HOROSPHERICAL INVARIANT MEASURES AND A RANK DICHOTOMY FOR ANOSOV GROUPS

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Abstract. Let $G = \prod_{i=1}^{\mathsf{r}} G_i$ be a product of simple real algebraic groups of rank one and Γ an Anosov subgroup of G with respect to a minimal parabolic subgroup. For each v in the interior of a positive Weyl chamber, let $\mathcal{R}_{\mathsf{v}} \subset \Gamma\backslash G$ denote the Borel subset of all points with recurrent $\exp(\mathbb{R}_+\mathsf{v})$-orbits. For a maximal horospherical subgroup N of G, we show that the N-action on $\mathcal{R}_{\mathsf{v}}$ is uniquely ergodic if $\mathsf{r} = \operatorname{rank}(G) \leq 3$ and v belongs to the interior of the limit cone of Γ, and that there exists no N-invariant Radon measure on $\mathcal{R}_{\mathsf{v}}$ otherwise.

Contents

1. Introduction

Let G be a connected semisimple real algebraic group, and $\Gamma < G$ be a Zariski dense discrete subgroup. Let N be a maximal horospherical subgroup of G, which is unique up to conjugation. We are interested in the study of N-invariant ergodic Radon measures on the quotient space $\Gamma\backslash G$ (from now on, all measures we will consider are implicitly assumed to be Radon measures). When Γ is a uniform lattice in G, the N-action on $\Gamma\backslash G$ is known to be uniquely ergodic, that is, there exists a unique N-invariant ergodic measure on $\Gamma\backslash G$, up to proportionality, which is the G-invariant measure. This result is due to Furstenberg [15] for $G = \mathrm{PSL}_2(\mathbb{R})$ and Veech [42] in general. Dani [10] classified all N-invariant ergodic measures for a general lattice Γ. Later, Ratner [33] gave a complete classification of all invariant ergodic measures for any unipotent subgroup action when Γ is a lattice of G.

This work was partially supported by ISF-Moked grant 2095/19 (Landesberg), ERC 2020 grant no. 833423 (Lindenstrauss) and the NSF grant 0003086 (Oh).

When G is of rank one and Γ is geometrically finite, there exists a unique MN-invariant ergodic measure on $\Gamma\backslash G$, not supported on a closed MN-orbit, where M is a maximal compact subgroup of the normalizer of N, called the Burger-Roblin measure. This result is due to Burger [6] for convex cocompact subgroups of $\mathrm{PSL}_2(\mathbb{R})$ with critical exponent bigger than $1/2$, and to Roblin [34] in general. For $G \not\simeq \mathrm{SL}_2(\mathbb{R})$, Winter [43] showed that the Burger-Roblin measure is N-ergodic, and hence the N-action on $\Gamma\backslash G$ is essentially uniquely ergodic. This relies on the fact that M is connected. Indeed, for $G \simeq \mathrm{SL}_2(\mathbb{R})$ where $M = \{\pm e\}$, the Burger-Roblin measure has one or two N-ergodic components depending on Γ (cf. [27, Thm. 7.14]).

For geometrically infinite groups, there may be a continuous family of N-invariant ergodic measures, as first discovered by Babillot and Ledrappier ([1], [2]). See ([36], [37], [24], [25], [30], [22], [23]) for partial classification results in the rank one case.

In this paper, we obtain a measure classification result for the N-action on Anosov homogeneous spaces $\Gamma\backslash G$ which surprisingly depends on the rank of G: on the recurrent set in an interior direction of the limit cone of Γ, the N-action is uniquely ergodic if $\operatorname{rank} G \le 3$, and admits no invariant measure if $\operatorname{rank} G > 3$.

When the rank of G is one, the class of Anosov subgroups coincides with that of Zariski dense convex cocompact subgroups. To define it in general, let P be a minimal parabolic subgroup of G. Let $\mathcal{F}$ denote the Furstenberg boundary G/P, and $\mathcal{F}^{(2)}$ the unique open G-orbit in $\mathcal{F} \times \mathcal{F}$. A Zariski dense discrete subgroup $\Gamma < G$ is called an *Anosov subgroup* (with respect to P) if it is a finitely generated word hyperbolic group which admits a Γ-equivariant embedding ζ of the Gromov boundary $\partial\Gamma$ into $\mathcal{F}$ such that $(\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)}$ for all $x \neq y$ in $\partial\Gamma$. First introduced by Labourie [21] as the images of Hitchin representations of surface groups, this definition is due to Guichard and Wienhard [16]. The class of Anosov groups in particular includes any Zariski dense Schottky subgroup (cf. [32], [13, Lem. 7.2]).

Let $P = AMN$ be the Langlands decomposition of P, so that A is a maximal real split torus of G, M is a compact subgroup which commutes with A and N is the unipotent radical of P. Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a} = \log A$, and denote by $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ the limit cone of Γ, i.e., $\mathcal{L}_\Gamma$ is the smallest closed cone of $\mathfrak{a}^+$ which contains the Jordan projection of Γ (see (2.1) for definition). It is known that if Γ is Zariski dense, $\mathcal{L}_\Gamma$ is a convex cone with non-empty interior [3, Thm. 1.2]. We denote by $\Lambda \subset \mathcal{F}$ the limit set of Γ, which is the unique Γ-minimal closed subset of $\mathcal{F}$. Then

$$\mathcal{E} := \{[g] \in \Gamma\backslash G : gP \in \Lambda\}$$

is the unique P-minimal closed subset of $\Gamma\backslash G$. For each vector $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+$, define the following directional recurrent subset of $\mathcal{E}$:

$$\mathcal{R}_{\mathsf{v}} = \{x \in \Gamma\backslash G : x \exp(t_i \mathsf{v}) \text{ is bounded for some } t_i \to +\infty\}. \tag{1.1}$$

It is easy to see that $\mathcal{R}_{\mathsf{v}} = \emptyset$ unless $\mathsf{v} \in \mathcal{L}_\Gamma$. Since $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+$ and AM centralizes $\exp(\mathbb{R}\mathsf{v})$, $\mathcal{R}_{\mathsf{v}}$ is a P-invariant dense Borel subset of $\mathcal{E}$ when it is non-empty. In particular, $\mathcal{R}_{\mathsf{v}}$ is either co-null or null for any N-invariant ergodic measure on $\Gamma\backslash G$. We are interested in understanding N-invariant ergodic measures supported on $\mathcal{R}_{\mathsf{v}}$.

In the rest of the introduction, we assume that

$$G = \prod_{i=1}^{\mathsf{r}} G_i$$

where each G_i is a rank one simple real algebraic group; hence $\mathsf{r} = \operatorname{rank} G$. While G_i can be isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, we exclude the case when G_i is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ in order to ensure that P is connected. We let $\Gamma < G$ be an Anosov subgroup. For each $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$, we denote by $m_{\mathsf{v}}^{\mathrm{BR}}$ the MN-invariant Burger-Roblin measure for the direction v (see (6.1)). For Anosov subgroups, it was shown by Lee and Oh that the family $\{\mathsf{m}_{\mathsf{v}}^{\mathrm{BR}} : \mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma\}$ gives all N-invariant ergodic and P quasi-invariant measures on $\mathcal{E}$, up to proportionality ([26], [27]).

The main result of this paper is as follows:

Theorem 1.1. *Let $\Gamma < G$ be an Anosov subgroup and $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+$.*

(1) For $\mathsf{r} \leq 3$ and $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$, the N-action on $\mathcal{R}_{\mathsf{v}}$ is uniquely ergodic. More precisely, $\mathsf{m}_{\mathsf{v}}^{\mathrm{BR}}$ is the unique N-invariant measure supported on $\mathcal{R}_{\mathsf{v}}$, up to proportionality.
(2) For $\mathsf{r} > 3$ or $\mathsf{v} \notin \operatorname{int} \mathcal{L}_\Gamma$, there exists no N-invariant measure supported on $\mathcal{R}_{\mathsf{v}}$.

This theorem uses the result by Burger, Landesberg, Lee and Oh [8] that $\mathcal{R}_{\mathsf{v}}$ is a co-null (resp. null) set for $\mathsf{m}_{\mathsf{v}}^{\mathrm{BR}}$ for $\mathsf{r} \leq 3$ (resp. $\mathsf{r} > 3$), which was developed simultaneously, in part for the purpose of this work.

We note that the unique ergodicity as in (1) implies that $\mathsf{m}_{\mathsf{v}}^{\mathrm{BR}}$ is N-ergodic, reproving some special cases of [27, Thm. 1.1]. When $\mathsf{r} = 1$ and Γ is a convex cocompact subgroup of G, this theorem recovers the unique ergodicity of the N-action on $\mathcal{E}$.

We deduce the following classification of N-ergodic measures supported on the directional recurrent set

$$\mathcal{R} := \cup_{\mathsf{v} \in \operatorname{int} \mathfrak{a}^+} \mathcal{R}_{\mathsf{v}}.$$

A measure μ on $\Gamma\backslash G$ is said to be supported on $\mathcal{R}$ if the complement of $\mathcal{R}$ is contained in a μ-null set.

Corollary 1.2. *The space $\mathcal{M}$ of all N-invariant ergodic measures supported on $\mathcal{R}$ is given by*

$$\mathcal{M} = \begin{cases} \{\mathsf{m}_{\mathsf{v}}^{\mathrm{BR}} : \mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma\} & \text{for } \mathsf{r} \leq 3 \\ \emptyset & \text{for } \mathsf{r} > 3. \end{cases}$$

We apply our theorem to some concrete examples considered in [7]. Let Σ be a surface subgroup with two convex cocompact realizations in rank one Lie groups G_1 and G_2. For each $i = 1, 2$, denote by $\pi_i : \Sigma \to G_i$ an injective homomorphism with Zariski dense image. We assume that $\pi_2 \circ \pi_1^{-1}$ does not extend to an algebraic group isomorphism $G_1 \to G_2$.

It is easy to check that $\Gamma_{\pi_1,\pi_2} := \{(\pi_1(\gamma), \pi_2(\gamma)) : \gamma \in \Sigma\}$ is an Anosov subgroup of $G := G_1 \times G_2$.

Corollary 1.3. *For $\Gamma = \Gamma_{\pi_1,\pi_2}$ as above, the N-action on $\mathcal{R}_{\mathsf{v}}$ is uniquely ergodic for each $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$.*

On the proof of Theorem 1.1. In the rank one case, i.e., when Γ is convex cocompact, Theorem 1.1 follows from the combined works of Roblin [34] and Winter [43] (see also [28] and [38] for $G = \mathrm{SO}^\circ(n, 1)$ case). These proofs are all based on the finiteness and the strong mixing property of the Bowen-Margulis-Sullivan measure. In the higher rank case, although there exists an analogous measure (which is also called the Bowen-Margulis-Sullivan measure) for each direction $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$, this is an infinite measure [26, Cor. 4.9] and it is not clear how to extend the approaches of the aforementioned papers. We henceforth follow an approach of the recent work of Landesberg and Lindenstrauss [22] for the case $G = \mathrm{SO}^\circ(n, 1)$ which is in the spirit of Ratner's work. The main technical result we prove in this paper is the following:

Proposition 1.4. *Let Γ be a Zariski dense discrete subgroup of G and $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+$. Then any N-invariant ergodic measure μ on $\mathcal{R}_{\mathsf{v}}$ is P-quasi-invariant.*

Remark 1.5. We refer to Theorem 4.1 for a more general version, analogous to the main theorem of [22] for $G = \mathrm{SO}^\circ(n, 1)$.

Following [22], our proof of Proposition 1.4 utilizes the geometry observed along the one-dimensional diagonal flow $\exp(\mathbb{R}\mathsf{v})$ of points in the support of μ to obtain an extra quasi-invariance of μ. Roughly speaking, if, for μ-a.e. $x \in \Gamma\backslash G$, we have $x \exp(t_n\mathsf{v})g_n = x \exp(t_n\mathsf{v})$ for some infinite sequence $t_n \to \infty$ and $g_n \in G$ converging to some loxodromic element $g_0 \in G$, we show that the generalized Jordan projection of g_0 preserves the measure class of μ, provided the attracting fixed point of g_0 is in general position with that of g_0^{-1}. The last condition always holds in the rank one setting as any two distinct points on $\mathcal{F}$ are in general position. In the higher rank setting, this property is needed to ensure that the high powers of g_0 attract some neighborhood of its attracting fixed point to itself, which is an underlying key point which makes our analysis possible.

For $G = \mathrm{SO}^\circ(n, 1)$, the conjugation action of an element of A on N is simply a scalar multiplication, and both the Besicovitch covering lemma and Hochman's ratio ergodic theorem for Euclidean norm balls in the abelian group $N \simeq \mathbb{R}^{\dim N}$ were used in [22], in order to control ergodic properties of N-orbits. In our setting where G is a product $\prod G_i$ of rank one Lie groups, the horospherical subgroup N is a product $\prod N_i$ of abelian and two-step nilpotent subgroups and the conjugation action by $\exp(t\mathsf{v})$ scales N_i's by different factors. The existence of $\exp(t\mathsf{v})$-invariant family of quasi-balls satisfying the Besicovitch covering property in this case is a consequence of the work of Le Donne and Rigot [11, Thm. 1.2]. This is precisely the main reason for our assumption that G is the product of rank one Lie groups. We note that in the higher rank case, the ratio ergodic theorem with respect to this family of quasi-balls in our $N = \prod N_i$, is available only when N is abelian [12].[1] To sidestep the lack of the ratio ergodic theorem in the generality we need, we use in this paper a modified argument relying only on the Besicovitch covering property. In addition to technical difficulties arising in the higher rank setting and from the fact that N is not necessarily abelian, our proof of Proposition 1.4 is different from [22] also in this aspect.

Theorem 1.1 is then deduced from Proposition 1.4 together with the classification of Γ-conformal measures on Λ of [26] (Theorem 6.1) and the dichotomy on the recurrence property of the Burger-Roblin measures according to the rank of G, obtained in [8] (Theorem 6.2).

Rank one groups. While the main emphasis in this paper is on the higher rank case, one can also deduce the following new result for all rank one groups. Given Theorem 4.1 and the description of N-ergodic invariant and P°-quasi invaiant measures (cf. [22, Lem. 5.2], [27, Prop. 7.2]), the following corollary can be proved almost verbatim as [22, Cor. 1.1, 1.2] and [23, Thm. 1.5] where similar statements were established for $G = \mathrm{SO}^\circ(n,1)$.

For $y \in \Gamma\backslash G$, we denote by $\mathrm{rad}_{\mathrm{inj}}(y)$ the supremal injectivity radius at y.

