ORBIT CLOSURES OF UNIPOTENT FLOWS FOR HYPERBOLIC MANIFOLDS WITH FUCHSIAN ENDS

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Abstract. We establish an analogue of Ratner's orbit closure theorem for any connected closed subgroup generated by unipotent elements in $\mathrm{SO}(d,1)$ acting on the space $\Gamma\backslash \mathrm{SO}(d,1)$, assuming that the associated hyperbolic manifold $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ is a convex cocompact manifold with Fuchsian ends. For $d = 3$, this was proved earlier by McMullen, Mohammadi and Oh. In a higher dimensional case, the possibility of accumulation on closed orbits of intermediate subgroups causes very serious obstacles, and surmounting these via the avoidance theorem (Theorem 7.13) is the heart of this paper.

Our results imply the following: for any $k \geq 1$,

(1) the closure of any k-horosphere in $\mathcal{M}$ is a properly immersed submanifold;
(2) the closure of any geodesic $(k+1)$-plane in $\mathcal{M}$ is a properly immersed submanifold;
(3) an infinite sequence of maximal properly immersed geodesic $(k+1)$-planes intersecting core $\mathcal{M}$ becomes dense in $\mathcal{M}$.

Contents

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1. Introduction

Let G be a connected simple linear Lie group and $\Gamma < G$ be a discrete subgroup. An element $g \in G$ is called *unipotent* if all of its eigenvalues are one, and a closed subgroup of G is called unipotent if all of its elements are unipotent. Let U be a connected unipotent subgroup of G, or more generally, any connected closed subgroup of G generated by unipotent elements in it. We are interested in the action of U on the homogeneous space $\Gamma\backslash G$ by right translations.

If the volume of the homogeneous space $\Gamma\backslash G$ is finite, i.e., if Γ is a lattice in G, then Moore's ergodicity theorem says that for almost all $x \in \Gamma\backslash G$, xU is dense in $\Gamma\backslash G$ [31]. While this theorem does not provide any information for a given point x, the celebrated Ratner's orbit closure theorem, which was a conjecture of Raghunathan, states that

$$\text{the closure of every } U\text{-orbit is homogeneous,} \tag{1.1}$$

that is, for any $x \in \Gamma\backslash G$, $\overline{xU} = xL$ for some connected closed subgroup $L < G$ containing U [37]. Ratner's proof is based on her classification of all U-invariant ergodic probability measures [36] and the work of Dani and Margulis [11] on the non-divergence of unipotent flow. Prior to her work, some important special cases of (1.1) were established by Margulis [21], Dani-Margulis ([9], [10]) and Shah ([43], [42]) by topological methods. This theorem is a fundamental result with numerous applications.

It is natural to ask if there exists a family of homogeneous spaces of infinite volume where an analogous orbit closure theorem holds. When the volume of $\Gamma\backslash G$ is infinite, the geometry of the associated locally symmetric space turns out to play an important role in this question. The first orbit closure theorem in the infinite volume case was established by McMullen, Mohammadi, and Oh ([26], [27]) for a class of homogeneous spaces $\Gamma\backslash \operatorname{SO}(3,1)$ which arise as the frame bundles of convex cocompact hyperbolic 3-manifolds with Fuchsian ends.

Our goal in this paper is to show that a similar type of orbit closure theorem holds in the higher dimensional analogues of these manifolds. We present a complete hyperbolic d-manifold $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ as the quotient of the hyperbolic space by the action of a discrete subgroup

$$\Gamma < G = \operatorname{SO}^\circ(d,1) \simeq \operatorname{Isom}^+(\mathbb{H}^d)$$

where $\operatorname{SO}^\circ(d,1)$ denotes the identity component of $\operatorname{SO}(d,1)$. The geometric boundary of $\mathbb{H}^d$ can be identified with the sphere $\mathbb{S}^{d-1}$. The limit set

$\Lambda \subset \mathbb{S}^{d-1}$ of Γ is the set of all accumulation points of an orbit Γx in the compactification $\mathbb{H}^d \cup \mathbb{S}^{d-1}$ for $x \in \mathbb{H}^d$.

