UNIQUENESS OF CONFORMAL MEASURES AND LOCAL MIXING FOR ANOSOV GROUPS

SAM EDWARDS, MINJU LEE, AND HEE OH

Abstract. In the late seventies, Sullivan showed that for a convex cocompact subgroup Γ of $\mathrm{SO}^\circ(n,1)$ with critical exponent $\delta > 0$, any Γ-conformal measure on $\partial\mathbb{H}^n$ of dimension δ is necessarily supported on the limit set Λ and that the conformal measure of dimension δ exists uniquely. We prove an analogue of this theorem for any Zariski dense Anosov subgroup Γ of a connected semisimple real algebraic group G of rank at most 3. We also obtain the local mixing for generalized BMS measures on $\Gamma\backslash G$ including Haar measures.

Dedicated to Gopal Prasad on the occasion of his 75th birthday with respect

1. Introduction

Let (X, d) be a Riemannian symmetric space of rank one and ∂X the geometric boundary of X. Let $G = \mathrm{Isom}^+ X$ denote the group of orientation preserving isometries and $\Gamma < G$ a non-elementary discrete subgroup. Fixing $o \in X$, a Borel probability measure ν on ∂X is called a Γ-conformal measure of dimension $s > 0$ if for all $\gamma \in \Gamma$ and $\xi \in \partial X$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{s(\beta_\xi(o,\gamma o))}$$

where $\beta_\xi(x, y) = \lim_{z\to\xi} d(x, z) - d(y, z)$ denotes the Busemann function.

Let $\delta > 0$ denote the critical exponent of Γ, i.e., the abscissa of the convergence of the Poincare series $\sum_{\gamma\in\Gamma} e^{-sd(\gamma o,o)}$. The well-known construction of Patterson and Sullivan ([9], [13]) provides a Γ-conformal measure of dimension δ supported on the limit set Λ, called the Patterson-Sullivan (PS) measure. A discrete subgroup $\Gamma < G$ is called *convex cocompact* if Γ acts cocompactly on some nonempty convex subset of X.

Theorem 1.1 (Sullivan)**.** [13] *If Γ is convex cocompact, then any Γ-conformal measure on ∂X of dimension δ is necessarily supported on Λ. Moreover, the* PS*-measure is the unique Γ-conformal measure of dimension δ.*

In this paper, we extend this result to Anosov subgroups, which may be regarded as higher rank analogues of convex cocompact subgroups of rank

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Edwards, Lee and Oh respectively supported by funding from the Heilbronn Institute for Mathematical Research, and the NSF grant No. DMS-1926686 (via the Institute for Advanced Study), and the NSF grant No. DMS-1900101.

one groups. Let G be a connected semisimple real algebraic group and P a minimal parabolic subgroup of G. Let $\mathcal{F} := G/P$ be the Furstenberg boundary, and $\mathcal{F}^{(2)}$ the unique open G-orbit in $\mathcal{F} \times \mathcal{F}$ under the diagonal action of G. In the whole paper, we let Γ be a Zariski dense Anosov subgroup of G with respect to P. This means that there exists a representation $\Phi : \Sigma \to G$ of a Gromov hyperbolic group Σ with $\Gamma = \Phi(\Sigma)$, which induces a continuous equivariant map ζ from the Gromov boundary $\partial\Sigma$ to $\mathcal{F}$ such that $(\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)}$ for all $x \neq y \in \partial\Sigma$. This definition is due to Guichard-Wienhard [5], generalizing that of Labourie [6].

Let $A < P$ be a maximal real split torus of G and $\mathfrak{a} := \mathrm{Lie}(A)$. Given a linear form $\psi \in \mathfrak{a}^*$, a Borel probability measure ν on $\mathcal{F}$ is called a (Γ, ψ)-conformal measure if, for any $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{\psi(\beta_\xi(e,\gamma))} \tag{1.2}$$

where β denotes the $\mathfrak{a}$-valued Busemann function (see (2.1) for the definition). Let $\Lambda \subset \mathcal{F}$ denote the limit set of Γ, which is the unique Γ-minimal subset (see [1], [7]). A (Γ, ψ)-conformal measure *supported on* Λ will be called a (Γ, ψ)-PS measure. Finally, a Γ-PS measure means a (Γ, ψ)-PS measure for some $\psi \in \mathfrak{a}^*$.

Fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and let $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ denote the limit cone of Γ. Benoist [1] showed that $\mathcal{L}_\Gamma$ is a convex cone with non-empty interior, using the well-known theorem of Prasad [10] on the existence of an $\mathbb{R}$-regular element in any Zariski dense subgroup of G. Let $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of Γ as defined in (2.2). Set

$$D_\Gamma^\star := \{\psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma,\ \psi(u) = \psi_\Gamma(u) \text{ for some } u \in \mathcal{L}_\Gamma \cap \operatorname{int} \mathfrak{a}^+\}. \tag{1.3}$$

As Γ is Anosov, for any $\psi \in D_\Gamma^\star$, there exist a unique unit vector $u \in \operatorname{int} \mathcal{L}_\Gamma$, such that $\psi(u) = \psi_\Gamma(u)$, and a unique (Γ, ψ)-PS measure ν_ψ. Moreover, this gives bijections among

$$D_\Gamma^\star \simeq \{u \in \operatorname{int} \mathcal{L}_\Gamma : \|u\| = 1\} \simeq \{\Gamma\text{-PS measures on } \Lambda\}$$

(see [4], [7]). When G has rank one, $D_\Gamma^\star = \{\delta\}$. Therefore the following generalizes Sullivan's theorem 1.1. We denote the real rank of G by $\operatorname{rank} G$, i.e., $\operatorname{rank} G = \dim \mathfrak{a}$.

Theorem 1.4. *Let* $\operatorname{rank} G \leq 3$. *For any* $\psi \in D_\Gamma^\star$, *any* (Γ, ψ)*-conformal measure on* $\mathcal{F}$ *is necessarily supported on* Λ. *Moreover, the* PS *measure* ν_ψ *is the unique* (Γ, ψ)*-conformal measure on* $\mathcal{F}$.

Our proof of Theorem 1.4 is obtained by combining the rank dichotomy theorem established by Burger, Landesberg, Lee, and Oh [2] and the local mixing property of a generalized Bowen-Margulis-Sullivan measure (Theorem 3.1), which generalizes our earlier work [4]. Indeed, our proof yields that under the hypothesis of Theorem 1.4, any (Γ, ψ)-conformal measure on $\mathcal{F}$ is supported on the u-directional radial limit set Λ_u (see (4.3)) where $\psi(u) = \psi_\Gamma(u)$.

We end the introduction by the following:
Open problem: Is Theorem 1.4 true without the hypothesis rank $G \leq 3$?

2. Local mixing of Generalized Bowen-Margulis-Sullivan measures

Let G be a connected semisimple real algebraic group and $\Gamma < G$ a Zariski dense discrete subgroup. Let $P = MAN$ be a minimal parabolic subgroup of G with fixed Langlands decomposition so that A is a maximal real split torus, M is the centralizer of A and N is the unipotent radical of P.

In [4, Prop. 6.8], we proved that local mixing of a BMS-measure on $\Gamma\backslash G/M$ implies local mixing of the Haar measure on $\Gamma\backslash G/M$. In this section, we provide a generalized version of this statement, where we replace the Haar measure by any generalized BMS-measure and also work on the space $\Gamma\backslash G$, rather than on $\Gamma\backslash G/M$. We refer to [4] for a more detailed description of a generalized BMS-measure, while only briefly recalling its definition here.

Let $\mathfrak{a} = \operatorname{Lie}(A)$ and fix a positive Weyl chamber $\mathfrak{a}^+ < \mathfrak{a}$ so that $\log N$ consists of positive root subspaces. We also fix a maximal compact subgroup $K < G$ so that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds. Denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection, i.e., for $g \in G$, $\mu(g) \in \mathfrak{a}^+$ is the unique element such that $g \in K \exp \mu(g) K$. Denote by $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ the limit cone of Γ, which is the asymptotic cone of $\mu(\Gamma)$, i.e., $\mathcal{L}_\Gamma = \{\lim t_i \mu(\gamma_i) \in \mathfrak{a}^+ : t_i \to 0, \gamma_i \in \Gamma\}$. The Furstenberg boundary $\mathcal{F} = G/P$ is isomorphic to K/M as K acts on $\mathcal{F}$ transitively with $K \cap P = M$.