Corollary 1.6. *Let Γ be a Zariski dense discrete subgroup of a simple real algebraic group G of rank one. Let μ be an N-invariant ergodic measure supported on $\mathcal{E}$.*

(1) If the injectivity radius on $\Gamma\backslash G$ is uniformly bounded away from 0*, then at least one of the following holds:*
 - *(a) μ is quasi-invariant under some loxodromic element of P,*
 - *(b) $\lim_{t\to\infty} \mathrm{rad}_{\mathrm{inj}}(x \exp t\mathsf{v}) = \infty$ for μ-a.e. x and $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+$.*

(2) If the injectivity radius on $\Gamma\backslash G$ is uniformly bounded from above *or if Γ is a normal subgroup of a geometrically finite subgroup of G, then either:*
 - *(a) μ is proportional to $\mathsf{m}_\nu^{\mathrm{BR}}|_Y$ for some Γ-conformal measure ν on Λ and a P°-minimal subset $Y \subset \Gamma\backslash G$ (see (6.1) for the definition of $\mathsf{m}_\nu^{\mathrm{BR}}$), or*
 - *(b) μ is supported on a closed MN-orbit.*

We remark that by a recent work of Fraczyk and Gelander [14], the injectivity radius on $\Gamma\backslash G$ is never bounded from above when G is simple with $\operatorname{rank} G \geq 2$ and $\mathrm{Vol}(\Gamma\backslash G) = \infty$.

[1] We mention that the only case when the ratio ergodic theorem is known and N is not abelian is when $G \simeq \mathrm{SU}(n,1)$ and N is Heisenberg [20].

Remark 1.7. For Γ geometrically finite, an atom of a Γ-conformal density is necessarily a parabolic limit point which yields a closed MN-orbit, and the so-called Patterson-Sullivan measure, say, ν_0, is the unique atom-free Γ-conformal measure on Λ [40]. Therefore Corollary 1.6(2) implies the essential unique ergodicity for the N-action as well as the N-ergodicity of $\mathsf{m}^{\mathrm{BR}}_{\nu_0}|_Y$ for each P°-minimal subset Y. Noting that the proofs given in [28] and [38] on the N-unique ergodicity for $\mathrm{SO}^\circ(n,1)$ rely on the ratio ergodic theorem for the abelian subgroup N which is not available for a general rank one group, our paper gives the only alternative proof for a general rank one case after Roblin and Winter ([34], [43]).

Organization. In section 2, we set up notations and recall basic definitions. In section 3, we deduce the Besicovitch covering lemma for our setting from [11] and state several consequences including the maximal ratio inequality. In section 4, we prove Theorem 4.1, which is the main technical result of this paper. In section 5, we prove Theorem 5.1 which in particular implies Proposition 1.4, using Theorem 4.1 together with some properties of Zariski dense subgroups. In section 6, we specialize to Anosov subgroups and prove Theorem 1.1.

We close the introduction with the following open problems.

Open problem 1.8. For $\mathsf{r} \leq 3$ and Γ Anosov, is any N-invariant ergodic measure on $\mathcal{E}$ necessarily supported on $\mathcal{R}_{\mathsf{v}}$ for some $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$?

Acknowledgement. We would like to thank Emmanuel Breuillard and Amir Mohammadi for useful conversations on this work.

2. Preliminaries

Let G be a connected, semisimple real algebraic group. We fix, once and for all, a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A := \exp \mathfrak{a}$ and $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M, and we set N^- and N^+ to be the contracting and expanding horospherical subgroup: for $a \in \operatorname{int} A^+$,

$$N^\pm = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to \mp\infty\}.$$

We set $P^\pm = MAN^\pm$, which are minimal parabolic subgroups. As we will be looking at the N^--action in this paper, we set $N := N^-$ and $P = P^-$ for notational simplicity. We also set $L = MA = P \cap P^+$.

Let $w_0 \in N_K(A)$ be the Weyl element satisfying $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. Then w_0 satisfies $w_0 P^- w_0^{-1} = P^+$. For each $g \in G$, we define

$$g^+ := gP \in G/P \quad \text{and} \quad g^- := g w_0 P \in G/P.$$

Let $\mathcal{F} = G/P$ and $\mathcal{F}^{(2)}$ denote the unique open G-orbit in $\mathcal{F} \times \mathcal{F}$:

$$\mathcal{F}^{(2)} = G(e^+, e^-) = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}.$$

We say that ξ, η are in general position if $(\xi, \eta) \in \mathcal{F}^{(2)}$.

Any element $g \in G$ can be written as the commuting product $g_h g_e g_u$, where g_h, g_e and g_u are unique elements which are conjugate to elements of A^+, K and N, respectively. We say g is *loxodromic* if $g_h \in \varphi(\operatorname{int} A^+)\varphi^{-1}$ for some $\varphi \in G$, and write

$$\lambda^A(g) := \varphi^{-1} g_h \varphi \in \operatorname{int} A^+ \tag{2.1}$$

calling it the Jordan projection of g. We set

$$y_g := \varphi^+; \tag{2.2}$$

this is well-defined independent of the choice of φ. We note that g fixes y_g and for any $h \in N^+$, $\lim_{k\to\infty} g^k(\varphi h e^+) = y_g$, uniformly on compact subsets of N^+, and for this reason, y_g is called the attracting fixed point of g.

Bruhat coordinates. The product map $N \times A \times M \times N^+ \to G$ is injective and its image is Zariski open in G. For $g \in G$ and $n \in N$ with $gn \in NAMN^+$, we write

$$gn = b^N(g,n) b^{AM}(g,n) b^{N^+}(g,n) \tag{2.3}$$

where $b^N(g,n) \in N$, $b^{AM}(g,n) \in AM$, $b^{N^+}(g,n) \in N^+$ are uniquely determined. For each subgroup $\star = N, AM$ or N^+, $b^\star(g,n)$ is a smooth function for each $g \in G$ and $n \in N$ whenever it is defined.

For convenience, for $\xi = ne^-$ with $n \in N$ and $g \in G$ with $g\xi \in Ne^-$, we set

$$b^\star(g,\xi) := b^\star(g,n).$$

If $g \in G$ is a loxodromic element with $y_g \in Ne^-$, the following generalized Jordan projection of g is well-defined:

$$\lambda(g) = b^{AM}(g, y_g).$$

We mention that the condition $y_g \in Ne^-$ implies that there exists $\varphi \in NN^+$ such that $g = \varphi a^{-1} m \varphi^{-1}$ for unique $a \in \operatorname{int} A^+$ and $m \in M$. In this case, $\lambda(g) = a^{-1}m$. In particular, the A-component of $\lambda(g)$ coincides with $\lambda^A(g^{-1})$. If g is not loxodromic, we set $\lambda(g) = e$.

3. Covering lemma for $\exp t\mathsf{v}$-conjugation invariant balls

In the rest of the paper, let $G := \prod_{i=1}^{\mathsf{r}} G_i$ where G_i is a connected simple real algebraic group of rank one. For each $1 \le i \le \mathsf{r}$, we identify G_i with the subgroup $\{(g_j)_j \in \prod_j G_j : g_j = e \text{ for all } j \neq i\} < G$ and we set $H_i := H \cap G_i$ for any subset $H \subset G$. We have $A = \prod_i A_i$ and $A^+ = \prod_i A_i^+$ where A_i is a one-parameter diagonalizable subgroup of G_i. Let α_i denote the simple root of G_i with respect to A_i. The subgroup $N = N^-$ is of the form $N = \prod_i N_i$, where N_i is the contracting horospherical subgroup of G for A_i^+ and $P = \prod P_i$ for $P_i = M_i A_i N_i$. We set $\mathcal{F}_i = G_i/P_i$.

As G_i has rank one, N_i is a connected simply connected nilpotent subgroup of at most 2-step. Let $\mathfrak{n}_i$ denote the Lie algebra of N_i. When $\mathfrak{n}_i$ is abelian, for each $a_i \in A_i$, $\mathrm{Ad}_{a_i}|_{\mathfrak{n}_i}$ is the multiplication by $e^{\alpha_i(\log a_i)}$. When $\mathfrak{n}_i$ is a 2-step nilpotent, we can write $\mathfrak{n}_i = \mathfrak{n}_{i_1} \oplus \mathfrak{n}_{i_2}$ where $[\mathfrak{n}_{i_1}, \mathfrak{n}_{i_1}] \subset \mathfrak{n}_{i_2}$ and $\mathfrak{n}_{i_2}$ is the center of $\mathfrak{n}_i$. We have that for $a_i \in A_i$, $\mathrm{Ad}_{a_i}|_{\mathfrak{n}_{i_1}} = e^{\alpha_i(\log a_i)}$ and $\mathrm{Ad}_{a_i}|_{\mathfrak{n}_{i_2}} = e^{2\alpha_i(\log a_i)}$ (cf. [29]).

We call a function $d : N \times N \to [0, \infty)$ a quasi-distance on N if it is symmetric, $d(x, y) = 0$ iff $x = y$, and there exists $C = C(d) \geq 1$ such that

$$d(x, y) \leq C(d(x, z) + d(z, y)) \quad \text{for all } x, y, z \in N. \tag{3.1}$$

For $s > 0$ and $x \in N$, we set $B_d(x, s) = \{y \in N : d(x, y) < s\}$. For simplicity, we write $B_d(s) := B_d(e, s)$. Note that whenever d is left-invariant, $B_d(x, s) = xB_d(s)$ for all $x \in N$ and $s > 0$.

When N is abelian, it is well-known that Euclidean norm-balls of N satisfy the Besicovitch covering property. In general, we deduce the following from [11].

Proposition 3.1. *For any $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+ \cup \{0\}$, there exists a continuous left-invariant quasi-distance $d = d_{\mathsf{v}}$ on N such that the family of balls $\{B_d(u, s) = uB_d(s) : u \in N, s > 0\}$ satisfies the Besicovitch covering property. That is, there exists a constant $\kappa_{\mathsf{v}} > 0$, depending only on d_{v}, such that for any bounded subset $S \subset N$, and any cover $\{uB_d(t_u) : u \in S\}$ of S, for some positive function $u \mapsto t_u$ on S, there exists a countable subset $F \subset S$ such that $\{uB_d(t_u) : u \in F\}$ covers S and*

$$\sum_{u \in F} \mathbb{1}_{uB_d(t_u)} \leq \kappa_{\mathsf{v}}.$$

Moreover, if $\mathsf{v} = 0$, we can take $d_{\mathsf{v}} = d_0$ to be a distance, and if $\mathsf{v} \neq 0$, we have

$$B_d(e^t r) = \exp(t\mathsf{v}) B_d(r) \exp(-t\mathsf{v}) \quad \textit{for all } t \in \mathbb{R} \textit{ and } r > 0. \tag{3.2}$$

Proof. For $\lambda \geq 1$, consider the Lie algebra homomorphism $\mathfrak{n} \to \mathfrak{n}$ given by $\delta_\lambda X = \mathrm{Ad}_{\exp((\log \lambda)\mathsf{v})} X$. Let $I := \{i : \mathfrak{n}_i \text{ abelian}\}$ and $J := \{i : \mathfrak{n}_i \text{ is of 2-step}\}$. Set $t_i := \alpha_i(\mathsf{v}) \geq 0$. For $i \in I$, set $V_{t_i} := \mathfrak{n}_i$ and for $i \in J$, set $V_{t_i} := \mathfrak{n}_{i_1}$ and $V_{2t_i} := \mathfrak{n}_{i_2}$. Since δ_λ acts on each V_{t_i} (resp. V_{2t_i}) by λ^{t_i} (resp. λ^{2t_i}), and $\sum_{i \in I} V_{t_i} + \sum_{i \in J} V_{2t_i}$ is the center of $\mathfrak{n}$, it follows that $\mathfrak{n} = (\oplus_{i \in I \cup J} V_{t_i}) \oplus (\oplus_{i \in J} V_{2t_i})$ provides commuting different layers for the family $\{\delta_\lambda | \lambda > 0\}$ in the terminology of [11]. Hence [11, Thm. 1.2] provides the required quasi-distance such that $d(\delta_\lambda(n_1), \delta_\lambda(n_2)) = \lambda d(n_1, n_2)$ where $\delta_\lambda(n) = e^{(\log \lambda)\mathsf{v}} n e^{-(\log \lambda)\mathsf{v}}$ also denotes the Lie group isomorphism of N induced from δ_λ. For $\lambda = e^t$, this implies (3.2). If $\mathsf{v} = 0$, then $t_i = 2t_i = 0$ for all i, and hence $\mathfrak{n} = V_0$. Now [11, Cor. 1.3, Def. 2.21] implies that d_0 can be taken to be a distance. $\square$

Indeed, an explicit construction of d_{v} has been given in [11]: for $\mathsf{v} \in \operatorname{int}\mathfrak{a}^+$, for $(X_i)_i, (Y_i)_i \in \prod_i N_i$, and

$$d_{\mathsf{v}}((X_i)_i, (Y_i)_i) = \max_i \mathsf{d}_i(X_i, Y_i)^{1/\alpha_i(\mathsf{v})} \tag{3.3}$$

where d_i is a left invariant metric on N_i induced from an Euclidean norm on $\mathfrak{n}_i$.

For each $\mathsf{v} \in \operatorname{int}\mathfrak{a}^+$ (resp. $\mathsf{v} = 0$), we fix a quasi-distance d_{v} as above (resp. a distance d_0), and write for any $\varepsilon > 0$ and $u \in N$,

$$B_{\mathsf{v}}(u, \varepsilon) := B_{d_{\mathsf{v}}}(u, \varepsilon), \quad \text{and} \quad B_{\mathsf{v}}(\varepsilon) := B_{d_{\mathsf{v}}}(\varepsilon). \tag{3.4}$$

We denote by m a Haar measure on N and by 2ρ the sum of all positive roots, i.e., $2\rho = \sum_{i=1}^{\mathsf{r}} \alpha_i(\dim N + \dim Z(N))$, where $Z(N)$ denotes the center of N. For $\mathsf{v} \neq 0$, we have from (3.2) that for any $R > 0$ and $u \in N$,

$$m(B_{\mathsf{v}}(u, R)) = R^{2\rho(\mathsf{v})} m(B_{\mathsf{v}}(u, 1)). \tag{3.5}$$

For $\mathsf{v} = 0$, d_0 is a left-invariant metric and by [17] (see also [5]), we have

$$m(B_0(u, R)) = O(R^{\dim N + \dim Z(N)}). \tag{3.6}$$

Lemma 3.2. *Fix $\mathsf{v} \in \operatorname{int}\mathfrak{a}^+$, $\beta > 0$, $0 < \eta_1 < \eta_2$ and let $u \mapsto t_u$ be a positive function on N. Consider the two collections of balls $\{B_{\mathsf{v}}(u, e^{t_u}\eta_i) : u \in N, t_u > 0\}$ for $i = 1, 2$. Then for any bounded subset $S \subset N$, there exists a countable subset $F \subset S$ such that $\{B_{\mathsf{v}}(u_i, e^{t_{u_i}}\eta_1) : u_i \in F\}$ covers S and the following holds: for each $u_j \in F$,*

$$\#\{u_i \in F : B_{\mathsf{v}}(u_i, e^{t_{u_i}}\eta_1) \subset B_{\mathsf{v}}(u_j, e^{t_{u_j}}\eta_2), |t_{u_i} - t_{u_j}| \leq \beta\} \leq \kappa_*(\mathsf{v}, \beta, \eta_1, \eta_2)$$

where $\kappa_*(\mathsf{v}, \beta, \eta_1, \eta_2) := \frac{m(B_{\mathsf{v}}(\eta_2))}{m(B_{\mathsf{v}}(\eta_1))} e^{\|2\rho\|\beta} \kappa_{\mathsf{v}}$.

