cap \operatorname{int} \mathcal{L}$ is dense in $\mathfrak{a}$. Hence for every $\varepsilon > 0$ there exist loxodromic elements $\gamma_1, ..., \gamma_q \in \Gamma$ such that

$$\lambda(\gamma_1), ..., \lambda(\gamma_q) \in \operatorname{Int}\mathcal{L}$$

and the group $\mathbb{Z}\lambda(\gamma_1) + \cdots + \mathbb{Z}\lambda(\gamma_q)$ is an ε-net in $\mathfrak{a}$, i.e., its ε-neighborhood covers all $\mathfrak{a}$. Denote $u_i = \lambda(\gamma_i)$ for $i = 1, ..., q$. By Proposition 3.7, the point ξ is u_1-horospherical. By Proposition 3.6, there exists a u_1-periodic point $x_1 \in \mathcal{E}$ contained in Y, set

$$Y_1 := \overline{x_1 NM} \subset Y.$$

By Lemma 3.5, x_1^+ is u_1-horospherical; in particular, it is a horospherical limit point. Therefore we can inductively find a u_i-periodic point x_i in $Y_{i-1} = \overline{x_{i-1}NM}$ for each $2 \le i \le q$. By periodicity $x_i(\exp u_i)M = x_i M$, and hence $Y_i \exp \mathbb{Z}u_i = Y_i$ for each $1 \le i \le q$. Therefore we obtain

$$Y \supset Y_1 \exp \mathbb{Z}u_1 \supset Y_2 \exp(\mathbb{Z}u_1 + \mathbb{Z}u_2) \supset \cdots \supset Y_q \exp\,(\sum_{i=1}^{q} \mathbb{Z}u_i).$$

Recalling the dependence of Y_q and $\sum_{i=1}^q \mathbb{Z}u_i$ on ε, set

$$Z_\varepsilon := Y_q MN \exp(\sum_{i=1}^{q} \mathbb{Z}u_i) \subset Y.$$

Since $MN \exp(\sum_{i=1}^q \mathbb{Z}u_i)$ is an ε-net of P and $\mathcal{E}$ is P-minimal, Z_ε is a 2ε-net of $\mathcal{E}$ for all $\varepsilon > 0$. Since Y contains a 2ε-net of $\mathcal{E}$ for all $\varepsilon > 0$ and Y is closed, it follows that $Y = \mathcal{E}$.

For the other direction, suppose that $[g]NM$ is dense in $\mathcal{E}$ for $g \in G$. Choose any $u \in \operatorname{int} \mathcal{L}$ and a closed cone $\mathcal{C} \subset \operatorname{int} \mathcal{L} \cup \{0\}$ which contains u. Then $\mathcal{H}_\xi = gN(\exp \mathcal{C})(o)$ is a Γ-tight horoball. Let $t > 1$. Since $ga_{-2tu} \in \mathcal{E}$, there exist $\gamma_i \in \Gamma$, $n_i \in N$, $m_i \in M$ and $q_i \to e$ in G such that for all $i \ge 1$, $\gamma_i g n_i m_i q_i = g a_{-2tu}$. Since $d(\gamma_i^{-1} g, g n_i m_i a_{2tu}) \le d(q_i a_{2tu}, a_{2tu}) \to 0$ as $i \to \infty$, it follows that for all sufficiently large $i \ge 1$, $\gamma_i^{-1} g o \in \mathcal{H}_\xi(t)$. Hence g^+ is a horospherical limit point by Definition 3.1. □

4. Topological mixing and directional limit points

There is a close connection between denseness of N-orbits and the topological mixing of one-parameter diagonal flows with direction in $\operatorname{int} \mathcal{L}$. This connection allows us to make use of recent topological mixing results by Chow-Sarkar [3]: recall the notation Ω_0 from (2.2).

Theorem 4.1. [3] *For any $u \in \operatorname{int} \mathcal{L}$, $\{a_{tu} : t \in \mathbb{R}\}$ is topologically mixing on Ω_0, i.e., for any open subsets $\mathcal{O}_1, \mathcal{O}_2$ of $\Gamma \backslash G$ intersecting Ω_0,*

$$\mathcal{O}_1 \exp tu \cap \mathcal{O}_2 \neq \emptyset \quad \textit{for all large } |t| \gg 1.$$

The above theorem was predated by a result of Dang [7] in the case where M is abelian.

N-orbits based at directional limit points along $\operatorname{int}\mathcal{L}$.

Definition 4.2. For $u \in \operatorname{int}\mathfrak{a}^+$, denote by Λ_u the set of all u-directional limit points, i.e., $\xi \in \Lambda_u$ if and only if $\limsup_{t\to+\infty} \Gamma g \exp(tu) \neq \emptyset$ for some (and hence any) $g \in G$ with $gP = \xi$.