The convex core of $\mathcal{M}$ is a submanifold of $\mathcal{M}$ given by the quotient

$$\operatorname{core} \mathcal{M} = \Gamma \backslash \operatorname{hull}(\Lambda)$$

where $\operatorname{hull}(\Lambda) \subset \mathbb{H}^d$ is the smallest convex subset containing all geodesics in $\mathbb{H}^d$ connecting points in Λ. When $\operatorname{core} \mathcal{M}$ is compact, $\mathcal{M}$ is called *convex cocompact*.

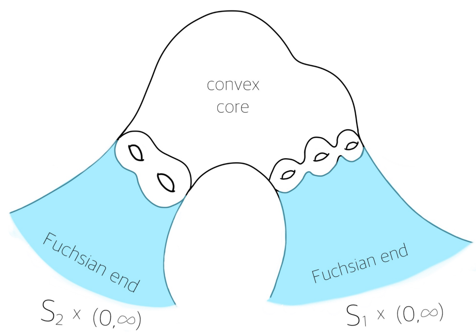

Figure 1. A convex cocompact hyperbolic manifold with non-empty Fuchsian ends

Convex cocompact manifolds with Fuchsian ends. Following the terminology introduced in [15], we define:

Definition 1.1. A convex cocompact hyperbolic d-manifold $\mathcal{M}$ is said to have Fuchsian ends if $\operatorname{core} \mathcal{M}$ has non-empty interior and has totally geodesic boundary.

The term *Fuchsian ends* reflects the fact that each component of the boundary of $\operatorname{core} \mathcal{M}$ is a $(d-1)$-dimensional closed hyperbolic manifold, and each component of the complement $\mathcal{M} - \operatorname{core} \mathcal{M}$ is diffeomorphic to the product $S \times (0, \infty)$ for some closed hyperbolic $(d-1)$-manifold S (see Figure 1).

Convex cocompact hyperbolic d-manifolds with non-empty Fuchsian ends can also be characterized as convex cocompact hyperbolic manifolds whose limit sets satisfy:

$$\mathbb{S}^{d-1} - \Lambda = \bigcup_{i=1}^{\infty} B_i$$

where B_i's are round balls with mutually disjoint closures (see Figure 2). Hence for $d = 2$, any non-elementary convex cocompact hyperbolic surface has Fuchsian ends. The double of the core of a convex cocompact hyperbolic d-manifold with non-empty Fuchsian ends is a closed hyperbolic d-manifold.

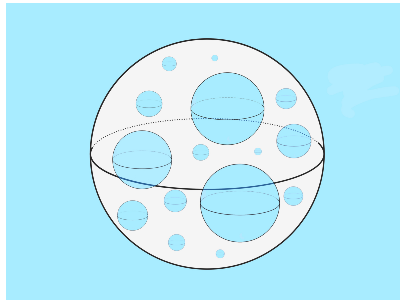

FIGURE 2. Limit set of a convex cocompact hyperbolic 4-manifold with non-empty Fuchsian ends

Any convex cocompact hyperbolic manifold with non-empty Fuchsian ends is constructed in the following way. Begin with a closed hyperbolic d-manifold $\mathcal{N}_0$ with a fixed collection of finitely many, mutually disjoint, properly embedded totally geodesic hypersurfaces. Cut $\mathcal{N}_0$ along those hypersurfaces to obtain a compact hyperbolic manifold W with totally geodesic boundary hypersurfaces. There is a canonical procedure of extending each boundary hypersurface to a Fuchsian end, which results in a convex cocompact hyperbolic manifold $\mathcal{M}$ (with Fuchsian ends) which is diffeomorphic to the interior of W.

By Mostow rigidity theorem, there are only countably infinitely many convex cocompact hyperbolic manifolds with Fuchsian ends of dimension at least 3. On the other hand, for a fixed closed hyperbolic d-manifold $\mathcal{N}_0$ with infinitely many properly immersed geodesic hypersurfaces,[1] one can produce infinitely many non-isometric convex compact hyperbolic d-manifolds with non-empty Fuchsian ends; for each properly immersed geodesic hypersurface $f_i(\mathbb{H}^{d-1})$ for a totally geodesic immersion $f_i : \mathbb{H}^{d-1} \to \mathcal{N}_0$, there is a finite covering $\mathcal{N}_i$ of $\mathcal{N}_0$ such that f_i lifts to $\mathbb{H}^{d-1} \to \mathcal{N}_i$ with image S_i being properly imbedded in $\mathcal{N}_i$ [17]. Cutting and pasting $\mathcal{N}_i$ along S_i as described above produces a hyperbolic manifold $\mathcal{M}_i$ with Fuchsian ends. When the volumes of S_i are distinct, $\mathcal{M}_i$'s are not isometric to each other.