Proof. Set $B_u := B_{\mathsf{v}}(u, e^{t_u}\eta_1)$ and $C_u := B_{\mathsf{v}}(u, e^{t_u}\eta_2)$. Let $F \subset S$ and $\{B_{u_i} : u_i \in F\}$ be respectively the countable subset and the corresponding countable subcover of S given by Proposition 3.1. Fix $u_j \in F$. Suppose that $B_{u_1} \cup \cdots \cup B_{u_p} \subset C_{u_j}$ and that $|t_{u_i} - t_{u_j}| \leq \beta$ for all $1 \leq i \leq p$. Since

$$\sum_{i=1}^{p} \mathbb{1}_{B_{u_i}} \leq \kappa_{\mathsf{v}} \cdot \mathbb{1}_{\cup_{i=1}^{p} B_{u_i}},$$

we have

$$m(C_{u_j}) \geq m(\cup_{i=1}^{p} B_{u_i}) \geq \frac{1}{\kappa_{\mathsf{v}}} \sum_{i=1}^{p} m(B_{u_i}). \tag{3.7}$$

Using (3.5), we get

$$m(B_{u_i}) \geq e^{-\|2\rho\|\beta} m(B_{u_j}), \text{ and } m(C_{u_j}) = \frac{m(B_{u_j}) m(B_{\mathsf{v}}(\eta_2))}{m(B_{\mathsf{v}}(\eta_1))}.$$

It then follows from (3.7):

$$\frac{m(B_{\mathsf{v}}(\eta_2))}{m(B_{\mathsf{v}}(\eta_1))} \geq \frac{p}{\kappa_{\mathsf{v}}} e^{-\|2\rho\|\beta}, \text{ and hence } p \leq \frac{m(B_{\mathsf{v}}(\eta_2))}{m(B_{\mathsf{v}}(\eta_1))} \kappa_{\mathsf{v}} e^{\|2\rho\|\beta},$$

proving the claim. □

The following is a consequence of the polynomial growth of the quasi-balls $B_{\mathsf{v}}(t)$ in N:

Lemma 3.3. *Let μ be an N-invariant ergodic measure on a Borel space Z and fix $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+ \cup \{0\}$. For any bounded Borel subset Ω of Z with $\mu(\Omega) > 0$, there exists a co-null subset Z' (depending on Ω) such that for all $x \in Z'$, we have the following: for any $r, \varepsilon > 0$, there exists a sequence $t_i \to \infty$ such that*

$$\frac{\int_{B_{\mathsf{v}}(t_i+r)} \mathbb{1}_\Omega(xn)\, dn}{\int_{B_{\mathsf{v}}(t_i)} \mathbb{1}_\Omega(xn)\, dn} \le 1 + \varepsilon. \tag{3.8}$$

Proof. For $x \in Z$ and a subset $\Omega \subset Z$, we write

$$\mathsf{T}_\Omega(x) = \{u \in N : xu \in \Omega\}. \tag{3.9}$$

By ergodicity of μ, we know that μ-almost every N-orbit intersects Ω non-trivially. Indeed, consider the set

$$E := \{x \in Z \mid m(\mathsf{T}_\Omega(x) \cap B_{\mathsf{v}}(s_x)) > 0 \text{ for some } s_x > 0\}.$$

If $x \in E$, then, for any $u \in N$, there exists $s > s_x$ satisfying

$$B_{\mathsf{v}}(s_x) \subset u B_{\mathsf{v}}(s)$$

and consequently

$$m(\mathsf{T}_\Omega(xu) \cap B_{\mathsf{v}}(s)) = m(\mathsf{T}_\Omega(x) \cap u B_{\mathsf{v}}(s)) \ge m(\mathsf{T}_\Omega(x) \cap B_{\mathsf{v}}(s_x)) > 0,$$

implying $xu \in E$. Hence the set E is N-invariant. Now, by ergodicity of μ, the set E is either null or conull. On the other hand, since

$$\int_Z m(\mathsf{T}_\Omega(x) \cap B_{\mathsf{v}}(1)) d\mu(x) = \int_{B_{\mathsf{v}}(1)} \int_Z \mathbb{1}_\Omega(xn) d\mu(x) dn = m(B_{\mathsf{v}}(1))\mu(\Omega) > 0,$$

the set $\{x \in Z : m(\mathsf{T}_\Omega(x) \cap B_{\mathsf{v}}(1)) > 0\}$ has positive measure. Therefore $\mu(E) > 0$, and hence E is conull. Set $Z' = E$. Let $x \in Z'$ and $s_x > 0$ be such that $m(\mathsf{T}_\Omega(x) \cap B_{\mathsf{v}}(s_x)) > 0$. Suppose that (3.8) does not hold for x. Then there exists $t_x > s_x$ such that for all $t \ge t_x$,

$$m(B_{\mathsf{v}}(t+r)) \ge m(\mathsf{T}_\Omega(x) \cap B_{\mathsf{v}}(t+r)) \ge (1+\varepsilon) m(\mathsf{T}_\Omega(x) \cap B_{\mathsf{v}}(t)).$$

It follows that for all $k \ge 1$,

$$m(B_{\mathsf{v}}(t_x + kr)) \ge (1+\varepsilon)^k m(\mathsf{T}_\Omega(x) \cap B_{\mathsf{v}}(t_x)).$$

Since $m(B_{\mathsf{v}}(t_x + kr))$ grows polynomially in k by (3.5) and (3.6), and since $m(\mathsf{T}_\Omega(x) \cap B_{\mathsf{v}}(t_x)) > 0$, this yields a contradiction. □

A standard consequence of the Besicovitch covering property is the maximal ratio inequality. These are in fact equivalent when considering symmetric averaging sets, see [19] and references therein. For completeness we include below a proof of this implication applicable to our setup:

Lemma 3.4 (Maximal ratio inequality)**.** *Let μ be an N-invariant ergodic measure on a Borel space Z. Fix $\mathsf{v} \in \operatorname{int}\mathfrak{a}^+ \cup \{0\}$ and $\alpha > 0$. For any bounded measurable subsets Ω_1 and Ω_2 of Z with $\mu(\Omega_2) < \infty$, we have*

$$\mu(\Omega_2 \cap E^\dagger) \leq 2\kappa_{\mathsf{v}}\alpha^{-1}\mu(\Omega_1)$$

where

$$E^\dagger := \left\{x \in Z : \exists R > 0 \ s.t \int_{B_{\mathsf{v}}(R)} \mathbb{1}_{\Omega_1}(xn)\, dn \geq \alpha \int_{B_{\mathsf{v}}(R)} \mathbb{1}_{\Omega_2}(xn)\, dn\right\}.$$

Proof. For $R_1 \geq 0$, set

$$E(R_1) := \left\{x \in Z : \exists 0 \leq R \leq R_1 \text{ s.t} \int_{B_{\mathsf{v}}(R)} \mathbb{1}_{\Omega_1}(xn)\, dn \geq \alpha \int_{B_{\mathsf{v}}(R)} \mathbb{1}_{\Omega_2}(xn)\, dn\right\}.$$

Since $E(R_1)$ is an increasing sequence of subsets whose union is $E^\dagger$ and $\mu(\Omega_2) < \infty$, it suffices to show that for any $R_1 \geq 0$,

$$\mu(\Omega_2 \cap E(R_1)) \leq 2\kappa_{\mathsf{v}}\alpha^{-1}\mu(\Omega_1).$$

Fix a compact subset $D = D(R_1) \subset N$ so that $0 < m(DB_{\mathsf{v}}(R_1)) \leq 2m(D)$, which is possible in view of (3.5) and (3.6). Let $\mathsf{T}_\Omega(x)$ be defined as in (3.9). For each $x \in Z$ with $xu \in E(R_1)$, there exists $0 \leq R_u \leq R_1$ such that

$$m(\mathsf{T}_{\Omega_2}(x) \cap B_{\mathsf{v}}(u, R_u)) \leq \alpha^{-1} m(\mathsf{T}_{\Omega_1}(x) \cap B_{\mathsf{v}}(u, R_u)).$$

Consider the cover $\mathcal{C}(x) = \{B_{\mathsf{v}}(u, R_u) : u \in D \cap \mathsf{T}_{E(R_1)}(x)\}$ of the subset $D \cap \mathsf{T}_{E(R_1)}(x)$. By Proposition 3.1, we can find a countable subset $I_x \subset N$ such that the family $\{B_{\mathsf{v}}(u, R_u) : u \in I_x\} \subset \mathcal{C}(x)$ covers $D \cap \mathsf{T}_{E(R_1)}(x)$ and

$$\sum_{u \in I_x} \mathbb{1}_{B_{\mathsf{v}}(u, R_u)} \leq \kappa_{\mathsf{v}} \mathbb{1}_{DB_{\mathsf{v}}(R_1)}.$$

We obtain:

$$\begin{aligned}
\mu(\Omega_2 \cap E(R_1)) &= \frac{1}{m(D)} \int_Z \int_D \mathbb{1}_{\Omega_2 \cap E(R_1)}(xn)\, dn d\mu(x) \\
&= \frac{1}{m(D)} \int_Z m(D \cap \mathsf{T}_{\Omega_2 \cap E(R_1)}(x) \cap (\cup_{u \in I_x} B_{\mathsf{v}}(u, R_u))) d\mu(x) \\
&\le \frac{1}{m(D)} \int_Z \sum_{u \in I_x} m(\mathsf{T}_{\Omega_2}(x) \cap B_{\mathsf{v}}(u, R_u)) d\mu(x) \\
&\le \frac{1}{\alpha \cdot m(D)} \int_Z \sum_{u \in I_x} \int_N \mathbb{1}_{B_{\mathsf{v}}(u, R_u)}(n) \cdot \mathbb{1}_{\Omega_1}(xn) dn\, d\mu(x) \\
&= \frac{1}{\alpha \cdot m(D)} \int_N \int_Z \left(\sum_{u \in I_x} \mathbb{1}_{B_{\mathsf{v}}(u, R_u)}(n) \right) \mathbb{1}_{\Omega_1}(xn) d\mu(x)\, dn \\
&\le \frac{\kappa_{\mathsf{v}}}{\alpha \cdot m(D)} \int_{D B_{\mathsf{v}}(R_1)} \int_Z \mathbb{1}_{\Omega_1}(xn) d\mu(x)\, dn \\
&= \frac{\kappa_{\mathsf{v}} \cdot m(D B_{\mathsf{v}}(R_1))}{\alpha \cdot m(D)} \mu(\Omega_1) \\
&\le 2 \frac{\kappa_{\mathsf{v}}}{\alpha} \mu(\Omega_1).
\end{aligned}$$

□

4. Scenery along $\exp(\mathbb{R}_+ \mathsf{v})$-flow and quasi-invariance

As before, let $G := \prod_{i=1}^{\mathsf{r}} G_i$ where G_i is a connected simple real algebraic group of rank one. Let Γ be a discrete subgroup of G. Let μ be an N-invariant ergodic measure on $\Gamma \backslash G$. In the whole section, we fix a vector $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+$, and set

$$a_t := \exp(t\mathsf{v}) \quad \text{ for } t \in \mathbb{R}.$$

For all $x \in \Gamma \backslash G$, define

$$\mathcal{S}_x(\mathsf{v}) := \limsup_{t \to +\infty} a_t^{-1} g^{-1} \Gamma g a_t = \limsup_{t \to +\infty} \operatorname{Stab}_G(x a_t).$$

The $\limsup_{t \to +\infty}$ above is the topological limit superior, i.e., the collection of all accumulation points; hence we may otherwise write

$$\mathcal{S}_x(\mathsf{v}) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{t > n} a_{-t} g^{-1} \Gamma g a_t}.$$

As $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+$, we have $\mathcal{S}_{xn}(\mathsf{v}) = \mathcal{S}_x(\mathsf{v})$ for all $n \in N$, and hence the measurable map $x \mapsto \mathcal{S}_x$ is N-invariant. Since μ is N-ergodic, there exists a closed subset $\mathcal{S}_\mu(\mathsf{v})$ of G for which $\mathcal{S}_x(\mathsf{v}) = \mathcal{S}_\mu(\mathsf{v})$ for μ-a.e. $x \in \Gamma \backslash G$.

For $\xi, \eta \in \mathcal{F}$, we set

$$\mathfrak{O}_{(\xi,\eta)} := \{ h \in G : \text{loxodromic}, (y_h, \xi), (y_{h^{-1}}, \eta) \in \mathcal{F}^{(2)} \}.$$

We remark that as G_i's are rank one groups, for a loxodromic element $h = (h_1, \cdots, h_{\mathsf{r}}) \in G$ with $h_i \in G_i$ and $\xi = (\xi_1, \cdots, \xi_{\mathsf{r}}) \in \mathcal{F}$ with $\xi_i \in \mathcal{F}_i$, we have $(y_h, \xi) \in \mathcal{F}^{(2)}$ if and only if $y_{h_i} \neq \xi_i$ for all $1 \le i \le \mathsf{r}$.

The main result of this section is the following:

Theorem 4.1. *We have*

$$\lambda\left(\mathcal{S}_\mu(\mathsf{v}) \cap (\mathfrak{O}_{(e^+,e^-)} \cup \mathfrak{O}_{(e^-,e^+)})\right) \subset \mathrm{Stab}_G([\mu]) \tag{4.1}$$

where $\mathrm{Stab}_G([\mu])$ *denotes the stabilizer in* G *of the measure class of* μ.

When G is of rank one, any loxodromic element of G belongs to either $\mathfrak{O}_{(e^+,e^-)}$ or $\mathfrak{O}_{(e^-,e^+)}$. Therefore (4.1) is same as saying

$$\lambda(\mathcal{S}_\mu(\mathsf{v})) \subset \mathrm{Stab}_G([\mu]);$$

this generalizes [22, Thm. 1.3] to all rank one Lie groups.

Since $\mathcal{S}_\mu(\mathsf{v})^{-1} = \mathcal{S}_\mu(\mathsf{v})$, $\mathfrak{O}_{(e^+,e^-)}^{-1} = \mathfrak{O}_{(e^-,e^+)}$, and $\mathrm{Stab}_G([\mu])$ is a subgroup of G, (4.1) follows if we show:

$$\lambda(\mathcal{S}_\mu(\mathsf{v}) \cap \mathfrak{O}_{(e^+,e^-)}) \subset \mathrm{Stab}_G([\mu]) \tag{4.2}$$

The rest of this section is devoted to the proof of (4.2). We fix the left-invariant quasi-distance d_{v} as in (3.3) and set

$$N_\eta := B_{\mathsf{v}}(\eta) \quad \text{for each } \eta > 0$$

where $B_{\mathsf{v}}(\eta)$ is defined as in (3.4). We set

$$t_i := \alpha_i(\mathsf{v}) > 0 \quad \text{for each } 1 \le i \le \mathsf{r}.$$

Since $d_{\mathsf{v}} = \max_i \mathsf{d}_i^{1/t_i}$ where d_i is a left-invariant metric on N_i, for any $\eta > 0$, the quasi-ball N_η is a product of balls in N_i:

$$N_\eta = \prod_{i=1}^{\mathsf{r}} N_i(\eta^{t_i}) \tag{4.3}$$

where $N_i(\eta^{t_i}) := \{x \in N_i : \mathsf{d}_i(e_i, x) < \eta^{t_i}\}$ and e_i denotes the identity element of G_i.[2]

Fix any loxodromic element

$$h_0 \in \mathcal{S}_\mu(\mathsf{v}) \cap \mathfrak{O}_{(e^+,e^-)}.$$

Our goal is to show that $\lambda(h_0) \in \mathrm{Stab}_G([\mu])$.