The $\mathfrak{a}$-valued Busemann function $\beta : \mathcal{F} \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi) \tag{2.1}$$

where the Iwasawa cocycle $\sigma(g^{-1}, \xi) \in \mathfrak{a}$ is defined by the relation $g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N$ for $\xi = kP$, $k \in K$.

The growth indicator function $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ is defined as a homogeneous function, i.e., $\psi_\Gamma(tu) = t\psi_\Gamma(u)$ for all $t > 0$, such that for any unit vector $u \in \mathfrak{a}^+$,

$$\psi_\Gamma(u) := \inf_{u \in \mathcal{C}, \text{open cones } \mathcal{C} \subset \mathfrak{a}^+} \tau_\mathcal{C} \tag{2.2}$$

where $\tau_\mathcal{C}$ is the abscissa of convergence of $\sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-t\|\mu(\gamma)\|}$ and the norm $\|\cdot\|$ on $\mathfrak{a}$ is the one induced from the Killing form on $\mathfrak{g}$.

Denote by $w_0 \in K$ a representative of the unique element of the Weyl group $N_K(A)/M$ such that $\operatorname{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The opposition involution $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ is defined by

$$\mathrm{i}(u) = -\operatorname{Ad}_{w_0}(u).$$

Note that i preserves $\operatorname{int} \mathcal{L}_\Gamma$.

The generalized BMS-measures m_{ν_1,ν_2}. For $g \in G$, we consider the following visual images:

$$g^+ = gP \in \mathcal{F} \quad \text{and} \quad g^- = gw_0P \in \mathcal{F}.$$

Then the map

$$gM \mapsto (g^+, g^-, b = \beta_{g^-}(e,g))$$

gives a homeomorphism $G/M \simeq \mathcal{F}^{(2)} \times \mathfrak{a}$, called the Hopf parametrization of G/M.

For a pair of linear forms $\psi_1, \psi_2 \in \mathfrak{a}^*$ and a pair of (Γ, ψ_1) and (Γ, ψ_2) conformal measures ν_1 and ν_2 respectively, define a locally finite Borel measure $\tilde{m}_{\nu_1,\nu_2}$ on G/M as follows: for $g = (g^+, g^-, b) \in \mathcal{F}^{(2)} \times \mathfrak{a}$,

$$d\tilde{m}_{\nu_1,\nu_2}(g) = e^{\psi_1(\beta_{g^+}(e,g)) + \psi_2(\beta_{g^-}(e,g))}\, d\nu_1(g^+) d\nu_2(g^-) db, \tag{2.3}$$

where $db = d\ell(b)$ is the Lebesgue measure on $\mathfrak{a}$. By abuse of notation, we also denote by $\tilde{m}_{\nu_1,\nu_2}$ the M-invariant measure on G induced by $\tilde{m}_{\nu_1,\nu_2}$. This is always left Γ-invariant and we denote by m_{ν_1,ν_2} the M-invariant measure on $\Gamma\backslash G$ induced by $\tilde{m}_{\nu_1,\nu_2}$.

The generalized BMS*-measures $m^*_{\nu_1,\nu_2}$. Similarly, with a different Hopf parametrization

$$gM \mapsto (g^+, g^-, b = \beta_{g^+}(e,g))$$

(that is, g^- replaced by g^+ in the subscript for β), we define the following measure

$$d\tilde{m}^*_{\nu_1,\nu_2}(g) = e^{\psi_1(\beta_{g^+}(e,g)) + \psi_2(\beta_{g^-}(e,g))}\, d\nu_1(g^+) d\nu_2(g^-) db \tag{2.4}$$

first on G/M and then the M-invariant measure $dm^*_{\nu_1,\nu_2}$ on $\Gamma\backslash G$. One can check

$$m^*_{\nu_1,\nu_2} = m_{\nu_2,\nu_1}.w_0. \tag{2.5}$$

Lemma 2.6. *If $\psi_2 = \psi_1 \circ \mathrm{i}$, then $m_{\nu_1,\nu_2} = m^*_{\nu_1,\nu_2}$.*

Proof. When $\psi_2 = \psi_1 \circ \mathrm{i}$, we can check that $m_{\nu_2,\nu_1}.w_0 = m_{\nu_1,\nu_2}$, which implies the claim by (2.5). □

PS-measures on $gN^{\pm}$. Let $N^- = N$ and $N^+ = w_0 N w_0^{-1}$. To a given (Γ, ψ)-conformal measure ν and $g \in G$, we define the following associated measures on $gN^{\pm}$: for $n \in N^+$ and $h \in N^-$,

$$d\mu_{gN^+,\nu}(n) := e^{\psi(\beta_{(gn)^+}(e,gn))} d\nu((gn)^+), \text{ and}$$
$$d\mu_{gN^-,\nu}(h) := e^{\psi(\beta_{(gh)^-}(e,gh))} d\nu((gh)^-).$$

Note that these are left Γ-invariant; for any $\gamma \in \Gamma$ and $g \in G$, $\mu_{\gamma g N^{\pm},\nu} = \mu_{gN^{\pm},\nu}$. For a given Borel subset $X \subset \Gamma\backslash G$, define the measure $\mu_{gN^+,\nu}|_X$ on N^+ by

$$d\mu_{gN^+,\nu}|_X(n) = \mathbb{1}_X([g]n)\, d\mu_{gN^+,\nu}(n);$$

note that here the notation $|_X$ is purely symbolic, as $\mu_{gN^+,\nu}|_X$ is not a measure on X. Set $P^\pm := MAN^\pm$. For $\varepsilon > 0$ and $\star = N, N^+, A, M$, let $\star_\varepsilon$ denote the ε-neighborhood of e in $\star$. We then set $P^\pm_\varepsilon = N^\pm_\varepsilon A_\varepsilon M_\varepsilon$.

We recall the following lemmas from [4]:

Lemma 2.7. [4, Lem. 5.6, Cor. 5.7] *We have:*

(1) *For any fixed $\rho \in C_c(N^\pm)$ and $g \in G$, the map $N^\mp \to \mathbb{R}$ given by $n \mapsto \mu_{gnN^\pm,\nu}(\rho)$ is continuous.*
(2) *Given $\varepsilon > 0$ and $g \in G$, there exist $R > 1$ and a non-negative $\rho_{g,\varepsilon} \in C_c(N_R)$ such that $\mu_{gnN,\nu}(\rho_{g,\varepsilon}) > 0$ for all $n \in N^+_\varepsilon$.*

Lemma 2.8. [4, Lem. 4.2] *For any $g \in G$, $a \in A$, $n_0, n \in N^+$, we have*

$$d(\theta_*^{-1}\mu_{gN^+,\nu})(n) = e^{-\psi(\log a)} d\mu_{gan_0N^+,\nu}(n),$$

where $\theta : N^+ \to N^+$ is given by $\theta(n) = an_0na^{-1}$.

Lemma 2.9. [4, Lem. 4.4 and 4.5] *For $i = 1, 2$, let $\psi_i \in \mathfrak{a}^*$ and ν_i a (Γ, ψ_i)-conformal measure. Then*

(1) *For $g \in G$, $f \in C_c(gN^+P)$, and $nham \in N^+NAM$,*

$$\tilde{m}_{\nu_1,\nu_2}(f) = \int_{N^+}\left(\int_{NAM} f(gnham)e^{(\psi_1-\psi_2\circ \mathrm{i})(\log a)}\, dm\, da\, d\mu_{gnN,\nu_2}(h)\right) d\mu_{gN^+,\nu_1}(n).$$

(2) *For $g \in G$, $f \in C_c(gPN^+)$, and $hamn \in NAMN^+$,*

$$\tilde{m}^*_{\nu_1,\nu_2}(f) = \int_{NAM}\left(\int_{N^+} f(ghamn)\, d\mu_{ghamN^+,\nu_1}(n)\right) e^{-\psi_2\circ \mathrm{i}(\log a)}\, dm\, da\, d\mu_{gN,\nu_2}(h).$$

Local mixing. Let P° denote the identity component of P and $\mathfrak{Y}_\Gamma$ denote the set of all P°-minimal subsets of $\Gamma\backslash G$. While there exists a unique P-minimal subset of $\Gamma\backslash G$ given by $\{[g] \in \Gamma\backslash G : g^+ \in \Lambda\}$, there may be more than one P°-minimal subset. Note that $\#\mathfrak{Y}_\Gamma \le [P : P^\circ] = [M : M^\circ]$. Set $\Omega = \{[g] \in \Gamma\backslash G : g^\pm \in \Lambda\}$ and write

$$\mathfrak{Z}_\Gamma = \{Y \cap \Omega \subset \Gamma\backslash G : Y \in \mathfrak{Y}_\Gamma\}.$$

Note that for each $Y \in \mathfrak{Y}_\Gamma$, we have $Y = (Y \cap \Omega)N$ and the collection $\{(Y \cap \Omega)N^+ : Y \in \mathfrak{Y}_\Gamma\}$ is in one-to-one correspondence with the set of $(M^\circ AN^+)$-minimal subsets of $\Gamma\backslash G$.