Orbit closures. In the rest of the introduction, we assume that for $d \geq 2$,

$\mathcal{M}$ is a convex cocompact hyperbolic d-manifold with Fuchsian ends.

The homogeneous space $\Gamma\backslash G$ can be regarded as the bundle $\mathrm{F}\,\mathcal{M}$ of oriented frames over $\mathcal{M}$. Let $A = \{a_t : t \in \mathbb{R}\} < G$ denote the one parameter subgroup of diagonalizable elements whose right translation actions on $\Gamma\backslash G$

[1]Any closed arithmetic hyperbolic manifold has infinitely many properly immersed geodesic hypersurfaces provided it has at least one. This is due to the presence of Hecke operators [39].

correspond to the frame flow. Let $N \simeq \mathbb{R}^{d-1}$ denote the contracting horospherical subgroup:

$$N = \{g \in G : a_{-t} g a_t \to e \text{ as } t \to +\infty\}.$$

We denote by $\mathrm{RF}\,\mathcal{M}$ the renormalized frame bundle of $\mathcal{M}$:

$$\mathrm{RF}\,\mathcal{M} := \{x \in \Gamma\backslash G : \ xA \text{ is bounded}\},$$

and also set

$$\mathrm{RF}_+\,\mathcal{M} := \{x \in \Gamma\backslash G : xA^+ \text{ is bounded}\}$$

where $A^+ = \{a_t : t \geq 0\}$. When $\mathrm{Vol}(M) < \infty$, we have

$$\mathrm{RF}\,\mathcal{M} = \mathrm{RF}_+\,\mathcal{M} = \Gamma\backslash G.$$

In general, $\mathrm{RF}\,\mathcal{M}$ projects into $\mathrm{core}\,\mathcal{M}$ (but not surjective in general) and $\mathrm{RF}_+\,\mathcal{M}$ projects onto $\mathcal{M}$ under the basepoint projection $\Gamma\backslash G \to \mathcal{M}$. The sets $\mathrm{RF}\,\mathcal{M}$ and $\mathrm{RF}_+\,\mathcal{M}$ are precisely non-wandering sets for the actions of A and N respectively [48].

For a connected closed subgroup $U < N$, we denote by $H(U)$ the smallest closed simple Lie subgroup of G which contains both U and A. If $U \simeq \mathbb{R}^k$, then $H(U) \simeq \mathrm{SO}^\circ(k+1,1)$. A connected closed subgroup of G generated by one-parameter unipotent subgroups is, up to conjugation, of the form $U < N$ or $H(U)$ for some $U < N$ (Corollary 3.8).

We set $F_{H(U)} := \mathrm{RF}_+\,\mathcal{M} \cdot H(U)$, which is a closed subset. It is easy to see that if $x \notin \mathrm{RF}_+\,\mathcal{M}$ (resp. $x \notin F_{H(U)}$), then xU (resp. $xH(U)$) is closed in $\Gamma\backslash G$. On the other hand, for almost all $x \in \mathrm{RF}_+\,\mathcal{M}$, xU is dense in $\mathrm{RF}_+\,\mathcal{M}$, with respect to a unique N-invariant locally finite measureon $\mathrm{RF}_+\,\mathcal{M}$, called the Burger-Roblin measure; this was shown by Mohammadi-Oh [30] for $d = 3$ and by Maucourant-Schapira for general $d \geq 3$ [29] (see section 12).