Writing $h_0 = (h_1, \cdots, h_{\mathsf{r}})$ component-wise, each h_i is a loxodromic element of G_i. We write $h_i = \varphi_i a_i^{-1} m_i \varphi_i^{-1}$ for some $a_i \in A_i^+ - \{e\}$, $m_i \in M_i$ and $\varphi_i \in G_i$ so that $\varphi_i^- = \varphi_i e_i^- \in \mathcal{F}_i$ and $\varphi_i^+ = \varphi_i e_i^+ \in \mathcal{F}_i$ are the unique attracting fixed points of h_i and h_i^{-1} respectively; here $e_i^\pm \in \mathcal{F}_i$ means

[2]We stress that the notation N_i with subscript i is used solely for the subgroup $G_i \cap N$, whereas N_η, N_ε, etc are used for quasi-balls in N.

the i-th component of $e^{\pm} \in \mathcal{F} = \prod_i \mathcal{F}_i$. As G_i is of rank one, we have $\mathcal{F}_i = N_i e_i^- \cup \{e_i^+\}$. Since $h_0 \in \mathfrak{O}_{(e^+,e^-)}$, we have, for all i,

$$\varphi_i^- \neq e_i^+ \text{ and } \varphi_i^+ \neq e_i^-.$$

We denote by n_i the unique element of N_i such that

$$\varphi_i^- = n_i e_i^- \in N_i e_i^-. \tag{4.4}$$

Using the diffeomorphism between N_i and $N_i e_i^-$ given by $n \mapsto n e_i^-$, we may regard d_i as a left-invariant metric on $N_i e_i^-$, so that

$$\mathsf{d}_i(n e_i^-, n' e_i^-) = \mathsf{d}_i(n, n') \quad \text{ for all } n, n' \in N_i. \tag{4.5}$$

Definition of η_0. Since $e_i^- \neq \varphi_i^+$ and hence $e_i^- \in \varphi_i N_i e_i^-$, there exist $\eta_0 > 0$ and $J > 0$ such that

$$N_{\eta_0} e^- \subset \prod_{i=1}^{\mathsf{r}} \varphi_i N_i(J) e_i^-. \tag{4.6}$$

Lemma 4.2. *There exists $p_0 = p_0(h_0) \in \mathbb{N}$ such that for all $p \geq p_0$, and $1 \leq i \leq \mathsf{r}$, we have*

$$\mathsf{d}_i(h_i^p z_i, h_i^p z_i') \leq \frac{1}{2^{(t_i+1)}} \cdot \mathsf{d}_i(z_i, z_i') \tag{4.7}$$

for all z_i, $z_i' \in \varphi_i N_i(J) e_i^-$.

Proof. Since $(a_i^{-1} m_i)^p n e_i^- = (a_i^{-p}(m_i^p n m_i^{-p}) a_i^p) e_i^-$ and M_i is a compact subgroup normalizing N_i, we have $(a_i^{-1} m_i)^p n e_i^- \to e_i^-$ as $p \to \infty$, uniformly for all $n \in N_i(J)$. Therefore $\varphi_i (a_i^{-1} m_i)^p N_i(J) e_i^-$ is contained in a compact subset of $N_i \varphi_i^- = N_i e_i^-$ for all sufficiently large p. Since $N_i e_i^-$ is endowed with a metric d_i, induced from a Euclidean norm on $\mathfrak{n}_i$, the Lipschitz constant $\operatorname{Lip}(\varphi_i|_{(a_i^{-1} m_i)^p N_i(J) e_i^-})$ is well defined and finite. Since $h_i^p = \varphi_i (a_i^{-1} m_i)^p \varphi_i^{-1}$, we have

$$\begin{aligned} &\operatorname{Lip}(h_i^p|_{\varphi_i N_i(J) e_i^-}) \\ &\leq \operatorname{Lip}(\varphi_i|_{(a_i^{-1} m_i)^p N_i(J) e_i^-}) \operatorname{Lip}((a_i^{-1} m_i)^p|_{N_i(J) e_i^-}) \operatorname{Lip}(\varphi_i^{-1}|_{\varphi_i N_i(J) e_i^-}). \end{aligned}$$

Since $\operatorname{Lip}((a_i^{-1} m_i)^p|_{N_i(J) e_i^-}) \to 0$ as $p \to \infty$ and $(a_i^{-1} m_i)^p N_i(J) e_i^- \to e_i^-$, we have $\operatorname{Lip}(h_i^p|_{\varphi_i N_i(J) e_i^-}) \to 0$ as $p \to \infty$. Therefore the lemma follows. $\square$

Since $h_0^p \prod_{i=1}^{\mathsf{r}} n_i N_i(\eta_0^{t_i}) e^- \to y_{h_0}$ uniformly, as $p \to \infty$, and $y_{h_0} \in N e^-$, by possibly increasing p_0 if necessary, we may assume that p_0 satisfies that for all $p \geq p_0$,

$$h_0^p \prod_{i=1}^{\mathsf{r}} n_i N_i(\eta_0^{t_i}) \subset NLN^+; \tag{4.8}$$

$$\sup_{u \in N_{\eta_0} y_{h_0}} |\operatorname{Jac}_u b^N(h_0^p, \cdot)| \leq 1/2; \tag{4.9}$$

$$h_0^p N_r y_{h_0} \subset N_{r/2} y_{h_0} \quad \text{for all } 0 < r < \eta_0. \tag{4.10}$$

We make use the following simple observation:

Lemma 4.3. *If there exists* $p_1 \geq 1$ *such that*

$$\{\lambda(h_0^p) : p \geq p_1\} \subset \operatorname{Stab}_G([\mu]),$$

then $\lambda(h_0) \in \operatorname{Stab}_G([\mu])$.

Proof. Since $\operatorname{Stab}_G([\mu])$ is a group and $\lambda(h_0)^p = \lambda(h_0^p)$, the above lemma implies that

$$\lambda(h_0) = \lambda(h_0)^{p+1} \lambda(h_0)^{-p} \in \operatorname{Stab}_G([\mu]).$$

□

Hence it suffices to show that for all $p \geq p_0$, $\lambda(h_0^p) \in \operatorname{Stab}_G([\mu])$. In the rest of this section, fix any $p \geq p_0$ and set

$$g_0 = h_0^p.$$

We now assume that

$$\ell_0 := \lambda(g_0) \notin \operatorname{Stab}_G([\mu]) \tag{4.11}$$

and will prove that this assumption leads to a contradiction.

We write $g_i = h_i^p$ so that

$$g_0 = (g_1, \cdots, g_{\mathsf{r}}).$$

Noting that φ_i^- and φ_i^+ are the attracting fixed points of g_i and g_i^{-1} respectively, we set $\varphi := (\varphi_1, \cdots, \varphi_{\mathsf{r}})$. Hence $\varphi^{\mp} = (\varphi_1^{\mp}, \cdots, \varphi_{\mathsf{r}}^{\mp})$ are the attracting fixed points of $g_0^{\pm 1}$ respectively. We set

$$y_{g_0} := \varphi^-.$$

Note that $y_{g_0} = y_{h_0}$. By (4.7), for all $k \in \mathbb{N}$, we have

$$\mathsf{d}_i(g_i^k z_i, g_i^k z_i') \leq \frac{1}{2^{(t_i+1)k}} \cdot \mathsf{d}_i(z_i, z_i') \tag{4.12}$$

for any z_i, $z_i' \in \varphi_i N_i(J) e_i^-$.

We begin by presenting a long list of constants and subsets in a carefully designed order to be used in getting two contradictory upper and lower bounds in Lemmas 4.15 and 4.16.

Definition of E, $\mathcal{O}_L$ and $\mathcal{O}_{\ell_0}$. We fix subsets $E \subset \Gamma\backslash G$ and $\mathcal{O}_L \subset L$ as given by the following lemma:

Lemma 4.4. *There exist an N-invariant μ-conull set $E \subset \Gamma\backslash G$ and a symmetric neighborhood $\mathcal{O}_L \subset L$ of e such that*

$$E \cap E\ell_0^{-1}\mathcal{O}_L = \emptyset.$$

Proof. Since μ is N-ergodic and $\ell_0 \notin \mathrm{Stab}[\mu]$, μ and $\mu.\ell_0$ are mutually singular. Hence there exists a μ-conull subset $E' \subset \Gamma\backslash G$ with $E' \cap E'\ell_0 = \emptyset$. Let $c = 1$ if $|\mu| = \infty$, and $c = |\mu|$ otherwise. Choose $x \in E' \cap \operatorname{supp}(\mu)$ and a bounded neighborhood $\mathcal{O} \subset G$ of e such that $\mu(x\mathcal{O}) > c/2$. Set $F := E' \cap x\mathcal{O}\ell_0^{-1}\mathcal{O}$. Since $F\ell_0 \subset E'\ell_0$ is a bounded null set, there exists a symmetric neighborhood $\mathcal{O}_L \subset L \cap \mathcal{O}$ of e such that $\mu(F\mathcal{O}_L\ell_0) < c/4$. Noting that $\mu(x\mathcal{O} - F\mathcal{O}_L\ell_0) > c/4$, we may choose a compact subset $C \subset x\mathcal{O} - F\mathcal{O}_L\ell_0$ with $\mu(C) > c/4$. Since $C\ell_0^{-1}\mathcal{O}_L \subset x\mathcal{O}\ell_0^{-1}\mathcal{O}$, we have

$$C\ell_0^{-1}\mathcal{O}_L \cap E' \subset x\mathcal{O}\ell_0^{-1}\mathcal{O} \cap E' = F.$$

Since $C\ell_0^{-1}\mathcal{O}_L \cap F = \emptyset$ by the choice of C, we get $C\ell_0^{-1}\mathcal{O}_L \cap E' = \emptyset$ and hence $\mu(C\ell_0^{-1}\mathcal{O}_L) = 0$. Consider the following N-invariant measurable subsets:

$$E_1 := \{z \in \Gamma\backslash G : \int_N \mathbb{1}_C(zn)dn > 0\} \quad \text{and}$$

$$E_2 := \{z \in \Gamma\backslash G : \int_N \mathbb{1}_{C\ell_0^{-1}\mathcal{O}_L}(zn)dn = 0\}.$$

Recall $B_{\mathsf{v}}(j)$ denotes the set $\{n \in N : d_{\mathsf{v}}(n,e) < j\}$ for each $j \in \mathbb{N}$. Since $\int_{z\in\Gamma\backslash G}\int_{B_{\mathsf{v}}(1)} \mathbb{1}_C(zn)dnd\mu(z) = \mu(C)m(B_{\mathsf{v}}(1)) > 0$, we have $\mu(E_1) > 0$ by Fubini's lemma. Since

$$\int_{z\in\Gamma\backslash G}\int_{B_{\mathsf{v}}(j)} \mathbb{1}_{C\ell_0^{-1}\mathcal{O}_L}(zn)dnd\mu(z) = \mu(C\ell_0^{-1}\mathcal{O}_L)m(B_{\mathsf{v}}(j)) = 0,$$

again by Fubini's lemma, $E_2(j)$ is μ-conull, where $E_2(j) := \{z \in \Gamma\backslash G : \int_{B_{\mathsf{v}}(j)} \mathbb{1}_{C\ell_0^{-1}\mathcal{O}_L}(zn)dn = 0\}$. Since $E_2 = \cap_{j=1}^{\infty} E_2(j)$, the set E_2 is μ-conull as well. Therefore, if we set $E = E_1 \cap E_2$, then E is an N-invariant measurable subset with $\mu(E) > 0$. Now the N-ergodicity of μ implies that E is a μ-conull subset. Moreover, we have $E \cap E\ell_0^{-1}\mathcal{O}_L = \emptyset$; to see this, suppose $z = y\ell_0^{-1}\ell$ for some $z, y \in E$ and $\ell \in \mathcal{O}_L$. Then $\int_N \mathbb{1}_{C\ell_0^{-1}\mathcal{O}_L}(y\ell_0^{-1}\ell n)dn = 0$. By changing the variable $\ell_0^{-1}\ell n(\ell_0^{-1}\ell)^{-1} \to n$, it implies that $\int_N \mathbb{1}_{C\ell_0^{-1}\mathcal{O}_L\ell^{-1}\ell_0}(yn)dn = 0$. Since $C \subset C\ell_0^{-1}\mathcal{O}_L\ell^{-1}\ell_0$, we get $\int_N \mathbb{1}_C(yn)dn = 0$, implying $y \notin E$, yielding contradiction. □

We set

$$\mathcal{O}_{\ell_0} := \ell_0\mathcal{O}_L, \tag{4.13}$$

so that $E \cap E\mathcal{O}_{\ell_0}^{-1} = \emptyset$.

For a differentiable map $f : N \to N$, let $\mathrm{D}_u f : T_uN \to T_{f(u)}N$ denote the differential of f at $u \in N$. Let $\tau_u : N \to N$ denote the left translation map, i.e., $\tau_u(n) = un$ for $n \in N$. Choosing a basis $\mathcal{B}_e := \{v_1, \cdots, v_m\}$ of $T_e N$, the collection $\mathcal{B}_w := \{\mathrm{D}_e\, \tau_w(v_1), \cdots, \mathrm{D}_e\, \tau_w(v_m)\}$ gives a basis for $T_w N$ for each $w \in N$. The following Jacobian of f at $u \in N$ is well-defined, independent of the choice of $\mathcal{B}_e$:

$$\mathrm{Jac}_u\, f := \det\, [\mathrm{D}_u\, f]^{\mathcal{B}_{f(u)}}_{\mathcal{B}_u}.$$

Here $[\mathrm{D}_u\, f]^{\mathcal{B}_{f(u)}}_{\mathcal{B}_u}$ denotes the matrix representation of $\mathrm{D}_u\, f$ with respect to the indicated bases.