In the rest of the section, we fix a unit vector $u \in \mathcal{L}_\Gamma \cap \operatorname{int}\mathfrak{a}^+$, and set

$$a_t = \exp(tu) \quad \text{for } t \in \mathbb{R}.$$

We also fix

$$\psi_1 \in \mathfrak{a}^* \quad \text{and} \quad \psi_2 := \psi_1 \circ \mathrm{i} \in \mathfrak{a}^*.$$

For each $i = 1, 2$, we fix a (Γ, ψ_i)-PS measure ν_i on $\mathcal{F}$. We will assume that the associated BMS-measure $\mathsf{m} = m_{\nu_1,\nu_2}$ satisfies the local mixing property for the $\{a_t : t \in \mathbb{R}\}$-action in the following sense:

Hypothesis on $\mathsf{m} = m_{\nu_1,\nu_2}$: there exists a proper continuous function $\Psi : (0, \infty) \to (0, \infty)$ such that for all $f_1,\ f_2 \in C_c(\Gamma\backslash G)$,

$$\lim_{t\to+\infty} \Psi(t) \int_{\Gamma\backslash G} f_1(xa_t) f_2(x)\, d\mathsf{m}(x) = \sum_{Z\in\mathfrak{Z}_\Gamma} \mathsf{m}|_Z(f_1)\, \mathsf{m}|_Z(f_2). \tag{2.10}$$

The main goal in this section is to obtain the following local mixing property for a generalized BMS-measure m_{λ_1,λ_2} from that of m (note that λ_1 and λ_2 are not assumed to be supported on Λ):

Theorem 2.11. *For $i = 1, 2$, let $\varphi_i \in \mathfrak{a}^*$ and λ_i be a (Γ, φ_i)-conformal measure on $\mathcal{F}$. Then for all $f_1, f_2 \in C_c(\Gamma\backslash G)$, we have*

$$\lim_{t\to+\infty} \Psi(t) e^{(\varphi_1-\psi_1)(tu)} \int_{\Gamma\backslash G} f_1(xa_t) f_2(x)\, dm^*_{\lambda_1,\lambda_2}(x)$$
$$= \sum_{Z\in\mathfrak{Z}_\Gamma} m_{\lambda_1,\nu_2}|_{ZN^+}(f_1)\, m^*_{\nu_1,\lambda_2}|_{ZN}(f_2).$$

Remark 2.12. If $\varphi_2 = \varphi_1 \circ \mathrm{i}$, we may replace $m^*_{\lambda_1,\lambda_2}$ by m_{λ_1,λ_2} in Theorem 2.11 by Lemma 2.6. For general φ_1, φ_2, we get, using the identity (2.5): for all $f_1, f_2 \in C_c(\Gamma\backslash G)$, we have

$$\lim_{t\to+\infty} \Psi(t) e^{(\varphi_1-\psi_1)(tu)} \int_{\Gamma\backslash G} f_1(xa_{-t}) f_2(x)\, dm_{\lambda_2,\lambda_1}(x)$$
$$= \sum_{Z\in\mathfrak{Z}_\Gamma} m^*_{\nu_2,\lambda_1}|_{ZN^+}(f_1)\, m_{\lambda_2,\nu_1}|_{ZN}(f_2).$$

In order to prove Theorem 2.11, we first deduce equidistribution of translates of μ_{gN^+,ν_1} from the local mixing property of m (Proposition 2.13), and then convert this into equidistribution of translates of μ_{gN^+,λ_1} (Proposition 2.17).

Proposition 2.13. *For any $x = [g] \in \Gamma\backslash G$, $f \in C_c(\Gamma\backslash G)$, and $\phi \in C_c(N^+)$,*

$$\lim_{t\to+\infty} \Psi(t) \int_{N^+} f\ (xna_t)\phi(n)\, d\mu_{gN^+,\nu_1}(n) = \sum_{Z\in\mathfrak{Z}_\Gamma} \mathsf{m}|_Z(f)\, \mu_{gN^+,\nu_1}|_{ZN}(\phi). \tag{2.14}$$

Proof. Let $x = [g]$, and $\varepsilon_0 > 0$ be such that $\phi \in C_c(N^+_{\varepsilon_0})$. For simplicity of notation, we write $d\mu_{\nu_1} = d\mu_{gN^+,\nu_1}$ throughout the proof. By Lemma 2.7, we can choose $R > 0$ and a nonnegative $\rho_{g,\varepsilon_0} \in C_c(N_R)$ such that

$$\mu_{gnN,\nu_2}(\rho_{g,\varepsilon_0}) > 0 \quad \text{for all } n \in N^+_{\varepsilon_0}.$$

Given any $\varepsilon > 0$, choose a non-negative function $q_\varepsilon \in C_c(A_\varepsilon M_\varepsilon)$ satisfying $\int_{AM} q_\varepsilon(am)\,da\,dm = 1$. Then

$$\int_{N^+} f(xna_t)\phi(n)\,d\mu_{\nu_1}(n) = \tag{2.15}$$
$$\int_{N^+} f(xna_t)\phi(n)\left(\frac{1}{\mu_{gnN,\nu_2}(\rho_{g,\varepsilon_0})}\int_{NA}\rho_{g,\varepsilon_0}(h)q_\varepsilon(am)\,da\,dm\,d\mu_{gnN,\nu_2}(h)\right)d\mu_{\nu_1}(n)$$
$$= \int_{N^+}\left(\int_{NA} f(xna_t)\frac{\phi(n)\rho_{g,\varepsilon_0}(h)q_\varepsilon(am)}{\mu_{gnN,\nu_2}(\rho_{g,\varepsilon_0})}\,da\,dm\,d\mu_{gnN,\nu_2}(h)\right)d\mu_{\nu_1}(n).$$

We now define $\tilde{\Phi}_\varepsilon \in C_c(gN^+_{\varepsilon_0}N_R A_\varepsilon M_\varepsilon) \subset C_c(G)$ and $\Phi_\varepsilon \in C_c(\Gamma\backslash G)$ by

$$\tilde{\Phi}_\varepsilon(g_0) := \begin{cases} \frac{\phi(n)\rho_{g,\varepsilon_0}(h)q_\varepsilon(am)}{\mu_{gnN,\nu_2}(\rho_{g,\varepsilon_0})} & \text{if } g_0 = gnham, \\ 0 & \text{otherwise,} \end{cases}$$

and $\Phi_\varepsilon([g_0]) := \sum_{\gamma\in\Gamma}\tilde{\Phi}_\varepsilon(\gamma g_0)$. Note that the continuity of $\tilde{\Phi}_\varepsilon$ follows from Lemma 2.7. We now assume without loss of generality that $f \geq 0$ and define, for all $\varepsilon > 0$, functions $f^\pm_\varepsilon$ as follows: for all $z \in \Gamma\backslash G$,

$$f^+_\varepsilon(z) := \sup_{b\in N^+_\varepsilon P_\varepsilon} f(zb) \quad\text{and}\quad f^-_\varepsilon(z) := \inf_{b\in N^+_\varepsilon P_\varepsilon} f(zb).$$

Since $u \in \operatorname{int}\mathfrak{a}^+$, for every $\varepsilon > 0$, there exists $t_0(R,\varepsilon) > 0$ such that

$$a_t^{-1}N_R a_t \subset N_\varepsilon \qquad \text{for all } t \geq t_0(R,\varepsilon).$$