Orbit closures are relatively homogeneous. We define the following collection of closed connected subgroups of G:

$$\mathcal{L}_U := \left\{ L = H(\widehat{U})C : \begin{array}{c} \text{for some } z \in \mathrm{RF}_+\,\mathcal{M},\ zL \text{ is closed in } \Gamma\backslash G \\ \text{and } \mathrm{Stab}_L(z) \text{ is Zariski dense in } L \end{array} \right\}.$$

where $U < \widehat{U} < N$ and C is a closed subgroup of the centralizer of $H(\widehat{U})$. We also define:

$$\mathcal{Q}_U := \{vLv^{-1} : L \in \mathcal{L}_U \text{ and } v \in N\}.$$

In view of the previous discussion, the following theorem gives a classification of orbit closures for all connected closed subgroups of G generated by unipotent one-parameter subgroups:

Theorem 1.2. *Let $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends, and let $U < N$ be a non-trivial connected closed subgroup.*

(1) ($H(U)$-orbit closures) *For any* $x \in \mathrm{RF}\,\mathcal{M} \cdot H(U)$,

$$\overline{xH(U)} = xL \cap F_{H(U)}$$

where xL *is a closed orbit of some* $L \in \mathcal{L}_U$.

(2) (U-orbit closures) *For any* $x \in \mathrm{RF}_+\,\mathcal{M}$,

$$\overline{xU} = xL \cap \mathrm{RF}_+\,\mathcal{M}$$

where xL *is a closed orbit of some* $L \in \mathcal{Q}_U$.

(3) (Equidistributions) *Let* x_iL_i *be a sequence of closed orbits intersecting* $\mathrm{RF}\,\mathcal{M}$, *where* $x_i \in \mathrm{RF}_+\,\mathcal{M}$ *and* $L_i \in \mathcal{Q}_U$. *Assume that no infinite subsequence of* x_iL_i *is contained in a subset of the form* y_0L_0D *where* y_0L_0 *is a closed orbit of* $L_0 \in \mathcal{L}_U$ *with* $\dim L_0 < \dim G$ *and* D *is a compact subset of the normalizer* $\mathrm{N}(U)$ *of* U. *Then*[2]

$$\lim_{i\to\infty} x_iL_i \cap \mathrm{RF}_+\,\mathcal{M} = \mathrm{RF}_+\,\mathcal{M}.$$

Remark 1.3. (1) If $x \in F_{H(U)} - \mathrm{RF}\,\mathcal{M} \cdot H(U)$, then $xH(U)$ is contained in an end component of $\mathcal{M}$ under the projection $\Gamma\backslash G \to \mathcal{M}$, and its closure is not relatively homogeneous in $F_{H(U)}$. More precisely,

$$\overline{xH(U)} = xLV^+H(U)$$

for some $L \in \mathcal{L}_U$, and some one-parameter semigroup $V^+ < N$ (see Theorem 11.5).

(2) If $\mathcal{M}$ has empty ends, i.e., if $\mathcal{M}$ is compact, Theorem 1.2(1) and (2) are special cases of Ratner's theorem [37], also proved by Shah [44] independently, and Theorem 1.2(3) follows from Mozes-Shah equidistribution theorem [32].

Theorem 1.2(1) and (2) can also be presented as follows in a unified manner:

Corollary 1.4. *Let* $H < G$ *be a connected closed subgroup generated by unipotent elements in it. Assume that* H *is normalized by* A. *For any* $x \in \mathrm{RF}\,\mathcal{M}$, *the closure of* xH *is homogeneous in* $\mathrm{RF}\,\mathcal{M}$, *that is,*

$$\overline{xH} \cap \mathrm{RF}\,\mathcal{M} = xL \cap \mathrm{RF}\,\mathcal{M} \tag{1.2}$$

where xL *is a closed orbit of some* $L \in \mathcal{Q}_{H\cap N}$.

Remark 1.5. If Γ is contained in $G(\mathbb{Q})$ for some $\mathbb{Q}$-structure of G, and $[g]L$ is a closed orbit appearing in Corollary 1.4, then L is defined by the condition that gLg^{-1} is the smallest connected $\mathbb{Q}$-subgroup of G containing gHg^{-1}.

[2]For a sequence of subsets Y_n in a topological space X such that $Y = \limsup_n Y_n = \liminf Y_n$, we write $Y = \lim_{n\to\infty} Y_n$, where $\limsup_n Y_n = \bigcup_n \overline{\bigcap_{m\geq n} Y_m}$ and $\liminf_n Y_n = \bigcap_n \overline{\bigcup_{m\geq n} Y_m}$.