Definition of r_1, r_0. Since $b^{AM}(g_0, y_{g_0}) = \ell_0$ and $b^{AM}(g_0, \cdot)$ is continuous at y_{g_0}, we can find $0 < r_1 < \min_i \frac{1}{2^{1+(1/t_i)}} \eta_0$ such that

$$b^{AM}(g_0, N_{r_1} y_{g_0}) \subset \mathcal{O}_{\ell_0}.$$

Set

$$r_0 := \frac{3}{4} r_1.$$

Definition of k, c, η. By (4.6), we have $g_0^j N_{\eta_0} e^- \to y_{g_0}$ uniformly as $j \to \infty$. Hence we may fix a large integer $k \geq 1$ which satisfies the following three conditions for all $1 \leq i \leq \mathsf{r}$:

$$N_{r_1/2} y_{g_0} \subset N_{r_0}\, g_0^k N_{\eta_0} e^- \subset N_{r_1} y_{g_0}; \tag{4.14}$$

$$b^{N_i}(g_i^k, N_i({\eta_0}^{t_i})) \subset n_i N_i(r_0^{t_i}/4); \tag{4.15}$$

$$g_0 b^N(g_0^k, N_{\eta_0}) N_{r_0} \subset N L N^+ \tag{4.16}$$

where n_i is given in (4.4). Since $g_i^k e_i^+ \neq g_i^k e_i^-$, we can choose $0 < \eta < \frac{1}{2}\eta_0$ satisfying

$$g_i^k e_i^+ \notin b^{N_i}(g_i^k, e_i) N_i(\eta^{t_i}) e_i^- \quad \text{ for all } i. \tag{4.17}$$

We fix a small number $0 < c < 1/2$ so that for all $1 \leq i \leq \mathsf{r}$ and $x, y \in N_i(\eta^{t_i}) e_i^-$,

$$(2c)^{t_i} \mathsf{d}_i(x, y) \leq \mathsf{d}_i(b^{N_i}(g_i^k, x), b^{N_i}(g_i^k, y)) \tag{4.18}$$

and

$$2c < \min\left(\inf_{u \in N_{r_1}} |\, \mathrm{Jac}_u\, b^N(g_0, \cdot)|, \inf_{u \in N_{r_1}} |\, \mathrm{Jac}_u\, b^N(g_0^k, \cdot)| \right).$$

Lemma 4.5. *We have*

$$b^N(g_0^k, e) N_{2c\eta} \subset b^N(g_0^k, N_\eta) \subset b^N(g_0^k, e) N_\eta, \tag{4.19}$$

and

$$b^N(g_0^k, N_{c\eta}) \subset b^N(g_0^k, e) N_{c\eta}. \tag{4.20}$$

Proof. Fix $1 \leq i \leq \mathsf{r}$. By (4.17), we have $b^{N_i}(g_i^k, e_i) N_i(\eta^{t_i}) e_i^- \subset g_i^k N_i e_i^-$ and hence $b^{N_i}(g_i^k, e_i) N_i(\eta^{t_i}) \subset b^{N_i}(g_i^k, N_i)$. Let $n \in N_i((2c\eta)^{t_i})$ be arbitrary. There exists $n' \in N_i$ such that $b^{N_i}(g_i^k, e_i) n = b^{N_i}(g_i^k, n')$. We have, by (4.18),

$$\begin{aligned}(2c)^{t_i} \mathsf{d}_i(e_i, n') &\leq \mathsf{d}_i(b^{N_i}(g_i^k, e_i), b^{N_i}(g_i^k, n')) \\ &= \mathsf{d}_i(b^{N_i}(g_i^k, e_i), b^{N_i}(g_i^k, e_i) n) = \mathsf{d}_i(e_i, n) \leq (2c\eta)^{t_i}\end{aligned}$$

and hence $\mathsf{d}_i(e_i, n') \le \eta^{t_i}$. It implies

$$b^{N_i}(g_i^k, e_i)n = b^{N_i}(g_i^k, n') \in b^{N_i}(g_i^k, N_i(\eta^{t_i})).$$

This proves the first inclusion in (4.19).

By (4.12) and (4.6), we have

$$\mathsf{d}_i(g_i^k n e_i^-, g_i^k n' e_i^-) \le 2^{-k}\mathsf{d}_i(n e_i^-, n' e_i^-) \quad \text{for all } n, n' \in N_i(\eta^{t_i}).$$

In other words, for all $n, n' \in N_i(\eta^{t_i})$,

$$\mathsf{d}_i(b^{N_i}(g_i^k, n), b^{N_i}(g_i^k, n')) \le 2^{-k}\mathsf{d}_i(n, n'). \tag{4.21}$$

Hence $b^{N_i}(g_i^k, \cdot)$ has Lipschitz constant less than 1 on $N_i(\eta^{t_i})$, the right inclusion in (4.19), as well as (4.20) follow. □

Lemma 4.6. *We have*

$$b^N(g_0, b^N(g_0^k, v)N_{r_0}) \subset b^N(g_0^k, v)N_{r_0} \quad \textit{for all } v \in N_\eta. \tag{4.22}$$

Proof. As d_i is left-invariant, the choice of k as in (4.15) implies that for any $v \in N_i(\eta^{t_i})$, we have

$$b^{N_i}(g_i^k, v)N_i(r_0^{t_i}) \supset n_i N_i(3r_0^{t_i}/4) \quad \text{and}$$

$$b^{N_i}(g_i^k, N_i(\eta^{t_i}))N_i({r_0}^{t_i}) \subset n_i N_i(3r_0^{t_i}/2).$$

Since $r_1 < \min_i \frac{1}{2^{1+(1/t_i)}}\eta_0$ and hence $3r_0^{t_i}/2 < \eta_0^{t_i}$ by the definition of r_0, it follows from (4.12) and the property $g_i\varphi_i^- = \varphi_i^-$ that

$$g_i n_i N_i(3r_0^{t_i}/2) \subset n_i N_i(3r_0^{t_i}/4).$$

Therefore, for any $v \in N_i(\eta^{t_i})$,

$$b^{N_i}(g_i, b^{N_i}(g_i^k, v)N_i({r_0}^{t_i})) \subset n_i N_i(3r_0^{t_i}/4) \subset b^{N_i}(g_i^k, v)N_i(r_0^{t_i}).$$

This proves the lemma. □

Definition of V_0. Since the following (4.23)- (4.30) are all open conditions which have been proved at $g = g_0$ in (4.9),(4.10), (4.14) and Lemmas 4.5 and 4.6, we may choose a bounded neighborhood V_0 of g_0 in G such that those conditions continue to hold for all $g \in V_0$, $u \in N_{r_0}b^N(g^k, N_\eta)$ and $v \in N_\eta$:

$$gN_{r_1}y_{g_0} \subset N_{r_1/2}y_{g_0}, \tag{4.23}$$

$$N_{r_1/2}y_{g_0} \subset N_{r_0}\, g^k N_\eta e^- \subset N_{r_1}y_{g_0}, \tag{4.24}$$

$$b^{AM}(g, u) \in \mathcal{O}_{\ell_0}, \tag{4.25}$$

$$b^N(g^k, e)N_{2c\eta} \subset b^N(g^k, N_\eta) \subset b^N(g^k, e)N_\eta \quad \text{and} \tag{4.26}$$

$$b^N(g^k, N_{c\eta}) \subset b^N(g^k, e)N_{c\eta}. \tag{4.27}$$

$$2c < |\operatorname{Jac}_u b^N(g, \cdot)| < 1, \quad 2c < |\operatorname{Jac}_v b^N(g^k, \cdot)| < 1 \tag{4.28}$$

$$b^N(g, b^N(g^k, v)N_{r_0}) \subset b^N(g^k, v)N_{r_0} \tag{4.29}$$

$$gb^N(g^k, N_\eta)N_{r_0} \subset NLN^+. \tag{4.30}$$

Definition of R, $\mathcal{B}_L$, and $\mathcal{B}_{N^+}$. Since the sets V_0, N_η and $\{b^N(g^k, N_\eta)N_{r_0} : g \in V_0\}$ are bounded, it follows from (4.30) that there exist $R > 0$ and bounded symmetric neighborhoods $\mathcal{B}_L \subset L$ and $\mathcal{B}_{N^+} \subset N^+$ of e such that for all $g \in V_0$,

$$g^k N_\eta \subset N_R \mathcal{B}_L \mathcal{B}_{N^+} \quad \text{and} \quad g b^N(g^k, N_\eta) N_{r_0} \subset N_R \mathcal{B}_L \mathcal{B}_{N^+}. \tag{4.31}$$

Definition of β, R' and κ_*. We fix $\beta > 0$ such that

$$a_t^{-1} N_R N_\eta a_t N_{c\eta} \subset N_{2c\eta} \quad \text{for all } t \geq \beta. \tag{4.32}$$

We also fix $R' > 0$ so that

$$\bigcup_{t \in [-\beta, \beta]} N_R N_\eta (a_t N_\eta N_R N_{c\eta} a_t^{-1}) \subset N_{R'}. \tag{4.33}$$

Recalling the notation from Lemma 3.2, we set

$$\kappa_* := \kappa_*(\mathsf{v}, \beta, c\eta, R') = \frac{m(N_{R'})}{m(N_{c\eta})} \kappa_{\mathsf{v}} e^{\|2\rho\|\beta}. \tag{4.34}$$

Definition of Ω, $\tilde{\Omega}$, $\mathcal{O}_{N^+}$, Q, $Q_\perp$ and T_0. Let E be an N-invariant μ-conull set as in Lemma 4.4. We fix a compact subset $\Omega \subset E$ with $\mu(\Omega) > 0$, and define

$$\tilde{\Omega} := \Omega \mathcal{B}_L \mathcal{B}_{N^+}. \tag{4.35}$$

Since $\mu(\tilde{\Omega}) = \mu(\tilde{\Omega} \cap E)$, we can find a compact set $\Omega \subset Q \subset \tilde{\Omega} \cap E$ satisfying

$$\mu(\tilde{\Omega} - Q) < \frac{c}{16\kappa_0 \kappa_*}. \tag{4.36}$$

Since $Q \subset E$, we know $\mu(Q\mathcal{O}_{\ell_0}^{-1}) = 0$. By the uniform convergence theorem, there exists a bounded symmetric neighborhood $\mathcal{O}_{N^+} \subset \mathcal{B}_{N^+}$ of e for which the set

$$Q_\perp := Q \mathcal{O}_{N^+} \mathcal{O}_{\ell_0}^{-1} \tag{4.37}$$

satisfies

$$\mu(Q_\perp) < \frac{c^2}{16\kappa_{\mathsf{v}} \kappa_0 \kappa_*} \mu(\Omega). \tag{4.38}$$

We fix $T_0 > 0$ such that

$$\operatorname{Ad}_{a_t} \mathcal{B}_{N^+} \subset \mathcal{O}_{N^+} \quad \text{for all } t \geq T_0. \tag{4.39}$$

Definition of T_1, Ω_1, Ω_2, Ξ and Θ. Since $\mathcal{S}_x(\mathsf{v}) = \mathcal{S}_\mu(\mathsf{v})$ for μ-a.e. $x \in \Gamma\backslash G$, we can find $T_1 > T_0$ so that the set

$$\tilde{\Omega}_1 := \{x \in \tilde{\Omega} : \operatorname{Stab}_G(x a_t) \cap V_0 \neq \emptyset \text{ for some } T_0 \leq t \leq T_1\} \tag{4.40}$$

satisfies

$$\mu(\tilde{\Omega} - \tilde{\Omega}_1) < \frac{1}{4}\mu(\Omega). \tag{4.41}$$

Set

$$\Omega_1 := \Omega \cap \tilde{\Omega}_1. \tag{4.42}$$

Since $\Omega \subset \tilde{\Omega}$, we have

$$\mu(\Omega_1) \geq \mu(\Omega) - \mu(\tilde{\Omega} - \tilde{\Omega}_1) > \frac{3}{4}\mu(\Omega). \tag{4.43}$$

We define

$$\Xi := \left\{ x \in \Gamma\backslash G : \exists t > 0 \text{ s.t } \int_{a_t N_{r_0} a_t^{-1}} \mathbb{1}_{Q_\perp}(xn)\, dn \geq 2c \int_{a_t N_{r_0} a_t^{-1}} \mathbb{1}_Q(xn)\, dn \right\}. \tag{4.44}$$

Set

$$\Omega_2 := \Omega_1 - \Xi. \tag{4.45}$$

Recall the notation for distance d_0 on N and the corresponding metric balls $B_0(r)$, $r > 0$, from Proposition 3.1. Consider the following set

$$\Theta := \left\{ x \in \Gamma\backslash G : \exists r > 0 \text{ s.t } \int_{B_0(r)} \mathbb{1}_{\tilde{\Omega}\cap\Xi}(xn)\, dn \geq \frac{c}{\kappa_*} \int_{B_0(r)} \mathbb{1}_{\Omega_2}(xn)\, dn \right\}. \tag{4.46}$$

Proposition 4.7. *We have*

$$\mu(\Omega_2 - \Theta) > \frac{1}{4}\mu(\Omega).$$

Proof. Since $a_t N_{r_0} a_t^{-1} = B_{\mathsf{v}}(e^t r_0)$ for any $t, r_0 > 0$, we may apply the maximal ratio inequality (Lemma 3.4) and (4.38) and get

$$\mu(Q \cap \Xi) \leq \frac{2\kappa_{\mathsf{v}}}{2c}\mu(Q_\perp) < \frac{\kappa_{\mathsf{v}}}{c} \cdot \frac{c^2}{16\kappa_{\mathsf{v}}\kappa_0\kappa_*}\mu(\Omega) = \frac{c}{16\kappa_0\kappa_*}\mu(\Omega).$$

Therefore, by (4.36),

$$\mu(\tilde{\Omega} \cap \Xi) \leq \mu(\tilde{\Omega} - Q) + \mu(Q \cap \Xi) < \frac{c}{8\kappa_0\kappa_*}\mu(\Omega).$$

By (4.43), we have

$$\mu(\Omega_2) = \mu(\Omega_1 - \Xi) \geq \mu(\Omega_1) - \mu(\tilde{\Omega} \cap \Xi) \geq \left(\frac{3}{4} - \frac{c}{8\kappa_0\kappa_*}\right)\mu(\Omega) > \frac{1}{2}\mu(\Omega).$$

Employing the maximal ratio inequality yet again, we deduce

$$\mu(\Omega_2 \cap \Theta) \leq \frac{2\kappa_0\kappa_*}{c}\mu(\tilde{\Omega} \cap \Xi) < \frac{2\kappa_0\kappa_*}{c} \cdot \frac{c}{8\kappa_0\kappa_*}\mu(\Omega) = \frac{1}{4}\mu(\Omega),$$

implying the claim by (4.43). ☐

Choice of x_0, R_1, R_2 and D. We fix $R_1, R_2 > 0$ so that $N_R \subset B_0(R_1)$ and

$$\bigcup_{0<t\leq T_1} a_t B_0(R_1) a_t^{-1} \subset B_0(R_2). \tag{4.47}$$

We choose x_0 and D as in the following lemma:

Lemma 4.8. *There exist $x_0 \in \Gamma\backslash G$ and a ball $D = B_0(R_{x_0})$ with $R_{x_0} > R_2$ such that*

$$\frac{\int_D \mathbb{1}_{\tilde{\Omega}\cap\Xi}(x_0 n)\, dn}{\int_D \mathbb{1}_{\Omega_2}(x_0 n)\, dn} < \frac{c}{\kappa_*}, \quad \text{and} \quad \frac{\int_{\partial_{R_2} D} \mathbb{1}_{\Omega_2}(x_0 n)\, dn}{\int_D \mathbb{1}_{\Omega_2}(x_0 n)\, dn} < \frac{1}{2}$$

where $\partial_r B_0(R_{x_0}) := B_0(R_{x_0}) - B_0(R_{x_0} - r)$.