Then, as $\operatorname{supp}(\tilde{\Phi}_\varepsilon) \subset gN^+_{\varepsilon_0}N_R A_\varepsilon M_\varepsilon$, we have

$$f(xna_t)\tilde{\Phi}_\varepsilon(gnham) \leq f^+_{3\varepsilon}(xnhama_t)\tilde{\Phi}_\varepsilon(gnham) \tag{2.16}$$

for all $nham \in N^+NAM$ and $t \geq t_0(R,\varepsilon)$. We now use $f^+_{3\varepsilon}$ to give an upper bound on the limit we are interested in; $f^-_{3\varepsilon}$ is used in an analogous way to provide a lower bound. Entering the definition of Φ_ε and the above inequality (2.16) into (2.15) gives

$$\limsup_{t\to+\infty}\Psi(t)\int_{N^+} f(xna_t)\phi(n)\,d\mu_{\nu_1}(n)$$
$$\leq \limsup_{t\to+\infty}\Psi(t)$$
$$\int_{N^+}\int_{NAM} f^+_{3\varepsilon}(xnhama_t)\tilde{\Phi}_\varepsilon(gnham)dm\,da\,d\mu_{gnN,\nu_2}(h)\,d\mu_{\nu_1}(n)$$
$$\leq \limsup_{t\to+\infty}\Psi(t)e^{\varepsilon\|\psi_1-\psi_2\circ\mathrm{i}\|}\int_{N^+}\int_{NAM} f^+_{3\varepsilon}(xnhama_t)\tilde{\Phi}_\varepsilon(gnham)$$
$$e^{(\psi_1-\psi_2\circ\mathrm{i})(\log a)}\,dm\,da\,d\mu_{gnN,\nu_2}(h)\,d\mu_{\nu_1}(n)$$
$$= \limsup_{t\to+\infty}\Psi(t)e^{\varepsilon\|\psi_1-\psi_2\circ\mathrm{i}\|}\int_G f^+_{3\varepsilon}([g_0]a_t)\tilde{\Phi}_\varepsilon(g_0)\,d\tilde{\mathsf{m}}(g_0)$$
$$= \limsup_{t\to+\infty}\Psi(t)e^{\varepsilon\|\psi_1-\psi_2\circ\mathrm{i}\|}\int_{\Gamma\backslash G} f^+_{3\varepsilon}([g_0]a_t)\Phi_\varepsilon([g_0])\,d\mathsf{m}([g_0]),$$

where $\|\cdot\|$ is the operator norm on $\mathfrak{a}^*$ and Lemma 2.9 was used in the second to last line of the above calculation. By the standing assumption (2.10), we have

$$\begin{aligned}\limsup_{t\to+\infty} \Psi(t) \int_N f(xna_t)\phi(n)\, d\mu_{gN,\nu_2}(n) \\ &\le e^{\varepsilon\|\psi_1-\psi_2\circ \mathrm{i}\|} \sum_{Z\in\mathfrak{Z}_\Gamma} \mathsf{m}|_Z(f_{3\varepsilon}^+)\mathsf{m}|_Z(\Phi_\varepsilon) \\ &= e^{\varepsilon\|\psi_1-\psi_2\circ \mathrm{i}\|} \sum_{Z\in\mathfrak{Z}_\Gamma} \mathsf{m}|_Z(f_{3\varepsilon}^+)\tilde{\mathsf{m}}|_{\tilde Z}(\tilde\Phi_\varepsilon),\end{aligned}$$

where $\tilde Z \subset G$ is a Γ-invariant lift of Z. Using Lemma 2.9, for all $0<\varepsilon \ll 1$,

$$\begin{aligned}&\tilde{\mathsf{m}}|_{\tilde Z}(\tilde\Phi_\varepsilon) \\ &= \int_{N^+}\left(\int_{NAM} \tilde\Phi_\varepsilon \mathbb{1}_{\tilde Z}(gnham)e^{(\psi_1-\psi_2\circ\mathrm{i})(\log a)}\, da\, dm\, d\mu_{gnN,\nu_2}(h)\right) d\mu_{\nu_1}(n) \le \\ &e^{\varepsilon\|\psi_1-\psi_2\circ \mathrm{i}\|}\int_{N^+} \frac{\phi(n)\mathbb{1}_{\tilde Z N}(gn)}{\mu_{gnN,\nu_2}(\rho_{g,\varepsilon_0})}\left(\int_{NAM} \rho_{g,\varepsilon_0}(h)q_\varepsilon(am)\, da\, dm\, d\mu_{gnN,\nu_2}(h)\right) d\mu_{\nu_1}(n) \\ &\le e^{\varepsilon\|\psi_1-\psi_2\circ \mathrm{i}\|}\mu_{\nu_1}|_{ZN}(\phi),\end{aligned}$$

where we have used the facts that $\tilde Z$ is invariant under the right translation of identity component M° of M, and $\operatorname{supp}\nu_2 = \Lambda$ as well as the identity $\mathbb{1}_{\tilde Z}(gnha) = \mathbb{1}_{\tilde Z N}(gn)\mathbb{1}_\Lambda(gnh^+)$ (we remark that $\operatorname{supp}\nu_2=\Lambda$ is not necessary for the upper bound as $\mathbb{1}_{\tilde Z}(gnha)\le \mathbb{1}_{\tilde Z N}(gn)$, but needed for the lower bound). Since $\varepsilon>0$ was arbitrary, taking $\varepsilon\to 0$ gives

$$\limsup_{t\to+\infty} \Psi(t) \int_{N^+} f(xna_t)\phi(n)\, d\mu_{\nu_1}(n) \le \sum_{Z\in\mathfrak{Z}_\Gamma} \mathsf{m}|_Z(f)\, \mu_{\nu_1}|_{ZN}(\phi).$$

The lower bound given by replacing $f_{3\varepsilon}^+$ with $f_{3\varepsilon}^-$ in the above calculations completes the proof. □

Proposition 2.17. *For any $x=[g]\in\Gamma\backslash G$, $f\in C_c(\Gamma\backslash G)$ and $\phi\in C_c(N^+)$,*

$$\begin{aligned}\lim_{t\to+\infty} \Psi(t)e^{(\varphi_1-\psi_1)(tu)} \int_{N^+} f\big(xna_t\big)\phi(n)\, d\mu_{gN^+,\lambda_1}(n) \\ = \sum_{Z\in\mathfrak{Z}_\Gamma} m_{\lambda_1,\nu_2}|_{ZN^+}(f)\, \mu_{gN^+,\nu_1}|_{ZN}(\phi).\end{aligned}$$

Proof. For $\varepsilon_0>0$, set $\mathcal{B}_{\varepsilon_0} = P_{\varepsilon_0}N^+_{\varepsilon_0}$. Given $x_0\in\Gamma\backslash G$, let $\varepsilon_0(x_0)$ denote the maximum number r such that the map $G\to\Gamma\backslash G$ given by $h\mapsto x_0h$ for $h\in G$ is injective on $\mathcal{B}_r$. By using a partition of unity if necessary, it suffices to prove that for any $x_0\in\Gamma\backslash G$ and $\varepsilon_0=\varepsilon_0(x_0)$, the claims of the proposition hold for any non-negative $f\in C(x_0\mathcal{B}_{\varepsilon_0})$, non-negative $\phi\in C(N^+_{\varepsilon_0})$, and $x=[g]\in x_0\mathcal{B}_{\varepsilon_0}$. Moreover, we may assume that f is given as

$$f([g]) = \sum_{\gamma\in\Gamma} \tilde{f}(\gamma g) \qquad \text{for all } g \in G,$$

for some non-negative $\tilde{f} \in C_c(g_0\mathcal{B}_{\varepsilon_0})$. For simplicity of notation, we write $\mu_{\lambda_1} = \mu_{gN^+,\lambda_1}$. Note that for $x = [g] \in [g_0]\mathcal{B}_{\varepsilon_0}$,

$$\int_{N^+} f([g]na_t)\phi(n)\, d\mu_{\lambda_1}(n) = \sum_{\gamma\in\Gamma} \int_{N^+} \tilde{f}(\gamma g n a_t)\phi(n)\, d\mu_{\lambda_1}(n). \tag{2.18}$$

Note that $\tilde{f}(\gamma gna_t) = 0$ unless $\gamma gna_t \in g_0\mathcal{B}_{\varepsilon_0}$. Together with the fact that $\operatorname{supp}(\phi) \subset N^+_{\varepsilon_0}$, it follows that the summands in (2.18) are non-zero only for finitely many elements $\gamma \in \Gamma \cap g_0\mathcal{B}_{\varepsilon_0} a_{-t} N^+_{\varepsilon_0} g^{-1}$.