Generic points. Denote by $\mathscr{G}(U)$ the set of all points $x \in \mathrm{RF}_+ \mathcal{M}$ such that x is not contained in any closed orbit of a proper reductive algebraic subgroup of G containing U. Theorem 1.2(2) implies that for any $x \in \mathscr{G}(U)$,

$$\overline{xU} = \mathrm{RF}_+ \mathcal{M}.$$

Geodesic planes, horospheres and spheres. We state implications of our main theorems on the closures of geodesic planes and horospheres of the manifold $\mathcal{M}$, as well as on the Γ-orbit closures of spheres in $\mathbb{S}^{d-1}$.

A geodesic k-plane P in $\mathcal{M}$ is the image of a totally geodesic immersion $f : \mathbb{H}^k \to \mathcal{M}$, or equivalently, the image of a geodesic k-subspace of $\mathbb{H}^d$ under the covering map $\mathbb{H}^d \to \mathcal{M}$. If f factors through the covering map $\mathbb{H}^k \to \Gamma_0 \backslash \mathbb{H}^k$ for a convex cocompact hyperbolic k-manifold with Fuchsian ends, we call $P = f(\mathbb{H}^k)$ a convex cocompact geodesic k-plane with Fuchsian ends.

Theorem 1.6. *Let $\mathcal{M} = \Gamma \backslash \mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends, and let P be a geodesic k-plane of $\mathcal{M}$ for some $k \geq 2$.*

(1) *If P intersects $\operatorname{core} \mathcal{M}$, then $\overline{P}$ is a properly immersed convex cocompact geodesic m-plane with Fuchsian ends for some $m \geq k$.*
(2) *Otherwise, P is contained in some Fuchsian end $E = S_0 \times (0, \infty)$ of $\mathcal{M}$, and either P is properly immersed or $\overline{P}$ is diffeomorphic to the product $S \times [0, \infty)$ for a closed geodesic m-plane S of S_0 for some $k \leq m \leq d - 1$.*

In particular, the closure of a geodesic plane of dimension at least 2 is a properly immersed submanifold of $\mathcal{M}$ (possibly with boundary).

We also obtain:

Theorem 1.7. (1) *Any infinite sequence of maximal properly immersed geodesic planes P_i of $\dim P_i \geq 2$ intersecting $\operatorname{core} \mathcal{M}$ becomes dense in $\mathcal{M}$, i.e.,*

$$\lim_{i \to \infty} P_i = \mathcal{M}$$

where the limit is taken in the Hausdorff topology on the space of all closed subsets in $\mathcal{M}$.
(2) *There are only countably many properly immersed geodesic planes of dimension at least 2 intersecting $\operatorname{core} \mathcal{M}$.*
(3) *If $\operatorname{Vol}(\mathcal{M}) = \infty$, there are only finitely many maximal properly immersed bounded geodesic planes of dimension at least 2.*

In fact, Theorem 1.7(3) holds for any convex cocompact hyperbolic d-manifold (see Remark 18.3).

A k-horosphere in $\mathbb{H}^d$ is a Euclidean sphere of dimension k which is tangent to a point in $\mathbb{S}^{d-1}$. A k-horosphere in $\mathcal{M}$ is simply the image of a k-horosphere in $\mathbb{H}^d$ under the covering map $\mathbb{H}^d \to \mathcal{M} = \Gamma \backslash \mathbb{H}^d$.

Theorem 1.8. *Let χ be a k-horosphere of $\mathcal{M}$ for $k \geq 1$. Then either*

(1) χ *is properly immersed; or*
(2) $\overline{\chi}$ *is a properly immersed* m*-dimensional submanifold, parallel to a convex cocompact geodesic* m*-plane of* $\mathcal{M}$ *with Fuchsian ends for some* $m \geq k+1$.