Proof. Choose any $x_0 \in \Omega_2 - \Theta$, which is possible by Proposition 4.7. By the definition of Θ, x_0 satisfies the first inequality for any ball $D = B_0(R)$. By Lemma 3.3, there exists $R_{x_0} > R_2$ satisfying the second inequality, as required. $\square$

For any $X \subset \Gamma\backslash G$, define the subset $\mathsf{T}_X \subset N$ by

$$\mathsf{T}_X := \{n \in N : x_0 n \in X\}.$$

Definition of t_u, a_u, g_u. By the definition of Ω_1 in (4.40), for each $u \in \mathsf{T}_{\Omega_1}$, we can choose $T_0 \le t_u \le T_1$ such that

$$\mathrm{Stab}_G(x_0 u a_{t_u}) \cap V_0 \neq \emptyset.$$

We set $a_u := a_{t_u}$ for the sake of simplicity, and choose

$$g_u \in \mathrm{Stab}_G(x_0 u a_u) \cap V_0.$$

Lemma 4.9. *For $u \in \mathsf{T}_{\Omega_1}$, we have $u a_u b^N(g_u^k, N_\eta) a_u^{-1} \subset \mathsf{T}_\Xi$.*

Proof. Let $u \in \mathsf{T}_{\Omega_1}$ and $v_0 \in N_\eta$ be arbitrary. Setting $v_0' := b^N(g_u^k, v_0)$, we need to show that $x_0 u a_u v_0' a_u^{-1} \in \Xi$. Observe that for all $v \in N$,

$$\begin{aligned} x_0 u(a_u v a_u^{-1}) &= x_0 u a_u (g_u v) a_u^{-1} \\ &= x_0 u a_u (b^N(g_u, v)\, b^{AM}(g_u, v)\, b^{N^+}(g_u, v)) a_u^{-1} \\ &= x_0 u (a_u b^N(g_u, v) a_u^{-1})\, b^{AM}(g_u, v)\, (a_u b^{N^+}(g_u, v) a_u^{-1}), \end{aligned} \tag{4.48}$$

whenever $b(g_u, v)$ is defined. For any $n \in N_{r_0}$, we can plug $v = v_0' n$ into (4.48) by (4.30), and get

$$x_0 u(a_u v_0' n a_u^{-1}) = x_0 u(a_u b^N(g_u, v_0' n) a_u^{-1})(\ell a_u b^{N^+}(g_u, v_0' n) a_u^{-1})$$

where $\ell := b^{AM}(g_u, v_0' n) \in \mathcal{O}_{\ell_0}$ by (4.25).

Recall that $b^{N^+}(g_u, v_0' n) \in \mathcal{B}_{N^+}$ by (4.31) and $\mathrm{Ad}_{a_t}(\mathcal{B}_{N^+}) \subset \mathcal{O}_{N^+}$ for all $t \ge T_0$ by (4.39). It follows that

$$a_u b^{N^+}(g_u, v_0' n) a_u^{-1} \in \mathcal{O}_{N^+}.$$

Since

$$x_0 u(a_u b^N(g_u, v_0' n) a_u^{-1}) = x_0 u(a_u v_0' n a_u^{-1})(a_u b^{N^+}(g_u, v_0' n) a_u^{-1})^{-1} \ell^{-1},$$

and $Q_\perp = Q \mathcal{O}_{N^+} \mathcal{O}_{\ell_0}^{-1}$ as defined in (4.37), we have for all $n \in N_{r_0}$,

$$\mathbb{1}_Q(x_0 u a_u v_0' n a_u^{-1}) \le \mathbb{1}_{Q_\perp}(x_0 u(a_u b^N(g_u, v_0' n) a_u^{-1})). \tag{4.49}$$

Note that

$$\begin{aligned}
&\int_{N_{r_0}} \mathbb{1}_Q(x_0 u a_u v_0' a_u^{-1}(a_u n a_u^{-1}))\, dn \\
&\le \int_{N_{r_0}} \mathbb{1}_{Q_\perp}(x_0 u a_u b^N(g_u, v_0' n) a_u^{-1})\, dn \quad \text{by (4.49)} \\
&\le (2c)^{-1} \int_{b^N(g_u, v_0' N_{r_0})} \mathbb{1}_{Q_\perp}(x_0 u(a_u n a_u^{-1}))\, dn \;\; \text{by (4.28) and Lemma 4.10} \\
&\le (2c)^{-1} \int_{v_0' N_{r_0}} \mathbb{1}_{Q_\perp}(x_0 u(a_u n a_u^{-1}))\, dn \quad \text{by (4.29)} \\
&= (2c)^{-1} \int_{N_{r_0}} \mathbb{1}_{Q_\perp}(x_0 u a_u v_0' a_u^{-1}(a_u n a_u^{-1}))\, dn.
\end{aligned}$$

Hence by the change of variable formula, we have

$$\int_{a_u N_{r_0} a_u^{-1}} \mathbb{1}_{Q_\perp}(x_0 u a_u v_0' a_u^{-1} n)\, dn \ge 2c \int_{a_u N_{r_0} a_u^{-1}} \mathbb{1}_Q(x_0 u a_u v_0' a_u^{-1} n)\, dn.$$

In view of definition (4.44), this proves that $x_0 u a_u v_0' a_u^{-1} \in \Xi$. $\square$

Although the following lemma, which was used in the above proof, should be a standard fact, we could not find a reference, so we provide a proof.

Lemma 4.10. *For any measurable function $f : N \to \mathbb{R}$ and a differentiable map $\phi : N \to N$, we have*

$$\int_N (f \circ \phi)(n)\, |\operatorname{Jac}_n \phi|\, dn = \int_N f(n)\, dn.$$

Proof. Since N is a simply connected nilpotent Lie group, the Haar measure dn on N is the push-forward of the Lebesgue measure $d\,\mathrm{Leb}$ on $\mathfrak{n} = \operatorname{Lie} N$ by the exponential map. Let $\tilde{\phi} := \log \circ \phi \circ \exp$. Note that $\mathrm{Id} + \frac{1}{2}\operatorname{ad}_x \in \mathrm{GL}(\mathfrak{n})$ is unipotent for all $x \in \mathfrak{n}$, as $\operatorname{ad}_x \in \operatorname{End}(\mathfrak{n})$ is a nilpotent element. We claim that $|\operatorname{Jac}_{e^x} \phi| = |\operatorname{Jac}_x \tilde{\phi}|$.

Since N is a nilpotent Lie group of at most 2-step, we have for any $n, n' \in N$,

$$\log(nn') = \log n + \log n' + \frac{1}{2}[\log n, \log n'].$$

Hence, we get via a direct computation:

$$
\begin{aligned}
&\frac{d}{dt}\log\phi(e^x)^{-1}\phi(e^xe^{ty})\\
&=\frac{d}{dt}\log\phi(e^x)^{-1}\phi(e^{x+ty+\frac{1}{2}t[x,y]})\\
&=\frac{d}{dt}\left(\log\phi(e^x)^{-1}+\log\phi(e^{x+ty+\frac{1}{2}t[x,y]})+\tfrac{1}{2}[\log\phi(e^x)^{-1},\log\phi(e^{x+ty+\frac{1}{2}t[x,y]})]\right)\\
&=(\mathrm{Id}_{\mathfrak{n}}+\tfrac{1}{2}\operatorname{ad}_{-\tilde{\phi}(x)})\left(\frac{d}{dt}\tilde{\phi}(x+t(y+\tfrac{1}{2}[x,y]))\right)\\
&=(\mathrm{Id}_{\mathfrak{n}}+\tfrac{1}{2}\operatorname{ad}_{-\tilde{\phi}(x)})\circ(\mathrm{D}_x\,\tilde{\phi})(y+\tfrac{1}{2}[x,y]).
\end{aligned}
$$

Now let $x\in\mathfrak{n}$ and $y\in T_{e^x}N$. In view of the identification $\mathfrak{n}=T_eN\simeq T_nN$ for $n=e^x$ and $\phi(e^x)$, we have

$$
\begin{aligned}
\mathrm{D}_{e^x}\,\phi(y)&=\frac{d}{dt}\Big|_{t=0}\phi(e^x)^{-1}\phi(e^xe^{ty})\\
&=\frac{d}{dt}\Big|_{t=0}\exp\circ\log\phi(e^x)^{-1}\phi(e^xe^{ty})\\
&=(\mathrm{D}_0\exp)\left(\frac{d}{dt}\Big|_{t=0}\log\phi(e^x)^{-1}\phi(e^xe^{ty})\right)\\
&=(\mathrm{D}_0\exp)\circ(\mathrm{Id}_{\mathfrak{n}}+\tfrac{1}{2}\operatorname{ad}_{-\tilde{\phi}(x)})\circ(\mathrm{D}_x\,\tilde{\phi})(y+\frac{1}{2}[x,y])\\
&=(\mathrm{D}_0\exp)\circ(\mathrm{Id}_{\mathfrak{n}}+\tfrac{1}{2}\operatorname{ad}_{-\tilde{\phi}(x)})\circ(\mathrm{D}_x\,\tilde{\phi})\circ(\mathrm{Id}_{\mathfrak{n}}+\tfrac{1}{2}\operatorname{ad}_x)(y)
\end{aligned}
$$

where we have used the convention $\frac{d}{dt}|_{t=0}\beta\in T_{\beta(0)}N$ to denote the element of $T_{\beta(0)}N$ represented by a smooth curve $\beta:(-\varepsilon,\varepsilon)\to N$. Since $\mathrm{D}_0\exp: T_0\mathfrak{n}\to T_eN=\mathfrak{n}$ is the identity map $\mathrm{Id}_{\mathfrak{n}}$ under the identification $T_0\mathfrak{n}\simeq\mathfrak{n}$ and $\mathrm{Id}_{\mathfrak{n}}+\frac{1}{2}\operatorname{ad}_z:\mathfrak{n}\to\mathfrak{n}$ has determinant one for any $z\in\mathfrak{n}$, being a unipotent matrix, we deduce that $\det(\mathrm{D}_{e^x}\,\phi)=\det(\mathrm{D}_x\,\tilde{\phi})$, proving the claim. Hence for any measurable function $f:N\to\mathbb{R}$, we have

$$
\begin{aligned}
&\int_N(f\circ\phi)(n)\,|\operatorname{Jac}_n\phi|\,dn=\int_{\mathfrak{n}}(\tilde{f}\circ\tilde{\phi})(x)\,|\operatorname{Jac}_{e^x}\phi|\,d\operatorname{Leb}(x)\\
&=\int_{\mathfrak{n}}(\tilde{f}\circ\tilde{\phi})(x)\,|\operatorname{Jac}_x\tilde{\phi}|\,d\operatorname{Leb}(x)=\int_{\mathfrak{n}}\tilde{f}(x)\,d\operatorname{Leb}(x)=\int_N f(n)\,dn,
\end{aligned}
$$

where we have used the change of variable formula for the Lebesgue measure in the second last equality. This proves the lemma. □

Definition of B_u, J_u. For each $u\in\mathsf{T}_{\Omega_1}$, we define

$$
\begin{aligned}
B_u&:=ua_uN_{c\eta}a_u^{-1},\ \text{and}\\
J_u&:=\{ua_ub^N(g_u^k,n)a_u^{-1}:n\in N_{c\eta},x_0ua_una_u^{-1}\in\Omega\}.
\end{aligned}
$$

Lemma 4.11. *For all $u\in\mathsf{T}_{\Omega_1}$, we have*

$$2c\cdot m(B_u\cap\mathsf{T}_\Omega)\le m(J_u).$$

Proof. Defining $\varphi_u : N \to N$ by $\varphi_u(n) = u(a_u n a_u^{-1})$, we have

$$J_u = (\varphi_u \circ b^N(g_u^k, \cdot) \circ \varphi_u^{-1})(B_u \cap \mathsf{T}_\Omega).$$

For all $v \in N_{c\eta} \subset N_\eta$, we have $2c \leq |\operatorname{Jac}_v b^N(g_u^k, \cdot)|$ by (4.28), and hence

$$2c \leq |\operatorname{Jac}_v(\varphi_u \circ b^N(g_u^k, \cdot) \circ \varphi_u^{-1})|.$$

The lemma follows from Lemma 4.10. □

Lemma 4.12. *For any $u \in \mathsf{T}_{\Omega_1} \cap (D - \partial_{R_2} D)$, we have*

$$J_u \subset \mathsf{T}_{\tilde{\Omega} \cap \Xi} \cap D.$$

Proof. Let $u \in \mathsf{T}_{\Omega_1}$ and $v \in J_u$ be arbitrary. Then $v = u(a_u b^N(g_u^k, n) a_u^{-1})$ for some $n \in N_{c\eta}$. Since $x_0 u \in \Omega_1$ we have for all $n \in N_{c\eta}$,

$$\begin{aligned} x_0 u(a_u n a_u^{-1}) &= x_0 u a_u (g_u^k n) a_u^{-1} \\ &= x_0 u a_u (b^N(g_u^k, n)\, b^{AM}(g_u^k, n)\, b^{N^+}(g_u^k, n)) a_u^{-1} \\ &= x_0 u (a_u b^N(g_u^k, n) a_u^{-1})\, b^{AM}(g_u^k, n)\, (a_u b^{N^+}(g_u^k, n) a_u^{-1}), \end{aligned} \tag{4.50}$$

with $b^{AM}(g_u^k, n) \in \mathcal{B}_L$ and $b^{N^+}(g_u^k, n) \in \mathcal{B}_{N^+}$, by (4.31). Since $t_u \geq T_0$, we have $a_u b^{N^+}(g_u^k, n) a_u^{-1} \subset \mathcal{O}_{N^+}$ by (4.39). Hence,

$$\begin{aligned} x_0 v &= x_0 u (a_u b^N(g_u^k, n) a_u^{-1}) \\ &= x_0 u (a_u n a_u^{-1})(a_u b^{N^+}(g_u^k, n)^{-1} a_u^{-1}) b^{AM}(g_u^k, n)^{-1} \in \Omega \mathcal{O}_{N^+} \mathcal{B}_L. \end{aligned}$$

Since $\mathcal{O}_{N^+} \subset \mathcal{B}_{N^+}$ we deduce

$$x_0 v \in \Omega \mathcal{B}_{N^+} \mathcal{B}_L \subset \tilde{\Omega}.$$

By Lemma 4.9, since $v = u(a_u b^N(g_u^k, n) a_u^{-1})$ with $u \in \Omega_1$ and $n \in N_{c\eta} \subset N_\eta$ we have $x_0 v \in \Xi$ implying $v \in \mathsf{T}_{\tilde{\Omega} \cap \Xi}$.