It is easy to see that $\Lambda_u \subset \Lambda$ for $u \in \operatorname{int}\mathfrak{a}^+$.

Proposition 4.3. *If $[g] \in \mathcal{E}_0$ satisfies $g^+ \in \Lambda_u$ for some $u \in \operatorname{int}\mathcal{L}$, then*

$$\overline{[g]N} = \mathcal{E}_0.$$

Proof. Since $\Omega_0 N = \mathcal{E}_0$, we may assume without loss of generality that $x = [g] \in \Omega_0$. There exist $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that $\gamma_i g a_{t_i u}$ converges to some $h \in G$. In particular, $x\exp(t_i u) \to [h]$. Since $x a_{t_i u} \in \Omega_0$ and Ω_0 is A-invariant and closed, we have $[h] \in \Omega_0$. We write $\gamma_i g a_{t_i u} = h q_i$ where $q_i \to e$ in G. Therefore $xN = [h]q_i N a_{-t_i u}$ for all $i \geq 1$. Let $\mathcal{O} \subset \Gamma\backslash G$ be any open subset intersecting Ω_0. It suffices to show that $xN \cap \mathcal{O} \neq \emptyset$. Let $\mathcal{O}_1$ be an open subset intersecting Ω_0 and $V \subset \check{P}$ be an open symmetric neighborhood of e such that $\mathcal{O}_1 V \subset \mathcal{O}$.

Since $q_i \to e$ and NV is an open neighborhood of e in G, there exists an open neighborhood, say, U of e in G and i_0 such that $U \subset q_i NV$ for all $i \geq i_0$. By Theorem 4.1, we can choose $i > i_0$ such that $[h]U \cap \mathcal{O}_1 a_{t_i u} \neq \emptyset$. It follows that $[h]q_i NV a_{-t_i u} \cap \mathcal{O}_1 \neq \emptyset$. Since $V \subset a_{-t_i u} V a_{t_i u}$ as $u \in \mathfrak{a}^+$, we have

$$[h]q_i NV a_{-t_i u} \cap \mathcal{O}_1 \subset [h]q_i N a_{-t_i u} V \cap \mathcal{O}_1.$$

Since $V = V^{-1}$, we get $[h]q_i N a_{-t_i u} \cap \mathcal{O}_1 V \neq \emptyset$. Therefore $xN \cap \mathcal{O} \neq \emptyset$, as desired. □

This immediately implies:

Corollary 4.4. *If $[g] \in \Omega_0$ is u-periodic for some $u \in \operatorname{int}\mathcal{L}$, then*

$$\overline{[g]N} = \mathcal{E}_0.$$

Proof. Since $[g](\exp ku) = [g]m_0^k$ for any integer k and M is compact, we have $g^+ \in \Lambda_u$. Therefore the claim follows from Proposition 4.3. □

We may now conclude our main theorem in its fullest form:

Theorem 4.5. *Let $[g] \in \mathcal{E}_0$. The following are equivalent:*

(1) $g^+ \in \Lambda$ is a horospherical limit point;
(2) $[g]N$ is dense in $\mathcal{E}_0$;
(3) $[g]NM$ is dense in $\mathcal{E}$.

Proof. The implication (2) ⇒ (3) is trivial and (3) ⇒ (1) was shown in Theorem 3.2. Hence let us prove (1) ⇒ (2).

Let $x = [g] \in \mathcal{E}_0$. Suppose that $g^+ \in \Lambda_h$. Fix any $u \in \lambda(\Gamma) \cap \operatorname{int}\mathcal{L}_\Gamma$. By Propositions 3.7 and 3.6, xN contains a u-periodic point, say, x_0. Hence by Corollary 4.4, $\overline{xN} \supset \overline{x_0 N} \supset \Omega_0 N = \mathcal{E}_0$. This proves (1) ⇒ (2). □

5. Conical limit points, Minimality and Jordan projection

A point $\xi \in \mathcal{F}$ is called a *conical* limit point of Γ if there exists a sequence $u_j \to \infty$ in $\mathfrak{a}^+$ such that for some (and hence every) $g \in G$ with $\xi = gP$

$$\limsup_{j\to\infty} \Gamma g a_{u_j} \neq \emptyset.$$

A conical limit point of Γ is indeed contained in Λ. We consider the following restricted notion:

Definition 5.1. We call $\xi \in \mathcal{F}$ a *strongly conical* limit point of Γ if there exists a closed cone $\mathcal{C} \subset \operatorname{int}\mathcal{L} \cup \{0\}$ and a sequence $u_j \to \infty$ in $\mathcal{C}$ such that for some (and hence every) $g \in G$ with $\xi = gP$,

$$\limsup_{j\to\infty} \Gamma g a_{u_j} \neq \emptyset.$$

Remarks 5.2. We mention that a conical limit point defined in [4] for $\Gamma < \mathrm{SL}_d(\mathbb{R})$ coincides with our strongly conical limit point.