By abuse of notation, let π denote both base point projection maps $G \to \mathbb{H}^d$ and $\Gamma\backslash G \to \mathcal{M}$ where we consider an element $g \in G$ as an oriented frame over $\mathbb{H}^d$. Let $H' = \mathrm{SO}^\circ(k+1,1)\,\mathrm{SO}(d-k-1)$, $1 \leq k \leq d-2$. The quotient space G/H' parametrizes all oriented k-spheres in $\mathbb{S}^{d-1}$, which we denote by $\mathcal{C}^k$. For each H'-orbit $gH' \subset G$, the image $\pi(gH') \subset \mathbb{H}^d$ is an oriented geodesic $(k+1)$-plane and the boundary $\partial(\pi(gH')) \subset \mathbb{S}^{d-1}$ is an oriented k-sphere. Passing to the quotient space $\Gamma\backslash G$, this gives bijections among:

(1) the space of all closed H'-orbits $xH' \subset \Gamma\backslash G$ for $x \in \mathrm{RF}\,\mathcal{M}$;
(2) the space of all oriented properly immersed geodesic $(k+1)$-planes P in $\mathcal{M}$ intersecting $\mathrm{core}\,\mathcal{M}$;
(3) the space of all closed Γ-orbits of oriented k-spheres $C \in \mathcal{C}^k$ with $\#C \cap \Lambda \geq 2$

If $U := H' \cap N$, then any k-horosphere in $\mathcal{M}$ is given by $\pi(xU)$ for some $x \in \Gamma\backslash G$.

In view of these correspondences, Theorems 1.6, 1.7 and 1.8 follow from Theorem 1.2, Theorem 11.5, and Corollary 5.8.

We also obtain the following description on Γ-orbits of spheres of any positive dimension.

Corollary 1.9. *Let* $1 \leq k \leq d-2$.

(1) *Let* $C \in \mathcal{C}^k$ *with* $\#C \cap \Lambda \geq 2$. *Then there exists a sphere* $S \in \mathcal{C}^m$ *such that* ΓS *is closed in* $\mathcal{C}^m$ *and*
$$\overline{\Gamma C} = \{D \in \mathcal{C}^k : D \cap \Lambda \neq \emptyset,\ D \subset \Gamma S\}.$$
(2) *Let* $C_i \in \mathcal{C}^k$ *be an infinite sequence of spheres with* $\#C_i \cap \Lambda \geq 2$ *such that* ΓC_i *is closed in* $\mathcal{C}^k$. *Assume that* ΓC_i *is maximal in the sense that there is no proper sphere* $S \subset \mathbb{S}^{d-1}$ *which properly contains* C_i *and that* ΓS *is closed. Then as* $i \to \infty$,
$$\lim_{i\to\infty} \Gamma C_i = \{D \in \mathcal{C}^k : D \cap \Lambda \neq \emptyset\}$$
where the limit is taken in the Hausdorff topology on the space of all closed subsets in $\mathcal{C}^k$.
(3) *If* $\Lambda \neq \mathbb{S}^{d-1}$, *there are only finitely many maximal closed* Γ*-orbits of spheres of positive dimension contained in* Λ.

Remark 1.10.
(1) The main results of this paper for $d = 3$ were proved by McMullen, Mohammadi, and Oh ([26], [27]). We refer to [26] for counterexamples to Theorem 1.2 for a certain family of quasi-Fuchsian 3-manifolds.
(2) A convex cocompact hyperbolic 3-manifold with Fuchsian ends (which was referred to as *a rigid acylindrical hyperbolic* 3*-manifold* in [26])

has a huge deformation space parametrized by the product of the Teichmuller spaces of the boundary components of core $\mathcal{M}$ (cf. [18]). Any convex cocompact acylindrical hyperbolic 3-manifold is a quasi-conformal conjugation of a rigid acylindrical hyperbolic 3-manifold [25]. An analogue of Theorem 1.2(1) was obtained for all convex cocompact acylindrical hyperbolic 3-manifolds in [28] and for all geometrically finite acylindrical hyperbolic 3-manifolds in [4].

(3) For $d \geq 4$, Kerckhoff and Storm showed that a convex cocompact hyperbolic manifold $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ with non-empty Fuchsian ends does not allow any non-trivial deformation, in the sense that the representation of Γ into G is infinitesimally rigid [15].

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2. Outline of the proof

We will explain the strategy of our proof of Theorem 1.2 with an emphasis on the difference between finite and infinite volume case and the difference between dimension 3 and higher case.