Further assuming that $u \in D - \partial_{R_2} D$, since $b^N(g_u^k, n) \in N_R \subset B_0(R_1)$, by (4.31), it follows from (4.47) that

$$a_u b^N(g_u^k, n) a_u^{-1} \in B_0(R_2).$$

Since d_0 is a distance, satisfying the triangle inequality, we deduce that $v \in D$, as claimed. □

Properties of coverings. For all $u \in \mathsf{T}_{\Omega_1}$, we have

$$\begin{aligned} &b^N(g_u^k, e) N_{2c\eta} \subset b^N(g_u^k, N_\eta) \subset b^N(g_u^k, e) N_\eta \text{ and} \\ &b^N(g_u^k, N_{c\eta}) \subset b^N(g_u^k, e) N_{c\eta}. \end{aligned} \tag{4.51}$$

Setting

$$w_u := u a_u b^N(g_u^k, e) a_u^{-1}, \tag{4.52}$$

we have

$$\begin{aligned} &J_u \subset w_u a_u N_{c\eta} a_u^{-1} \text{ and} \\ &w_u a_u N_{2c\eta} a_u^{-1} \subset u a_u b^N(g_u^k, N_\eta) a_u^{-1} \subset w_u a_u N_\eta a_u^{-1}. \end{aligned} \tag{4.53}$$

Since $b^N(g_u^k, e) \in N_R$ by (4.31), we have $w_u \in u a_u N_R a_u^{-1}$. Hence

$$J_u \subset w_u a_u N_{2c\eta} a_u^{-1} \subset u a_u b^N(g_u^k, N_\eta) a_u^{-1} \subset u a_u N_R N_\eta a_u^{-1}. \tag{4.54}$$

Lemma 4.13. *If $u_i, u_j \in \mathsf{T}_{\Omega_2}$ satisfy that $J_{u_i} \cap J_{u_j} \neq \emptyset$, then*

(1) $a_{u_i}^{-1} a_{u_j} N_R N_\eta a_{u_j}^{-1} a_{u_i} N_{c\eta} \not\subset N_{2c\eta}$,
(2) $u_i^{-1} u_j \in a_{u_i} N_R N_\eta a_{u_i}^{-1} a_{u_j} N_\eta N_R a_{u_j}^{-1}$,
(3) $B_{u_j} \subset u_i a_{u_i} N_R N_\eta (a_{u_i}^{-1} a_{u_j} N_\eta N_R N_{c\eta} a_{u_j}^{-1} a_{u_i}) a_{u_i}^{-1}$, *and*
(4) $a_{u_i}^{-1} a_{u_j} \subset \exp([-\beta, \beta]\mathsf{v})$.

Proof. To prove (1), let $v \in J_{u_i} \cap J_{u_j}$. By (4.54), we have $u_j^{-1} v \in a_{u_j} N_R N_\eta a_{u_j}^{-1}$ and by (4.53), we have $v^{-1} w_{u_i} \in a_{u_i} N_{c\eta} a_{u_i}^{-1}$, using the fact that $N_{c\eta}$ is symmetric. Hence,

$$u_j^{-1} w_{u_i} = (u_j^{-1} v)(v^{-1} w_{u_i}) \in a_{u_j} N_R N_\eta a_{u_j}^{-1} a_{u_i} N_{c\eta} a_{u_i}^{-1}. \tag{4.55}$$

Since $u_i \in \mathsf{T}_{\Omega_1}$ and $u_j \notin \mathsf{T}_\Xi$, we have $u_j \notin u_i a_{u_i} b^N(g_{u_i}^k, N_\eta) a_{u_i}^{-1}$ by Lemma 4.9. It follows from (4.53) that $u_j \notin w_{u_i} a_{u_i} N_{2c\eta} a_{u_i}^{-1}$, or equivalently,

$$u_j^{-1} w_{u_i} \notin a_{u_i} N_{2c\eta} a_{u_i}^{-1}.$$

Note that by (4.55),

$$a_{u_i}^{-1} u_j^{-1} w_{u_i} a_{u_i} \in a_{u_i}^{-1} a_{u_j} N_R N_\eta a_{u_j}^{-1} a_{u_i} N_{c\eta} - N_{2c\eta},$$

proving (1). We now prove (2). Since $J_{u_i} \cap J_{u_j} \neq \emptyset$, by (4.53) and (4.54),

$$u_i a_{u_i} N_R N_\eta a_{u_i}^{-1} \cap u_j a_{u_j} N_R N_\eta a_{u_j}^{-1} \neq \emptyset.$$

Since N_η and N_R are symmetric, we get

$$u_i^{-1} u_j \in a_{u_i} N_R N_\eta a_{u_i}^{-1} a_{u_j} N_\eta N_R a_{u_j}^{-1}.$$

To check (3), observe that

$$\begin{aligned} B_{u_j} &= u_j a_{u_j} N_{c\eta} a_{u_j}^{-1} = u_i (u_i^{-1} u_j) a_{u_j} N_{c\eta} a_{u_j}^{-1} \\ &\subset u_i (a_{u_i} N_R N_\eta a_{u_i}^{-1} a_{u_j} N_\eta N_R a_{u_j}^{-1}) a_{u_j} N_{c\eta} a_{u_j}^{-1} \\ &= u_i a_{u_i} N_R N_\eta (a_{u_i}^{-1} a_{u_j} N_\eta N_R N_{c\eta} a_{u_j}^{-1} a_{u_i}) a_{u_i}^{-1}, \end{aligned} \tag{4.56}$$

where the inclusion $\subset$ follows from Claim (2). Claim (4) follows from (1) by the choice of β as in (4.32). $\square$

Lemma 4.14. *For a bounded subset $S \subset \mathsf{T}_{\Omega_2}$, consider the covering $\{B_u : u \in S\}$. There exists a countable subset $F \subset S$ such that $\{B_{u_i} : u_i \in F\}$ covers S and*

$$\sum_i \mathbb{1}_{J_{u_i}} \leq \kappa_*. \tag{4.57}$$

where κ_* is given in (4.34).

Proof. Let $\{B_{u_i} : u_i \in F\}$ be a countable subcover of S given by Lemma 3.2 with respect to the parameters $\beta, \eta_1 = c\eta$ and $\eta_2 = R'$. Since $S \subset \mathsf{T}_{\Omega_2}$, note that whenever $J_{u_i} \cap J_{u_j} \neq \emptyset$, we have $|t_{u_i} - t_{u_j}| \leq \beta$ by Lemma 4.13(4). Moreover by Lemma 4.13(3), and the definition of $R' > 0$ as given in (4.33), we also have

$$B_{u_j} = u_j a_{u_j} N_{c\eta} a_{u_j}^{-1} \subset C_{u_i} := u_i a_{u_i} N_{R'} a_{u_i}^{-1}.$$

Therefore, if $J_{u_1} \cap \cdots \cap J_{u_q} \neq \emptyset$ for some $q \geq 2$, then

$$\bigcup_{j=1}^{q} B_{u_j} \subset C_{u_i}$$

and $|t_{u_i} - t_{u_j}| \leq \beta$ for all $1 \leq i, j \leq q$. Hence by Lemma 3.2, we get $q \leq \kappa_*$. Hence the claim follows. □

Lemma 4.15 (Lower bound)**.** *We have*

$$m\left(\bigcup_{u \in \mathsf{T}_{\Omega_2}} J_u \cap D\right) \geq \frac{c}{\kappa_*} \cdot m(\mathsf{T}_{\Omega_2} \cap D)$$

Proof. First, note that the union in the statement is indeed measurable as this is a union of open sets in N. Consider the cover

$$\mathcal{F} := \{B_u : u \in \mathsf{T}_{\Omega_2} \cap (D - \partial_{R_2} D)\}$$

of the bounded subset $\mathsf{T}_{\Omega_2} \cap (D - \partial_{R_2} D)$, where R_2 is given (4.47). By Lemma 4.14, we can find a countable subset $F \subset \mathsf{T}_{\Omega_2} \cap (D - \partial_{R_2} D)$ such that the collection $\{B_{u_i} : u_i \in F\}$ covers $\mathsf{T}_{\Omega_2} \cap (D - \partial_{R_2} D)$ and

$$\sum_{u_i \in F} \mathbb{1}_{J_{u_i}} \leq \kappa_*. \tag{4.58}$$

By Lemma 4.12, we have $J_{u_i} \subset D$ for all $u_i \in F \subset \mathsf{T}_{\Omega_2} \cap (D - \partial_{R_2} D)$. Hence, using (4.58), we get

$$m\left(\bigcup_{u \in \mathsf{T}_{\Omega_2}} J_u \cap D\right) \geq m\left(\bigcup_{u_i \in F} J_{u_i}\right) \geq \frac{1}{\kappa_*} \sum_{u_i \in F} m\left(J_{u_i}\right).$$

Since $m(J_{u_i}) \geq 2c \cdot m(B_{u_i} \cap \mathsf{T}_\Omega)$ by Lemma 4.11 (recall that $\Omega_2 \subset \Omega$), we have

$$m\left(\bigcup_{u \in \mathsf{T}_{\Omega_2}} J_u \cap D\right) \geq \frac{2c}{\kappa_*} \sum_{u_i \in F} m\left(B_{u_i} \cap \mathsf{T}_\Omega\right) \geq \frac{2c}{\kappa_*} m\left(\mathsf{T}_{\Omega_2} \cap (D - \partial_{R_2} D)\right),$$

where the last inequality holds as $\{B_{u_i} : u_i \in F\}$ is a cover of $\mathsf{T}_{\Omega_2} \cap (D - \partial_{R_2} D)$. Since

$$2 \cdot m\left(\mathsf{T}_{\Omega_2} \cap (D - \partial_{R_2} D)\right) \geq m(\mathsf{T}_{\Omega_2} \cap D)$$

by the second inequality of Lemma 4.8, the claim follows. □

Lemma 4.16 (Upper bound)**.** *We have*

$$m\left(\bigcup_{u\in\mathsf{T}_{\Omega_2}} J_u \cap D\right) < \frac{c}{\kappa_*} m(\mathsf{T}_{\Omega_2}\cap D).$$

Proof. By Lemma 4.12 and the fact that $\Omega_2 \subset \Omega_1$, we have

$$\bigcup_{u\in\mathsf{T}_{\Omega_2}} J_u \cap D \subset \mathsf{T}_{\tilde{\Omega}\cap\Xi}\cap D.$$

By the choice of x_0 satisfying the first inequality in Lemma 4.8, we have

$$m(\mathsf{T}_{\tilde{\Omega}\cap\Xi}\cap D) < \frac{c}{\kappa_*} m(\mathsf{T}_{\Omega_2}\cap D),$$

implying the claim. $\square$

These two lemmas yield a contradiction to the hypothesis (4.11) that $\lambda(g_0) = \lambda(h_0^p) \notin \mathrm{Stab}_G([\mu])$. As $p \geq p_0$ was arbitrary, we deduce that $\lambda(h_0) \in \mathrm{Stab}_G([\mu])$ by Lemma 4.3. Therefore we have proved (4.2) and hence Theorem 4.1.

5. Measures supported on directional recurrent sets

Let $G = \prod_{i=1}^{\mathsf{r}} G_i$ be a product of simple real algebraic groups of rank one. Let $\Gamma_0 < G$ be a Zariski dense discrete subgroup of G, and Γ be a Zariski dense normal subgroup of Γ_0.

For $\mathsf{v} \in \mathrm{int}\,\mathfrak{a}^+$, define

$$\mathcal{R}_{\mathsf{v}}^* = \{\Gamma\backslash\Gamma g \in \Gamma\backslash G : \limsup_{t\to\infty} \Gamma_0\backslash\Gamma_0 g \exp(t\mathsf{v}) \neq \emptyset\}. \tag{5.1}$$

As Γ is normal in Γ_0, $\mathcal{R}_{\mathsf{v}}^*$ is well-defined.

The main goal of this section is to deduce the following theorem and corollary from Theorem 4.1:

Theorem 5.1. *For $\mathsf{v} \in \mathrm{int}\,\mathfrak{a}^+$, any N-invariant, ergodic measure μ supported on $\mathcal{R}_{\mathsf{v}}^*$ is P° quasi-invariant.*

Corollary 5.2. *Set $\mathcal{R}^*(\mathrm{int}\,\mathfrak{a}^+) := \cup_{\mathsf{v}\in\mathrm{int}\,\mathfrak{a}^+}\mathcal{R}_{\mathsf{v}}^*$. Any N-invariant, ergodic measure μ supported on $\mathcal{R}^*(\mathrm{int}\,\mathfrak{a}^+)$ is P° quasi-invariant.*

We remark that any N-invariant, ergodic and P°-invariant measure on $\mathcal{E}$ is of the form $\mathsf{m}_\nu^{\mathrm{BR}}|_Y$ for some Γ-conformal measure ν on Λ and P°-minimal subset $Y \subset \Gamma\backslash G$ (see (6.1) and [27, Prop. 7.2]).

Proposition 1.4 is a special case of Theorem 5.1 when $\Gamma = \Gamma_0$ and M is connected. We recall that as long as none of G_i is isomorphic $\mathrm{SL}_2(\mathbb{R})$, M is always connected [43, Lem. 2.4].

Properties of Zariski dense groups. In the following Theorem 5.3, and Lemmas 5.4 and 5.5, we let Σ be a Zariski dense discrete subgroup of any semisimple real algebraic group G. Note that Σ contains a Zariski dense subset of loxodromic elements [3]. The following theorem can be deduced from the work of Guivarch and Raugi [18].

Theorem 5.3. [27, Cor. 3.6] *Any closed subgroup of MA containing the generalized Jordan projection $\lambda(\Sigma)$ contains $M^\circ A$.*

We denote by $\Lambda(\Sigma) \subset \mathcal{F}$ the limit set of Σ, which is the unique Σ-minimal subset.

We refer to [13, Def. 7.1] for the definition of a Schottky subgroup of G.

Lemma 5.4. *Let $\mathcal{O}$ be a Zariski open subset of $\mathcal{F}$. Any Zariski dense subgroup Σ of G contains a Zariski dense Schottky subgroup Σ_1 with $\Lambda(\Sigma_1) \subset \mathcal{O}$.*

Proof. This can be proved similarly to the proof of [3, Prop. 4.3] (see also proof of [13, Lem. 7.3]). First, we may assume that Σ is finitely generated. Hence there exists an integer $n := n_\Sigma \geq 1$ such that the subgroup $\langle \gamma^n \rangle$ generated by γ^n is Zariski connected for all $\gamma \in \Sigma$ [41].

Since $\mathcal{O}$ and $\mathcal{F}^{(2)}$ are Zariski open in $\mathcal{F}$ and $\mathcal{F} \times \mathcal{F}$ respectively, we can choose open subsets $b_i^\pm$, $i = 1, 2$ whose closures are contained in $\mathcal{O}$ and which are pairwise in general position.[3] By [3, Lemma 3.6], for each $i = 1, 2$, the subset $\{\gamma \in \Sigma : \text{loxodromic}, (y_\gamma, y_{\gamma^{-1}}) \in b_i^+ \times b_i^-\}$ is Zariski dense. Hence there exists $g_1 \in \Sigma$ such that $\gamma_1 := g_1^n$ is loxodromic and $(y_{\gamma_1}, y_{\gamma_1^{-1}}) \in b_1^+ \times b_1^-$. By [41, Proposition 4.4], there exists a proper Zariski closed subset $F_{\gamma_1} \subset G$ containing all proper Zariski closed and Zariski connected subgroups of G containing γ_1. Hence we can find a loxodromic element $g_2 \in \Sigma - F_{\gamma_1}$ such that $(y_{g_2}, y_{g_2^{-1}}) \in b_2^+ \times b_2^-$. Set $\gamma_2 := g_2^n$. By definition of n and F_{γ_1}, the subgroup $\Sigma_k := \langle \gamma_1^k, \gamma_2^k \rangle$ is Zariski dense for any $k \geq 1$.