Lemma 5.3. *Any strongly conical limit point of* Γ *is horospherical.*

Proof. Suppose that $\xi = gP$ is strongly conical, that is, there exist $\gamma_j \in \Gamma$ and $u_j \to \infty$ in some closed cone $\mathcal{C} \subset \operatorname{int}\mathcal{L} \cup \{0\}$ such that $\gamma_j g a_{u_j}$ converges to some $h \in G$. Write $\gamma_j g a_{u_j} = h q_j$ where $q_j \to e$ in G. Let $\mathcal{C}'$ be a closed cone contained in $\operatorname{int}\mathcal{L} \cup \{0\}$ whose interior contains $\mathcal{C} \setminus \{0\}$.

Then $\gamma_j^{-1} = g a_{u_j} q_j^{-1} h^{-1}$ and

$$\beta_{gP}(e, \gamma_j^{-1}) = \beta_P(g^{-1}, a_{u_j} q_j^{-1} h^{-1}) = \beta_P(g^{-1}, q_j^{-1} h^{-1}) + \beta_P(e, a_{u_j}).$$

Since $\beta_P(e, a_{u_j}) = u_j$ and $q_j^{-1} h^{-1}$ are uniformly bounded, the sequence

$$\beta_{gP}(e, \gamma_j^{-1}) - u_j$$

is uniformly bounded. Since $u_j \in \mathcal{C}$ and $\mathcal{C} \subset \operatorname{int}\mathcal{C}' \cup \{0\}$, it follows that

$$\beta_{gP}(e, \gamma_j^{-1}) \in \mathcal{C}'$$

for all sufficiently large j. This proves that $\xi \in \Lambda_h$.

□

Corollary 5.4. *For any* $g \in G$ *with strongly conical* $g^+ \in \mathcal{F}$*, we have*

$$\overline{[g]NM} = \mathcal{E}.$$

Directionally conical limit points. If $v \in \operatorname{int}\mathcal{L}$, then clearly Λ_v is contained in the horospherical limit set of Γ, and hence any NM-orbit based at a point of Λ_v is dense in $\mathcal{E}$. On the other hand, we would like to show in this section that the existence of a point in Λ_v for $v \in \partial\mathcal{L}_\Gamma$ implies the existence of a nondense NM-orbit in $\mathcal{E}$.

The flow $\exp(\mathbb{R}u)$ is said to be topologically transitive on $\Omega/M = \{\Gamma g M : g^\pm \in \Lambda\}$ if, for any open subsets $\mathcal{O}_1, \mathcal{O}_2$ intersecting Ω/M, there exists a sequence $t_n \to +\infty$ such that $\mathcal{O}_1 \cap \mathcal{O}_2 a_{t_n u} \neq \emptyset$.

We make the following simple observation:

Lemma 5.5. *For $g \in \Omega$, we have*

$$\overline{gNM} \supset \Omega \quad \textit{if and only if} \quad \overline{gw_0\check{N}M} \supset \Omega.$$

Proof. We have $\check{N} = w_0 N w_0^{-1}$. Note that $[g] \in \Omega$ if and only if $[gw_0] \in \Omega$, since $(gw_0)^{\pm} = g^{\mp}$. So $\Omega w_0 = \Omega$. Hence gNM is dense in Ω if and only if $gw_0\check{N}Mw_0^{-1}$ is dense in Ω if and only if $[g]w_0\check{N}M$ is dense in $\Omega w_0 = \Omega$. $\square$

Since the opposition involution preserves $\mathcal{L}$ and $\lambda(g^{-1}) = \mathrm{i}\,\lambda(g)$ for any loxodromic element, it follows that $\lambda(\gamma) \in \partial\mathcal{L}$ if and only if $\lambda(\gamma^{-1}) \in \partial\mathcal{L}$.

Proposition 5.6.

(1) If $\Lambda = \Lambda_h$, then $\exp(\mathbb{R}v)$ is topologically transitive on Ω/M for any $v \in \operatorname{int}\mathfrak{a}^+$ such that $\Lambda_v \neq \emptyset$.

(2) For any loxodromic element $\gamma \in \Gamma$ with $\{y_\gamma, y_{\gamma^{-1}}\} \subset \Lambda_h$, the flow $\exp(\mathbb{R}\lambda(\gamma))$ is topologically transitive on Ω/M.