Thick recurrence of unipotent flows. Let $U_0 = \{u_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of N. The main obstacle of carrying out unipotent dynamics in a homogeneous space of *infinite* volume is the scarcity of recurrence of unipotent flow. In a compact homogeneous space, every U_0-orbit stays in a compact set for the obvious reason. Already in a *noncompact* homogeneous space of finite volume, understanding the recurrence of U_0-orbit is a non-trivial issue. Margulis showed that any U_0-orbit is recurrent to a compact subset [19], and Dani-Margulis showed that for any $x \in \Gamma\backslash G$, and for any $\varepsilon > 0$, there exists a compact subset $\Omega \subset \Gamma\backslash G$ such that

$$\ell\{t \in [0, T] : xu_t \in \Omega\} \geq (1 - \varepsilon)T$$

for all large $T \gg 1$, where ℓ denotes the Lebesgue measure on $\mathbb{R}$ [11]. This non-divergence of unipotent flows is an important ingredient of Ratner's orbit closure theorem [37].

In contrast, when $\Gamma\backslash G$ has infinite volume, for any compact subset $\Omega \subset \Gamma\backslash G$, and for almost all x (with respect to any Borel measure μ on $\mathbb{R}$),

$$\mu\{t \in [0, T] : xu_t \in \Omega\} = o(T)$$

for all $T \gg 1$ [1].

Nonetheless, the pivotal reason that we can work with convex cocompact hyperbolic manifolds of non-empty Fuchsian ends is the following *thick* recurrence property that they possess: there exists $k > 1$, depending only on the systole of the double of core $\mathcal{M}$, such that for any $x \in \mathrm{RF}\,\mathcal{M}$, the return time

$$\mathsf{T}(x) := \{t \in \mathbb{R} : xu_t \in \mathrm{RF}\,\mathcal{M}\}$$

is k-thick, in the sense that for any $\lambda > 0$,

$$\mathsf{T}(x) \cap ([-k\lambda, -\lambda] \cup [\lambda, k\lambda]) \neq \emptyset. \tag{2.1}$$

This recurrence property was first observed by McMullen, Mohammadi and Oh [26] in the case of dimension 3 in order to get an additional invariance of a relative U_0-minimal subset with respect to $\mathrm{RF}\,\mathcal{M}$ by studying the polynomial divergence property of U_0-orbits of two nearby $\mathrm{RF}\,\mathcal{M}$-points.

Beyond $d = 3$. In a higher dimensional case, the possible presence of closed orbits of intermediate subgroups introduces a variety of serious hurdles. Roughly speaking, calling the collection of all such closed orbits as the singular set and its complement as the generic set, one of the main new ingredients of this paper is the *avoidance of the singular set* along the k-thick recurrence of U_0-orbits to $\mathrm{RF}\,\mathcal{M}$ for a sequence of $\mathrm{RF}\,\mathcal{M}$-points limiting at a generic point. Its analogue in the finite volume case was proved by Dani-Margulis [12] and also independently by Shah [41] based on the *linearization methods.*

Road map for induction. Roughly speaking,[3] Theorem 1.2 is proved by induction on the co-dimension of U in N. For each $i = 1, 2, 3$, let us say that $(i)_m$ holds, if Theorem 1.2(i) is true for all U satisfying $\text{co-dim}_N(U) \leq m$. We show that the validity of $(2)_m$ and $(3)_m$ implies that of $(1)_{m+1}$, the validity of $(1)_{m+1}$, $(2)_m$, and $(3)_m$ implies that of $(2)_{m+1}$ and the validity of $(1)_{m+1}$, $(2)_{m+1}$, and $(3)_m$ implies that of $(3)_{m+1}$. In order to give an outline of the proof of $(1)_{m+1}$, we suppose that $\text{co-dim}_N(U) \leq m + 1$. Let

$$F := \mathrm{RF}_+\,\mathcal{M} \cdot H(U), \quad F^* := \mathrm{Interior}(F), \text{ and } \partial F := F - F^*.$$

Let $x \in F^* \cap \mathrm{RF}\,\mathcal{M}$, and consider

$$X := \overline{xH(U)} \subset F.$$

The strategy in proving $(1)_{m+1}$ for X consists of two steps:

(1) (Find) Find a closed L-orbit $x_0 L$ *with* $x_0 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ such that $x_0 L \cap F$ contained in X for some $L \in \mathcal{L}_U$;
(2) (Enlarge) If $X \not\subset x_0 L\, \mathrm{C}(H(U))$,[4] then enlarge $x_0 L$ to a bigger closed orbit $x_1 \widehat{L}$ so that $x_1 \widehat{L} \cap F \subset X$ where $x_1 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and $\widehat{L} \in \mathcal{L}_{\widehat{U}}$ for some $\widehat{U} < N$ containing $L \cap N$ properly.