Suppose $\gamma g N^+_{\varepsilon_0} a_t \cap g_0\mathcal{B}_{\varepsilon_0} \neq \emptyset$. Then $\gamma g a_t \in g_0 P_{\varepsilon_0} N^+$, and there are unique elements $p_{t,\gamma} \in P_{\varepsilon_0}$ and $n_{t,\gamma} \in N^+$ such that

$$\gamma g a_t = g_0 p_{t,\gamma} n_{t,\gamma} \in g_0 P_{\varepsilon_0} N^+.$$

Let Γ_t denote the subset $\Gamma \cap g_0(P_{\varepsilon_0}N^+)a_t^{-1}g^{-1}$. Note that although Γ_t may possibly be infinite, only finitely many of the terms in the sums we consider will be non-zero. This together with Lemma 2.8 gives

$$\begin{aligned}
&\int_{N^+} f([g]na_t)\phi(n)\, d\mu_{\lambda_1}(n) = \sum_{\gamma\in\Gamma} \int_{N^+} \tilde{f}(\gamma gna_t)\phi(n)\, d\mu_{\lambda_1}(n)\\
&= \sum_{\gamma\in\Gamma_t} \int_{N^+} \tilde{f}(\gamma g a_t(a_t^{-1}na_t))\phi(n)\, d\mu_{\lambda_1}(n)\\
&= e^{-\varphi_1(\log a_t)} \sum_{\gamma\in\Gamma_t} \int_{N^+} \tilde{f}(\gamma g a_t n)\phi(a_t n a_t^{-1})\, d\mu_{ga_tN^+,\lambda_1}(n)\\
&= e^{-\varphi_1(\log a_t)} \sum_{\gamma\in\Gamma_t} \int_{N^+} \tilde{f}\big(g_0 p_{t,\gamma} n_{t,\gamma} n\big)\phi(a_t n a_t^{-1})\, d\mu_{ga_tN^+,\lambda_1}(n)\\
&= e^{-\varphi_1(\log a_t)} \sum_{\gamma\in\Gamma_t} \int_{N^+} \tilde{f}\big(g_0 p_{t,\gamma} n\big)\phi\big(a_t\, n_{t,\gamma}^{-1} n\ a_t^{-1}\big)\, d\mu_{g_0p_{t,\gamma}N^+,\lambda_1}(n).
\end{aligned}$$

Since $\operatorname{supp}(\tilde{f}) \subset g_0\mathcal{B}_{\varepsilon_0}$, we have

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_t} \int_{N^+} \tilde{f}\big(g_0 p_{t,\gamma} n\big)\phi\big(a_t\, n_{t,\gamma}^{-1} n\ a_t^{-1}\big)\, d\mu_{g_0p_{t,\gamma}N^+,\lambda_1}(n)\\
&\quad\le \sum_{\gamma\in\Gamma_t} \left(\sup_{n\in N^+_{\varepsilon_0}} \phi\big(a_t\, n_{t,\gamma}^{-1}\ a_t^{-1}(a_t n a_t^{-1})\big)\right) \cdot \int_{N^+} \tilde{f}\big(g_0 p_{t,\gamma} n\big)\, d\mu_{g_0p_{t,\gamma}N^+,\lambda_1}(n).
\end{aligned}$$

Since u belongs to $\operatorname{int}\mathcal{L}_\Gamma$, there exist $t_0 > 0$ and $\alpha > 0$ such that

$$a_t N^+_r a_t^{-1} \subset N^+_{re^{-\alpha t}} \qquad \text{for all } \ r > 0 \text{ and } t > t_0.$$

Therefore, for all $n \in N^+_{\varepsilon_0}$ and $t > t_0$, we have

$$\phi\big(a_t\, n_{t,\gamma}^{-1} a_t^{-1}(a_t n a_t^{-1})\big) \le \phi^+_{\varepsilon_0 e^{-\alpha t}}\big(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}\big), \tag{2.19}$$

where

$$\phi^+_\varepsilon(n) := \sup_{b\in N^+_\varepsilon} \phi(nb) \qquad \text{for all } n \in N^+,\ \varepsilon > 0.$$

We now have the following inequality for $t > t_0$:

$$\begin{aligned} &e^{\varphi_1(\log a_t)} \int_{N^+} f([g]na_t)\phi(n)\, d\mu_{\lambda_1}(n) \\ &\le \sum_{\gamma\in\Gamma_t} \phi^+_{\varepsilon_0 e^{-\alpha t}}\big(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}\big) \int_{N^+_{\varepsilon_0}} \tilde f\big(g_0 p_{t,\gamma} n\big)\, d\mu_{g_0 p_{t,\gamma} N^+,\lambda_1}(n). \end{aligned} \tag{2.20}$$

By Lemma 2.7, we can now choose $R > 0$ and $\rho \in C_c(N^+_R)$ such that $\rho(n) \ge 0$ for all $n \in N^+$, and $\mu_{g_0pN^+,\nu_1}(\rho) > 0$ for all $p \in P_{\varepsilon_0}$. Define $\tilde F \in C_c(g_0 P_{\varepsilon_0} N^+_R)$ by

$$\tilde F(g) = \begin{cases} \frac{\rho(n)}{\mu_{g_0pN^+,\nu_1}(\rho)} \int_{N^+_{\varepsilon_0}} \tilde f\big(g_0pv\big)\, d\mu_{g_0pN^+,\lambda_1}(v) & \text{if } g = g_0pn \in g_0P_{\varepsilon_0}N^+_R \\ 0 & \text{if } g \notin g_0P_{\varepsilon_0}N^+_R. \end{cases}$$

We claim that for all $p \in P_{\varepsilon_0}$ and $Z \in \mathfrak{Z}_\Gamma$ such that $g_0p^- \in \Lambda$,

$$\begin{aligned} \int_{N^+} \tilde F(g_0pn)\, d\mu_{g_0pN^+,\nu_1}|_Z(n) &= \int_{N^+_R} \tilde F(g_0pn)\, d\mu_{g_0pN^+,\nu_1}|_Z(n) \\ &= \int_{N^+_{\varepsilon_0}} (\tilde f \mathbb{1}_{ZN^+})(g_0pn)\, d\mu_{g_0pN^+,\lambda_1}(n). \end{aligned} \tag{2.21}$$

Indeed, by the assumption $\operatorname{supp}\nu_1 = \Lambda$ and the fact $\Omega \cap ZN^+ = Z$, we have the identity $\mathbb{1}_Z(g_0pn)\, d\mu_{g_0pN^+,\nu_1}(n) = \mathbb{1}_{ZN^+}(g_0p)\, d\mu_{g_0pN^+,\nu_1}(n)$ and hence

$$\begin{aligned} &\int_{N^+} \tilde F(g_0pn)\, d\mu_{g_0pN^+,\nu_1}|_Z(n) \\ &= \int_{N^+} \tilde F(g_0pn)\mathbb{1}_Z(g_0pn)\, d\mu_{g_0pN^+,\nu_1}(n) \\ &= \int_{N^+} \frac{\rho(n)\mathbb{1}_{ZN^+}(g_0p)}{\mu_{g_0pN^+,\nu_1}(\rho)} \left(\int_{N^+_{\varepsilon_0}} \tilde f\big(g_0pv\big)\, d\mu_{g_0pN^+,\lambda_1}(v) \right) d\mu_{g_0pN^+,\nu_1}(n) \\ &= \int_{N^+} \frac{\rho(n)}{\mu_{g_0pN^+,\nu_1}(\rho)} \left(\int_{N^+_{\varepsilon_0}} (\tilde f \mathbb{1}_{ZN^+})\big(g_0pv\big)\, d\mu_{g_0pN^+,\lambda_1}(v) \right) d\mu_{g_0pN^+,\nu_1}(n) \\ &= \int_{N^+_{\varepsilon_0}} (\tilde f \mathbb{1}_{ZN^+})\big(g_0pv\big)\, d\mu_{g_0pN^+,\lambda_1}(v). \end{aligned}$$

Summing up (2.21) for all $Z \in \mathfrak{Z}_\Gamma$ and using $\operatorname{supp}\nu_1 = \Lambda$, we get

$$\begin{aligned}
&\int_{N^+} \tilde{F}(g_0pn)\, d\mu_{g_0pN^+,\nu_1}(n) \\
&= \sum_{Z\in\mathfrak{Z}_\Gamma} \int_{N^+} \tilde{F}(g_0pn)\, d\mu_{g_0pN^+,\nu_1}|_Z(n) \\
&= \sum_{Z\in\mathfrak{Z}_\Gamma} \int_{N^+_{\varepsilon_0}} (\tilde{f}\mathbb{1}_{ZN^+})(g_0pn)\, d\mu_{g_0pN^+,\lambda_1}(n).
\end{aligned}$$