Proof. Assume that $\Lambda = \Lambda_h$; so the NM-action on $\mathcal{E}$ is minimal. Suppose that $\Lambda_v \neq \emptyset$ for some $v \in \operatorname{int}\mathfrak{a}^+$. We claim that for any $\mathcal{O}_1, \mathcal{O}_2$ be two right M-invariant open subsets intersecting Ω, $\mathcal{O}_1 \exp(t_i v) \cap \mathcal{O}_2 \neq \emptyset$ for some sequence $t_i \to +\infty$. Choose $x = [g] \in \Omega$ so that $g^+ \in \Lambda_v$. Then there exists $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that $\gamma_i g a_{t_i v}$ converges to some g_0. Note that $x_0 := [g_0] \in \Omega$. So write $\gamma_i g a_{t_i v} = g_0 h_i$ with $h_i \to e$. By the NM-minimality assumption, xNM intersects every open subset of Ω. Since $v \in \operatorname{int}\mathfrak{a}^+$ and hence $a_{-tv} n a_{tv} \to e$ as $t \to +\infty$, we may assume without loss of generality that $x \in \mathcal{O}_1$. Choose an open neighborhood U of e in G so that $\mathcal{O}_1 \supset xUM$. Note that there exists a sequence $T_i \to \infty$ as $i \to \infty$ such that for all i,

$$xUMa_{t_i v} \supset x a_{t_i v} a_{-t_i v} \check{N}_\varepsilon M a_{t_i v} \supset x_0 h_i \check{N}_{T_i},$$

where $\check{N}_R = \check{N} \cap B_R^G$ is the the set of elements of $\check{N}$ of norm$\leq R$. So $\mathcal{O}_1 a_{t_i v} \supset x_0 h_i \check{N}_{T_i}$.

Choose an open neighborhood V of e in G and some open subset $\mathcal{O}_2'$ intersecting Ω so that $\mathcal{O}_2 \supset \mathcal{O}_2' V$. Since $x_0\check{N}M$ is dense in Ω, $x_0 n \in \mathcal{O}_2'$ for some $n \in \check{N}$. Hence $x_0 h_i n = x_0 n(n^{-1} h_i n) \in \mathcal{O}_2' V \subset \mathcal{O}_2$ for all i large enough so that $n^{-1} h_i n \in V$. Therefore for all i such that $n \in \check{N}_{T_i}$, we get

$$x_0 h_i n \in \mathcal{O}_1 a_{t_i v} \cap \mathcal{O}_2 \neq \emptyset.$$

This proves the first claim.

Now suppose that $\gamma \in \Gamma$ is a loxodromic element with $y_\gamma, y_{\gamma^{-1}} \in \Lambda_h$. Write $\gamma = g m a_v g^{-1}$ for some $g \in G$ and $m \in M$. Since $y_\gamma = g^+$ and $y_{\gamma^{-1}} = g w_0^+$, we have each $[g]NM$ and $[g]w_0NM$ contains Ω in its closure. Now in the notation of the proof of the first claim, note that $x_0 = [g_0] \in [g]M$ since $[g]\exp(\mathbb{R}v)M$ is closed. Therefore each $\overline{x_0NM}$ and $\overline{x_0\check{N}M}$ contains Ω. Based on this, the same argument as above shows the topological transitivity of $\exp\mathbb{R}v$, which finishes the proof since $v = \lambda(\gamma)$. $\square$

Since $\mathcal{L}$ is invariant under the opposition involution i and $\lambda(\gamma) = \mathrm{i}\,\lambda(\gamma^{-1})$ for any loxodromic element $\gamma \in \Gamma$, the Jordan projection $\lambda(\gamma)$ belongs to $\partial\mathcal{L}$ if and only if the Jordan projection $\lambda(\gamma^{-1})$ belongs to $\partial\mathcal{L}$. Together with the result of Dang and Gloriuex [8, Proposition 4.7] which say that $\exp(\mathbb{R}u)$ is not topologically transitive on Ω/M for any $u \in \partial\mathcal{L} \cap \operatorname{int}\mathfrak{a}^+$, Proposition 5.6 implies the following:

Corollary 5.7.

(1) If $\Lambda_v \neq \emptyset$ for some $v \in \partial\mathcal{L} \cap \operatorname{int}\mathfrak{a}^+$, then

$$\Lambda \neq \Lambda_h.$$

(2) For any loxodromic element $\gamma \in \Gamma$, we have $\lambda(\gamma) \in \partial\mathcal{L}$ if and only if

$$\{y_\gamma, y_{\gamma^{-1}}\} \not\subset \Lambda_h.$$

Hence, if $\Lambda = \Lambda_h$, then $\lambda(\Gamma) \subset \operatorname{int}\mathcal{L}$.

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