We can find open subsets $B_i^\pm \subset \mathcal{F}$, $i = 1, 2$ such that $\cap_{i=1}^2 (B_i^+ \cap B_i^-) \neq \emptyset$ and $\gamma_i^{\pm k}(B_i^\pm) \subset b_i^\pm$ for all sufficiently large $k \geq 1$. Fix one such k. If we take $\xi_0 \in \cap_{i=1}^2 (B_i^+ \cap B_i^-)$, then $\Sigma_k \xi_0$ is contained in the union $\cup_{i=1,2}(b_i^+ \cup b_i^-) \subset \mathcal{O}$. Since the closure of $\Sigma_k \xi_0$ contains $\Lambda(\Sigma_k)$, which is the minimal Σ_k-subset, it follows that $\Lambda(\Sigma_k) \subset \mathcal{O}$. $\square$

Lemma 5.5. *For any $\xi, \eta \in \mathcal{F}$, set*

$$\mathfrak{O}_{(\xi,\eta)} := \{g \in G : \textit{loxodromic}, (y_g, \xi), (y_{g^{-1}}, \eta) \in \mathcal{F}^{(2)}\}. \tag{5.2}$$

For any Zariski dense subgroup Σ of G, the intersection $\Sigma \cap \mathfrak{O}_{(\xi,\eta)}$ contains a Zariski dense Schottky subgroup of G.

Proof. For $\xi \in \mathcal{F}$, the subset $\mathcal{O}_\xi := \{\xi' \in \mathcal{F} : (\xi, \xi') \in \mathcal{F}^{(2)}\}$ is Zariski open. By Lemma 5.4, Σ contains a Zariski dense Schottky subgroup Σ_1 consisting

[3] Two subsets A and B of $\mathcal{F}$ are in general position if $A \times B \subset \mathcal{F}^{(2)}$.

of loxodromic elements and with $\Lambda(\Sigma_1) \subset \mathcal{O}_\xi$. Now Σ_1 contains a Zariski dense Schottky subgroup Σ_2 with $\Lambda(\Sigma_2) \subset \mathcal{O}_\eta$. Then $\Sigma_2 \subset \mathfrak{O}_{(\xi,\eta)}$ since

$$\{y_{\gamma^{\pm 1}} \in \mathcal{F} : \gamma \in \Sigma_2\} \subset \Lambda(\Sigma_2) \subset \mathcal{O}_\eta \cap \mathcal{O}_\xi.$$

□

Proof of Theorem 5.1. As μ is supported on $\mathcal{R}^*_{\mathsf{v}}$, there exists $x = [g] \in \mathcal{R}^*_{\mathsf{v}}$ such that $\mathcal{S}_\mu(\mathsf{v}) = \mathcal{S}_x(\mathsf{v})$. By the definition of $\mathcal{R}^*_{\mathsf{v}}$, there exist $\gamma_i \in \Gamma_0$ and $t_i \to +\infty$ such that $\gamma_i g \exp(t_i \mathsf{v})$ converges to some $h_0 \in G$. Since Γ is normal in Γ_0, it follows that $\mathcal{S}_x(\mathsf{v})$ contains $\Sigma := h_0^{-1} \Gamma h_0$, and hence

$$\mathcal{S}_\mu(\mathsf{v}) \supset \Sigma.$$

Hence by Theorem 4.1,

$$\lambda(\Sigma \cap \mathfrak{O}_{(e^+,e^-)}) \subset \mathrm{Stab}_G([\mu]).$$

Since Σ is Zariski dense, by Lemma 5.5, the intersection $\Sigma \cap \mathfrak{O}_{(e^+,e^-)}$ contains a Zariski dense discrete subgroup, say Σ'. Since the closure of the subgroup generated by $\lambda(\Sigma')$ contains AM° by Theorem 5.3, we get $AM^\circ \subset \mathrm{Stab}_G([\mu])$, proving the claim.

Proof of Corollary 5.2. By Theorem 5.1, it suffices to show the following lemma:

Lemma 5.6. *Any N-invariant, ergodic measure μ supported on $\mathcal{R}^*(\operatorname{int}\mathfrak{a}^+)$ is supported on $\mathcal{R}^*_{\mathsf{v}}$ for some $\mathsf{v} \in \operatorname{int}\mathfrak{a}^+$.*

Proof. For any subset $U \subset \operatorname{int}\mathfrak{a}^+$, we set

$$\mathcal{R}^*(U) := \cup_{\mathsf{u}\in U} \mathcal{R}^*_{\mathsf{u}} \subset \Gamma\backslash G.$$

Note that $\mathcal{R}^*(U)$ is N-invariant, since $\mathcal{R}^*_{\mathsf{u}}$ itself is N-invariant for each $\mathsf{u} \in \operatorname{int}\mathfrak{a}^+$. Note that $\mathcal{R}^*(\operatorname{int}\mathfrak{a}^+) = \bigcup_{\mathsf{u}\in S} \mathcal{R}^*_{\mathsf{u}}$ where $S := \{\mathsf{u} \in \operatorname{int}\mathfrak{a}^+ : \|\mathsf{u}\| = 1\}$. Let $(\Gamma\backslash G, \mathcal{A}, \mu)$ be the completion of the measure space $(\Gamma\backslash G, \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-algebra of $\Gamma\backslash G$.

Claim. *For any open set $U \subset S$, the set $\mathcal{R}^*(U)$ belongs to $\mathcal{A}$ and is either μ-null or co-null.*

Given U, denote $X_U = \Gamma\backslash G \times U$ equipped with the product σ-algebra $\mathcal{B} \otimes \mathcal{B}_U$ with respect to the Borel σ-algebras on $\Gamma\backslash G$ and U. Define the function $\psi : X_U \to [0, \infty]$ by

$$\psi(x, \mathsf{u}) = \liminf_{t\to\infty} d_{\Gamma\backslash G}(x, x\exp(t\mathsf{u})),$$

where $d_{\Gamma\backslash G}$ is the metric induced from the left-invariant metric on G. The function ψ is clearly $\mathcal{B} \otimes \mathcal{B}_U$-measurable and therefore so is the set $W := \psi^{-1}([0, \infty))$. Note that $\mathcal{R}^*(U) = \pi_{\Gamma\backslash G}(W)$ is the image of W under the projection map $\pi_{\Gamma\backslash G} : X_U \to \Gamma\backslash G$. We would have liked to conclude that $\mathcal{R}^*(U)$ is itself Borel measurable but this might not be true. Fortunately, we have the following Measurable Projection Theorem [9, III.23]:

Let $(Y, \mathcal{F})$ be a measure space and let $(U, \mathcal{B}_U)$ be a Polish space, i.e. a separable completely metrizable space, together with its Borel σ-algebra. Let $X = Y \times U$ together with $\mathcal{F} \otimes \mathcal{B}_U$ be the product measure space. Then for any set $W \in \mathcal{F} \otimes \mathcal{B}_U$, the projection $\pi_Y(W) \subset Y$ is universally measurable, that is, $\pi_Y(W)$ is contained in the completion of $\mathcal{F}$ with respect to any probability measure ν on $(Y, \mathcal{F})$.

The space U is clearly Polish whenever U is open in S. Since μ is equivalent to a probability measure, say, $f d\mu$ for some $0 < f \in L^1(\mu)$ of norm$= 1$, this theorem implies that $\mathcal{R}^*(U) = \pi_{\Gamma\backslash G}(W) \in \mathcal{A}$. By the properties of the completion σ-algebra, there exist Borel measurable sets $Q_1 \subset \mathcal{R}^*(U) \subset Q_2$ satisfying $\mu(Q_2 - Q_1) = 0$. Since $\mathcal{R}^*(U)$ is N-invariant we have

$$Q_1 N \subset \mathcal{R}^*(U) N = \mathcal{R}^*(U) \subset Q_2$$

and hence $\mu(Q_1 \Delta Q_1 N) = 0$, where Δ denotes symmetric difference. By ergodicity, this implies that Q_1, and hence also $\mathcal{R}^*(U)$, are either μ-null or co-null, proving the claim.

Now take a countable basis $\{U_{1,i}\}$ of S consisting of open balls of diameter at most $1/2$. By the claim above, the sets $\mathcal{R}^*(U_{1,i})$ are either μ-null or co-null. Since μ is supported on

$$\mathcal{R}^*(\operatorname{int} \mathfrak{a}^+) = \mathcal{R}^*(S) = \bigcup_{i \geq 1} \mathcal{R}^*(U_{1,i}),$$

there exists some i_1 for which $\mathcal{R}^*(U_{1,i_1})$ is co-null. Take a countable basis $\{U_{2,i}\}$ of U_{1,i_1} consisting of open balls of diameter at most $1/4$. Then there exists $U_{2,i_2} \subset U_{1,i_1}$ for which $\mathcal{R}^*(U_{2,i_2})$ is co-null. Continuing inductively, we get a decreasing sequence of balls $U_{1,i_1} \supset U_{2,i_2} \supset \cdots$ of diameters $\operatorname{diam} U_{k,i_k} \leq 2^{-k}$ satisfying that $\mathcal{R}^*(U_{k,i_k})$ are μ-co-null. Hence $\bigcap_k U_{k,i_k} = \{\mathsf{v}\}$ for some $\mathsf{v} \in S$ and $\mathcal{R}^*_{\mathsf{v}} = \bigcap_k \mathcal{R}^*(U_{k,i_k})$ is co-null for μ. $\square$

6. Unique ergodicity and Anosov groups

We begin by recalling the definition of Burger-Roblin measures given in [13]. Let Γ be a Zariski dense discrete subgroup of a connected semisimple real algebraic group G. Denote by $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ the growth indicator function of Γ defined by Quint [31]. Let ψ be a linear form on $\mathfrak{a}$ and ν a (Γ, ψ)-conformal measure supported on the limit set Λ. This implies $\psi \geq \psi_\Gamma$ [31, Thm. 1.2]. When the rank of G is one, ψ is simply a real number and ψ_Γ is equal to the critical exponent of Γ. The Burger-Roblin measure $\mathsf{m}_\nu^{\mathrm{BR}}$ associated to ν is the MN-invariant Borel measure on $\Gamma\backslash G$ which is induced from the following measure $\tilde{\mathsf{m}}_\nu^{\mathrm{BR}}$ on G/M: using the Hopf parametrization $G/M = \mathcal{F}^{(2)} \times \mathfrak{a}$ given by $gM \to (g^+, g^-, \beta_{g^+}(e, g))$,

$$d\tilde{\mathsf{m}}_\nu^{\mathrm{BR}}(g) = e^{\psi(\beta_{g^+}(e,g)) + 2\rho(\beta_{g^-}(e,g))}\, d\nu(g^+) dm_o(g^-) db, \tag{6.1}$$

where db is the Lebesgue measure on $\mathfrak{a}$, m_o is the K-invariant probability measure on $\mathcal{F}$ and $\beta_{g^+}(e, g) \in \mathfrak{a}$ and $\beta_{g^-}(e, g) \in \mathfrak{a}$ are respectively given by the conditions

$$g \in K \exp(\beta_{g^+}(e,g))N \quad \text{and} \quad g \in K \exp(\mathrm{Ad}_{w_0}(\beta_{g^-}(e,g)))N^+.$$

Now, let Γ be an Anosov subgroup of G, as defined in the introduction. For each $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$, there exist a unique linear form $\psi_{\mathsf{v}} \in \mathfrak{a}^*$ such that $\psi_{\mathsf{v}} \geq \psi_\Gamma$ and $\psi_{\mathsf{v}}(\mathsf{v}) = \psi_\Gamma(\mathsf{v})$ and a unique $(\Gamma, \psi_{\mathsf{v}})$-conformal probability measure supported on the limit set Λ, which we denote by ν_{v} (see [35] and [13, Theorem 7.9]). We set

$$\mathsf{m}^{\mathrm{BR}}_{\mathsf{v}} := \mathsf{m}^{\mathrm{BR}}_{\nu_{\mathsf{v}}}. \tag{6.2}$$

Note that if $\mathbb{R}\mathsf{v} = \mathbb{R}\mathsf{u}$, then $\psi_{\mathsf{u}} = \psi_{\mathsf{v}}$ and hence $\mathsf{m}^{\mathrm{BR}}_{\mathsf{v}} = \mathsf{m}^{\mathrm{BR}}_{\mathsf{u}}$.

We recall the following result of Lee and Oh, which is based on their classification of Γ-conformal measures on Λ [26, Thm. 7.7]:

Theorem 6.1. [27, Prop. 7.2] *Any N-invariant ergodic and P°-quasi invariant measure on $\mathcal{E}$ is of the form $\mathsf{m}^{\mathrm{BR}}_{\mathsf{v}}|_Y$ for some $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$ and some P°-minimal subset $Y \subset \Gamma\backslash G$, up to proportionality.*

Indeed in [26], it was also shown that each $\mathsf{m}^{\mathrm{BR}}_{\mathsf{v}}|_Y$ in the above theorem is N-ergodic; however we will not need this result.

For $\mathsf{v} \in \operatorname{int} \mathfrak{a}^+$, set

$$\mathcal{R}_{\mathsf{v}} := \{x \in \mathcal{E} : \limsup_{t \to +\infty} x \exp t\mathsf{v} \neq \emptyset\}.$$

We also recall the following recent result obtained by Burger, Landersberg, Lee and Oh:

Theorem 6.2. [8] *Let $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$ and $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$.*

- *If $\operatorname{rank} G \leq 3$, then $\mathsf{m}^{\mathrm{BR}}_{\mathsf{v}}(\Gamma\backslash G - \mathcal{R}_{\mathsf{v}}) = 0$.*
- *If $\operatorname{rank} G > 3$ or $\mathbb{R}\mathsf{u} \neq \mathbb{R}\mathsf{v}$, then $\mathsf{m}^{\mathrm{BR}}_{\mathsf{v}}(\mathcal{R}_{\mathsf{u}}) = 0$.*

Proof of Theorem 1.1. Let μ be an N-invariant measure supported on $\mathcal{R}_{\mathsf{u}}$ for some $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$. In view of the ergodic decomposition, we may assume without loss of generality that μ is ergodic. By Proposition 1.4, μ is P quasi-invariant. Since $P = P^\circ$ under the hypothesis that none of G_i is isomorphic to $\mathrm{SL}_2(\mathbb{R})$, it follows from Theorem 6.1 that $\mu = \mathsf{m}^{\mathrm{BR}}_{\mathsf{v}}$ for some $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$. By Theorem 6.2, this implies that $\operatorname{rank} G \leq 3$ and $\mathbb{R}\mathsf{v} = \mathbb{R}\mathsf{u}$ and hence $\mathsf{u} \in \operatorname{int} \mathcal{L}_\Gamma$; in other cases, such μ cannot exist. This proves the claim.

Proof of Corollary 1.2. By Corollary 5.2, any N-invariant ergodic measure supported on $\mathcal{R}$ is supported on $\mathcal{R}_{\mathsf{u}}$ for some $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$. Hence the claim follows from Theorem 1.1.

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