Hence we can write

$$\begin{aligned}
&\int_{N^+_{\varepsilon_0}} \tilde{f}(g_0pn)\, d\mu_{g_0pN^+,\lambda_1}(n) \\
&= \int_{N^+} \tilde{F}(g_0pn)\, d\mu_{g_0pN^+,\nu_1}(n) + \int_{N^+_{\varepsilon_0}} \tilde{h}(g_0pn) d\mu_{g_0pN^+,\lambda_1}(n)
\end{aligned}$$

for some $\tilde{h}$ that vanishes on $\bigcup_{Z\in\mathfrak{Z}_\Gamma} ZN^+$. Returning to (2.20), we now give an upper bound. We observe:

$$\begin{aligned}
&e^{\varphi_1(\log a_t)} \int_{N^+} f([g]na_t)\phi(n)\, d\mu_{\lambda_1}(n) \\
&\le \sum_{\gamma\in\Gamma_t} \phi^+_{\varepsilon_0 e^{-\alpha t}}\big(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}\big) \int_{N^+_{\varepsilon_0}} \tilde{f}\big(g_0p_{t,\gamma}n\big)\, d\mu_{\lambda_1}(n) \\
&= \sum_{\gamma\in\Gamma_t} \phi^+_{\varepsilon_0 e^{-\alpha t}}\big(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}\big) \int_{N^+_R} (\tilde{F}+\tilde{h})(g_0p_{t,\gamma}n)\, d\mu_{g_0p_{t,\gamma}N^+,\nu_1}(n) \\
&= \sum_{\gamma\in\Gamma_t} \int_{N^+_R} (\tilde{F}+\tilde{h})(g_0p_{t,\gamma}n)\phi^+_{\varepsilon_0 e^{-\alpha t}}\big(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}\big)\, d\mu_{g_0p_{t,\gamma}N^+,\nu_1}(n).
\end{aligned}$$

Similarly as before, we have, for all $t > t_0$ and $n \in N^+_R$,

$$\begin{aligned}
\phi^+_{\varepsilon_0 e^{-\alpha t}}\big(a_t\, n_{t,\gamma}^{-1}\, a_t^{-1}\big) &= \phi^+_{\varepsilon_0 e^{-\alpha t}}\big(a_t\, n_{t,\gamma}^{-1} n(n)^{-1}\, a_t^{-1}\big) \\
&\le \phi^+_{(R+\varepsilon_0)e^{-\alpha t}}\big(a_t\, n_{t,\gamma}^{-1} n\, a_t^{-1}\big). \qquad (2.22)
\end{aligned}$$

Hence (2.20) is bounded above by

$$\begin{aligned}
&\le \sum_{\gamma\in\Gamma_t} \int_{N^+_R} (\tilde{F}+\tilde{h})(g_0p_{t,\gamma}n)\phi^+_{(R+\varepsilon_0)e^{-\alpha t}}\big(a_t\, n_{t,\gamma}^{-1} n\, a_t^{-1}\big)\, d\mu_{g_0p_{t,\gamma}N^+,\nu_1}(n) \\
&= \sum_{\gamma\in\Gamma_t} \int_{N^+} (\tilde{F}+\tilde{h})\big(g_0p_{t,\gamma}n_{t,\gamma}a_t^{-1}na_t\big)\phi^+_{(R+\varepsilon_0)e^{-\alpha t}}(n)\, d((\theta_{t,\gamma})_*^{-1}\mu_{g_0p_{t,\gamma}N^+,\nu_1})(n)
\end{aligned}$$

where $\theta_{t,\gamma}(n) = n_{t,\gamma}a_t^{-1}na_t$. By Lemma 2.8,

$$d((\theta_{t,\gamma})_*^{-1}\mu_{g_0p_{t,\gamma}N^+,\nu_1})(n) = e^{\psi_1(\log a_t)} d\mu_{g_0p_{t,\gamma}n_{t,\gamma}a_t^{-1}N^+,\nu_1}(n).$$

Since $g_0 p_{t,\gamma} n_{t,\gamma} a_t^{-1} = \gamma g$, it follows that for all $t > t_0$,

$$\begin{aligned}
&e^{(\varphi_1-\psi_1)(\log a_t)} \int_{N^+} f([g]na_t)\phi(n)\, d\mu_{\lambda_1}(n) \\
&\le \sum_{\gamma\in\Gamma_t} \int_{N^+} (\tilde{F}+\tilde{h})(\gamma g n a_t)\phi^+_{(R+\varepsilon_0)e^{-\alpha t}}(n)\, d\mu_{\gamma g N^+,\nu_1}(n) \\
&\le \int_{N^+} \left(\sum_{\gamma\in\Gamma} (\tilde{F}+\tilde{h})(\gamma g n a_t) \right) \phi^+_{(R+\varepsilon_0)e^{-\alpha t}}(n)\, d\mu_{\nu_1}(n).
\end{aligned}$$

Define functions F and h on $\Gamma\backslash G$ by

$$F([g]) := \sum_{\gamma\in\Gamma} \tilde{F}(\gamma g) \quad \text{and} \quad h([g]) := \sum_{\gamma\in\Gamma} \tilde{h}(\gamma g).$$

Then for any $\varepsilon > 0$ and for all $t > t_0$ such that $(R+\varepsilon_0)e^{-\alpha t} \le \varepsilon$,

$$\begin{aligned}
&\Psi(t) e^{(\varphi_1-\psi_1)(\log a_t)} \int_{N^+} f([g]na_t)\phi(n)\, d\mu_{\lambda_1}(n) \\
&\qquad\qquad \le \Psi(t) \int_{N^+} (F+h)([g]na_t)\phi^+_\varepsilon(n)\, d\mu_{\nu_1}(n).
\end{aligned}$$

By Proposition 2.13, letting $\varepsilon \to 0$ gives

$$\begin{aligned}
&\limsup_{t\to+\infty} \Psi(t) e^{(\varphi_1-\psi_1)(\log a_t)} \int_{N^+} f([g]na_t)\phi(n)\, d\mu_{\lambda_1}(n) \\
&\qquad\qquad \le \sum_{Z\in\mathfrak{Z}_\Gamma} \mathsf{m}|_Z(F+h)\, \mu_{\nu_1}|_{ZN}(\phi).
\end{aligned}$$

Note that $\mathsf{m}^* = \mathsf{m}$ by Lemma 2.6. Now, by Lemma 2.9 and the fact $\tilde{\mathsf{m}}(\tilde{h}) = 0$, we have

$$\begin{aligned}
&\mathsf{m}|_Z(F+h) = \tilde{\mathsf{m}}|_{\tilde{Z}}(\tilde{F}+\tilde{h}) = \tilde{\mathsf{m}}|_{\tilde{Z}}(\tilde{F}) = \tilde{\mathsf{m}}^*|_{\tilde{Z}}(\tilde{F}) \\
&= \int_P \left(\int_{N^+} \tilde{F}\mathbb{1}_{\tilde{Z}}(g_0 hamn)\, d\mu_{g_0 hamN^+,\nu_1}(n) \right) e^{-\psi_2\circ \mathrm{i}(\log a)}\, dm\, da\, d\mu_{g_0 N,\nu_2}(h) \\
&= \int_P \left(\int_{N^+} (\tilde{f}\mathbb{1}_{ZN^+})(g_0 hamn)\, d\mu_{g_0 hamN^+,\lambda_1}(n) \right) e^{-\psi_2\circ \mathrm{i}(\log a)}\, dm\, da\, d\mu_{g_0 N,\nu_2}(h) \\
&= \tilde{m}_{\lambda_1,\nu_2}|_{\tilde{Z}N^+}(\tilde{f}) = m_{\lambda_1,\nu_2}|_{ZN^+}(f).
\end{aligned}$$

This gives the desired upper bound. Note that we have used the assumption $\operatorname{supp}\nu_2 = \Lambda$ in the fourth equality above to apply (2.21). The lower bound can be obtained similarly, finishing the proof. □

With the help of Proposition 2.13, we are now ready to give:

Proof of Theorem 2.11 By the compactness hypothesis on the supports of f_i, we can find $\varepsilon_0 > 0$ and $x_i \in \Gamma\backslash G$, $i = 1, \cdots, \ell$ such that the map $G \to \Gamma\backslash G$ given by $g \to x_i g$ is injective on $R_{\varepsilon_0} = P_{\varepsilon_0} N^+_{\varepsilon_0}$, and $\bigcup_{i=1}^{\ell} x_i R_{\varepsilon_0/2}$ contains both $\operatorname{supp} f_1$ and $\operatorname{supp} f_2$. We use continuous partitions of unity to write f_1 and f_2 as finite sums $f_1 = \sum_{i=1}^{\ell} f_{1,i}$ and $f_2 = \sum_{j=1}^{\ell} f_{2,j}$ with $\operatorname{supp} f_{1,i} \subset x_i R_{\varepsilon_0/2}$ and $\operatorname{supp} f_{2,j} \subset x_j R_{\varepsilon_0/2}$. Writing $p = ham \in NAM$ and using Lemma 2.9,

$$dm^*_{\lambda_1,\lambda_2}(hamn) = d\mu_{hamN^+,\lambda_1}(n) e^{-\psi_2\circ \mathrm{i}(\log a)}\, dm\, da\, d\mu_{N,\lambda_2}(h).$$

We have

$$\int_{\Gamma\backslash G} f_1(xa_t) f_2(x)\, dm^*_{\lambda_1,\lambda_2}(x) = \tag{2.23}$$
$$\sum_{i,j} \int_{R_{\varepsilon_0}} f_{1,i}(x_j p n a_t) f_{2,j}(x_j p n) d\mu_{hamN^+,\lambda_1}(n) e^{-\psi_2\circ \mathrm{i}(\log a)}\, dm\, da\, d\mu_{N,\lambda_2}(h)$$
$$= \sum_{i,j} \int_{N_{\varepsilon_0} A_{\varepsilon_0} M_{\varepsilon_0}} \left(\int_{N^+_{\varepsilon_0}} f_{1,i}(x_j p n a_t) f_{2,j}(x_j p n)\, d\mu_{hamN^+,\lambda_1}(n) \right) \times e^{-\psi_2\circ \mathrm{i}(\log a)}\, dm\, da\, d\mu_{N,\lambda_2}(h).$$

Applying Proposition 2.17, it follows:

$$\lim_{t\to\infty} \Psi(t) e^{(\varphi_1-\psi_1)(\log a_t)} \int_{\Gamma\backslash G} f_1(xa_t) f_2(x)\, dm^*_{\lambda_1,\lambda_2}(x)$$
$$= \sum_j \sum_{Z\in\mathfrak{Z}_\Gamma} m_{\lambda_1,\nu_2}|_{ZN^+}(f_{1,j}) \sum_i \int_{N_{\varepsilon_0} A_{\varepsilon_0} M_{\varepsilon_0}} \mu_{x_i pN^+,\nu_1}|_{ZN}(f_{2,i}(x_j p\,\cdot\,))\, e^{-\psi_2\circ \mathrm{i}(\log a)}\, dm\, da\, d\mu_{N,\lambda_2}(h)$$
$$= \sum_{Z\in\mathfrak{Z}_\Gamma} m_{\lambda_1,\nu_2}|_{ZN^+}(f_1) \sum_i \int_{N_{\varepsilon_0} A_{\varepsilon_0} M_{\varepsilon_0}} \mu_{x_i pN^+,\nu_1}(f_{2,i} \mathbb{1}_{ZN}(x_j p\,\cdot\,))\, e^{-\psi_2\circ \mathrm{i}(\log a)}\, dm\, da\, d\mu_{N,\lambda_2}(h)$$
$$= \sum_{Z\in\mathfrak{Z}_\Gamma} m_{\lambda_1,\nu_2}|_{ZN^+}(f_1) \sum_i m^*_{\nu_1,\lambda_2}(f_{2,i}\mathbb{1}_{ZN}) = \sum_{Z\in\mathfrak{Z}_\Gamma} m_{\lambda_1,\nu_2}|_{ZN^+}(f_1) m^*_{\nu_1,\lambda_2}|_{ZN}(f_2)$$

where the second last equality is valid by Lemma 2.9. This completes the proof. □

3. Local mixing for Anosov groups

Let $\Gamma < G$ be a Zariski dense Anosov subgroup with respect to P. For any $u \in \operatorname{int} \mathcal{L}_\Gamma$, there exists a unique

$$\psi = \psi_u \in D^\star_\Gamma$$

such that $\psi(u) = \psi_\Gamma(u)$ [7, Prop. 4.4]. Let ν_ψ denote the unique (Γ, ψ)-PS measure [7, Thm. 1.3]. Similarly, $\nu_{\psi\circ \mathrm{i}}$ denotes the unique $(\Gamma, \psi\circ \mathrm{i})$-PS-measure.

In this section, we deduce ($r := \dim \mathfrak{a}$):

Theorem 3.1 (Local mixing)**.** *For $i = 1, 2$, let $\varphi_i \in \mathfrak{a}^*$ and λ_{φ_i} be any (Γ, φ_i)-conformal measure on $\mathcal{F}$. For any $u \in \operatorname{int} \mathcal{L}_\Gamma$, there exists $\kappa_u > 0$ such that for any $f_1, f_2 \in C_c(\Gamma\backslash G)$, we have*

$$\lim_{t\to+\infty} t^{(r-1)/2} e^{(\varphi_1-\psi_u)(tu)} \int_{\Gamma\backslash G} f_1(x\exp(tu)) f_2(x)\, dm^*_{\lambda_{\varphi_1},\lambda_{\varphi_2}}(x)$$
$$= \kappa_u \sum_{Z\in\mathfrak{Z}_\Gamma} m_{\lambda_{\varphi_1},\nu_{\psi_u\circ \mathrm{i}}}|_{ZN^+}(f_1)\, m^*_{\nu_{\psi_u},\lambda_{\varphi_2}}|_{ZN}(f_2).$$

Theorem 3.1 is a consequence of Theorem 2.11, since the measure $\mathsf{m} = m_{\nu_{\psi_u},\nu_{\psi_u\circ\mathrm{i}}}$ satisfies the Hypothesis 2.10 by the following theorem of Chow and Sarkar.

Theorem 3.2. [3] *Let $u \in \operatorname{int} \mathcal{L}_\Gamma$. There exists $\kappa_u > 0$ such that for any $f_1, f_2 \in C_c(\Gamma\backslash G)$, we have*

$$\lim_{t\to+\infty} t^{(r-1)/2} \int_{\Gamma\backslash G} f_1(x\exp(tu)) f_2(x)\, dm_{\nu_{\psi_u},\nu_{\psi_u\circ\mathrm{i}}}(x)$$
$$= \kappa_u \sum_{Z\in\mathfrak{Z}_\Gamma} m_{\nu_{\psi_u},\nu_{\psi_u\circ\mathrm{i}}}|_Z(f_1)\, m_{\nu_{\psi_u},\nu_{\psi_u\circ\mathrm{i}}}|_Z(f_2).$$

Let m_o denote the K-invariant probability measure on $\mathcal{F} = G/P$. Then m_o coincides with the $(G, 2\rho)$-conformal measure on $\mathcal{F}$ where 2ρ denotes the sum of positive roots for $(\mathfrak{g}, \mathfrak{a}^+)$. The corresponding BMS measure $dx = dm_{m_o,m_o}$ is a G-invariant measure on $\Gamma\backslash G$. The measure $dm^{\mathrm{BR}}_{\nu_{\psi\circ\mathrm{i}}} = dm_{m_o,\nu_{\psi\circ\mathrm{i}}}$ was defined and called the N^+M-invariant Burger-Roblin measure in [4]. Similarly, the NM-invariant Burger-Roblin measure was defined as $dm^{\mathrm{BR}\,*}_{\nu_\psi}$. In these terminologies, the following is a special case of Theorem 3.1:

Corollary 3.3 (Local mixing for the Haar measure)**.** *For any $u \in \operatorname{int} \mathcal{L}_\Gamma$, and for any $f_1, f_2 \in C_c(\Gamma\backslash G)$, we have*

$$\lim_{t\to+\infty} t^{(r-1)/2} e^{(2\rho-\psi_u)(tu)} \int_{\Gamma\backslash G} f_1(x\exp(tu)) f_2(x)\, dx$$
$$= \kappa_u \sum_{Z\in\mathfrak{Z}_\Gamma} m^{\mathrm{BR}}_{\nu_{\psi_u\circ\mathrm{i}}}|_{ZN^+}(f_1)\, m^{\mathrm{BR}\,*}_{\nu_{\psi_u}}|_{ZN}(f_2)$$

where κ_u is as in Theorem 3.2.

In fact, we get the following more elaborate version of the above corollary by combining the proof of [4, Theorem 7.12] and the proof of Corollary 3.3.

Theorem 3.4. *Let $u \in \operatorname{int} \mathcal{L}_\Gamma$. For any $f_1, f_2 \in C_c(\Gamma\backslash G)$ and $v \in \ker \psi_u$,*

$$\lim_{t\to+\infty} t^{(r-1)/2} e^{(2\rho-\psi_u)(tu+\sqrt{t}v)} \int_{\Gamma\backslash G} f_1(x\exp(tu+\sqrt{t}v)) f_2(x) dx$$
$$= \kappa_u\, e^{-I(v)/2} \sum_{Z\in\mathfrak{Z}_\Gamma} m^{\mathrm{BR}}_{\nu_{\psi_u\circ\mathrm{i}}}|_{ZN^+}(f_1)\, m^{\mathrm{BR}\,*}_{\nu_{\psi_u}}|_{ZN}(f_2)$$

where $I : \ker \psi_u \to \mathbb{R}$ *is given by*

$$I(v) := c \cdot \frac{\|v\|_*^2 \|u\|_*^2 - \langle v, u\rangle_*^2}{\|u\|_*^2} \tag{3.5}$$

for some inner product $\langle \cdot, \cdot \rangle_*$ *and some* $c > 0$. *Moreover the left-hand sides of the above equalities are uniformly bounded for all* $(t, v) \in (0, \infty) \times \ker \psi_u$ *with* $tu + \sqrt{t}v \in \mathfrak{a}^+$.

4. Proof of Theorem 1.4

Let $\Gamma < G$ be a Zariski dense Anosov subgroup with respect to P.

The u-balanced measures. Let $\Omega = \{[g] \in \Gamma\backslash G : g^{\pm} \in \Lambda\}$. Following [2], given $u \in \operatorname{int} \mathcal{L}_\Gamma$, we say that a locally finite Borel measure m_0 on $\Gamma\backslash G$ is u-balanced if

$$\limsup_{T \to +\infty} \frac{\int_0^T \mathsf{m}_0(\mathcal{O}_1 \cap \mathcal{O}_1 \exp(tu))\, dt}{\int_0^T \mathsf{m}_0(\mathcal{O}_2 \cap \mathcal{O}_2 \exp(tu))\, dt} < \infty,$$

for all bounded M-invariant Borel subsets $\mathcal{O}_i \subset \Gamma\backslash G$ with $\Omega \cap \operatorname{int} \mathcal{O}_i \neq \emptyset$, $i = 1, 2$.

As an immediate corollary of Theorem 3.1, we get

Corollary 4.1. *Let* $\varphi \in \mathfrak{a}^*$. *For any pair* $(\lambda_\varphi,\ \lambda_{\varphi \circ \mathrm{i}})$ *of* (Γ, φ) *and* $(\Gamma, \varphi \circ \mathrm{i})$*-conformal measures on* $\mathcal{F}$ *respectively, the corresponding BMS-measure* $m_{\lambda_\varphi, \lambda_{\varphi\circ \mathrm{i}}}$ *is* u*-balanced for any* $u \in \operatorname{int} \mathcal{L}_\Gamma$.

Proof. Let $\mathcal{O}_1, \mathcal{O}_2$ be M-invariant Borel subsets such that $\Omega \cap \operatorname{int} \mathcal{O}_i \neq \emptyset$ for each $i = 1, 2$. Let $f_1, f_2 \in C_c(\Gamma\backslash G)$ be non-negative functions such that $f_1 \geq 1$ on $\mathcal{O}_1$ and $f_2 \leq 1$ on $\mathcal{O}_2$ and 0 outside $\mathcal{O}_2$. Since $\operatorname{int} \mathcal{O}_2 \cap \Omega \neq \emptyset$, we may choose f_2 so that $m^*_{\nu_{\psi_u}, \lambda_{\varphi\circ \mathrm{i}}}(f_2) > 0$. For simplicity, we set $\mathsf{m}_0 = m_{\lambda_\varphi, \lambda_{\varphi\circ\mathrm{i}}}$. By Theorem 3.1 and using the fact that m_0 is A-quasi-invariant, we obtain that for any $u \in \operatorname{int} \mathcal{L}_\Gamma$,

$$\begin{aligned}
&\limsup_{t \to +\infty} \frac{\mathsf{m}_0(\mathcal{O}_1 \cap \mathcal{O}_1 \exp(tu))}{\mathsf{m}_0(\mathcal{O}_2 \cap \mathcal{O}_2 \exp(tu))} \\
&\leq \limsup_{t \to +\infty} \frac{\int f_1(x) f_1(x \exp(-tu)) d\mathsf{m}_0(x)}{\int f_2(x) f_2(x \exp(-tu)) d\mathsf{m}_0(x)} \\
&= \limsup_{t \to +\infty} \frac{\int f_1(x) f_1(x \exp(tu)) d\mathsf{m}_0(x)}{\int f_2(x) f_2(x \exp(tu)) d\mathsf{m}_0(x)} \\
&= \limsup_{t \to +\infty} \frac{t^{(r-1)/2} e^{(\varphi - \psi_u)(tu)} \int f_1(x) f_1(x \exp(tu)) d\mathsf{m}_0(x)}{t^{(r-1)/2} e^{(\varphi - \psi_u)(tu)} \int f_2(x) f_2(x \exp(tu)) d\mathsf{m}_0(x)} \\
&= \frac{m_{\lambda_\varphi, \nu_{\psi_u}\circ \mathrm{i}}(f_1)}{m^*_{\nu_{\psi_u}, \lambda_{\varphi\circ\mathrm{i}}}(f_2)} < \infty.
\end{aligned}$$

This shows that m_0 is u-balanced. ☐

Recall Theorem 1.4 from the introduction:

Theorem 4.2. *Let* $\operatorname{rank} G \leq 3$. *For any* $\psi \in D_\Gamma^\star$, *any* (Γ,ψ)*-conformal measure on* $\mathcal{F}$ *is necessarily supported on* Λ. *Moreover, the* PS *measure* ν_ψ *is the unique* (Γ,ψ)*-conformal measure on* $\mathcal{F}$.

Proof. Let $u \in \operatorname{int} \mathcal{L}_\Gamma$ denote the unique unit vector such that $\psi(u) = \psi_\Gamma(u)$, that is, $\psi = \psi_u$. Let λ_ψ be any (Γ,ψ)-conformal measure on $\mathcal{F}$. We claim that λ_ψ is supported on Λ. The main ingredient is the higher rank Hopf-Tsuji-Sullivan dichotomy established in [2]. The main point is that all seven conditions of Theorem 1.4 of [2] are equivalent to each other for Anosov groups and $u \in \operatorname{int} \mathcal{L}_\Gamma$, since all the measures considered there are u-balanced by Corollary 4.1. In this proof, we only need the equivalence of (6) and (7), which we now recall.

Consider the following u-directional conical limit set of Γ:

$$\Lambda_u := \{g^+ \in \Lambda : \gamma_i \exp(t_i u) \text{ is bounded for some } t_i \to +\infty \text{ and } \gamma_i \in \Gamma\}. \tag{4.3}$$

Note that $\Lambda_u \subset \Lambda$. For $R > 0$, we set $\Gamma_{u,R} := \{\gamma \in \Gamma : \|\mu(\gamma) - \mathbb{R}u\| < R\}$. Applying the dichotomy [2, Thm. 1.4] to a u-balanced measure $m_{\lambda_\psi,\nu_{\psi\circ \mathrm{i}}}$, we deduce

Proposition 4.4. *The following conditions are equivalent for* λ_ψ*:*

1. $\lambda_\psi(\Lambda_u) = 1$*;*
2. $\sum_{\gamma\in\Gamma_{u,R}} e^{-\psi(\mu(\gamma))} = \infty$ *for some* $R > 0$.

On the other hand, if $\operatorname{rank} G \leq 3$, we have

$$\sum_{\gamma\in\Gamma_{u,R}} e^{-\psi(\mu(\gamma))} = \infty$$

for some $R > 0$ [2, Thm. 6.3]. Therefore, by Proposition 4.4, we have $\lambda_\psi(\Lambda_u) = 1$ and hence λ_ψ is supported on Λ in this case. This finishes the proof of the first part of Theorem 1.4. The second claim follows from the first one by [7, Thm. 1.3]. □

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School of Mathematics, University of Bristol, BS8 1QU, Bristol

School of mathematics, Institute for Advanced Study, Princeton, NJ 08540

Mathematics department, Yale university, New Haven, CT 06520

Email address: samuel.edwards@bristol.ac.uk
Email address: minju@ias.edu
Email address: hee.oh@yale.edu