[3]To be precise, we need to carry out induction on the co-dimension of U in $\widehat{L} \cap N$ whenever xU is contained in a closed orbit $x_0 \widehat{L}$ for some $\widehat{L} \in \mathcal{L}_U$ as formulated in Theorem 14.1.

[4]The notation $\mathrm{C}(S)$ denotes the identity component of the centralizer of S

The enlargement process must end after finitely many steps because of dimension reason. Finding a closed orbit as in (1) is based on the study of the relative U-minimal sets and the unipotent blow up argument using the polynomial divergence of U-orbits of nearby $\mathrm{RF}\,\mathcal{M}$-points. To explain the enlargement step, suppose that we are given an intermediate closed L-orbit with $x_0L\cap F\subset X$ by the step (1), and a one-parameter subgroup $U_0=\{u_t\}$ of U such that x_0U_0 is dense in $x_0L\cap \mathrm{RF}_+\,\mathcal{M}$. As L is reductive, the Lie algebra of G can be decomposed into the $\mathrm{Ad}(L)$-invariant subspaces $\mathfrak{l}\oplus\mathfrak{l}^\perp$ where $\mathfrak{l}$ denotes the Lie algebra of L. Suppose that we could arrange a sequence $x_0g_i\to x_0$ in X for some $g_i\to e$ such that writing $g_i=\ell_ir_i$ with $\ell_i\in L$ and $r_i\in\exp(\mathfrak{l}^\perp)$, the following conditions are satisfied:

- $x_0\ell_i\in \mathrm{RF}\,\mathcal{M}$;
- $r_i\notin \mathrm{N}(U_0)$.

Then the k-thick return property of $x_0\ell_i\in\mathrm{RF}\,\mathcal{M}$ along U_0 yields a sequence $u_{t_i}\in U_0$ such that

$$x_0\ell_iu_{t_i}\to x_1\in \mathrm{RF}\,\mathcal{M}\cap x_0L \quad\text{and}\quad u_{t_i}^{-1}r_iu_{t_i}\to v$$

for some element $v\in N-L$. This gives us a point

$$x_1v\in X.$$

(2.2) *If we could guarantee that x_1 is a generic point for U in x_0L,*

then $\overline{x_1U}$ must be equal to $x_0L\cap\mathrm{RF}_+\,\mathcal{M}$ by induction hypothesis $(2)_m$, since the co-dimension of U inside $L\cap N$ is at most m. Then

$$\overline{x_1vU}=\overline{x_1U}v=x_0Lv\cap\mathrm{RF}_+\,\mathcal{M}\subset X.$$

Using the A-invariance of X and the fact that the double coset AvA contains a one-parameter unipotent subsemigroup V^+, we can put $x_0LV^+\cap F$ inside X.

(2.3) *Assuming that $x_0\in F^*\cap\mathrm{RF}_+\,\mathcal{M}$,*

we can promote V^+ to a one-parameter subgroup V, and find an orbit of a bigger unipotent subgroup $\widehat{U}:=(L\cap N)V$ inside X. This enables us to use the induction hypothesis $(2)_m$ to complete the enlargement step. Note that if x_1 is not generic for U in x_0L, the closure of x_1U may be stuck in a smaller closed orbit inside x_0L, in which case $\overline{x_1U}v$ may not be bigger than x_0L in terms of the dimension, resulting in no progress.

We now explain how we establish (2.2).[5]

[5] For the dimension $d=3$, L is either the entire $\mathrm{SO}^\circ(3,1)$ in which case we are done, or $L=H(U)=\mathrm{SO}^\circ(2,1)$. In the latter case, (2.2) is automatic as U is a horocyclic subgroup of L.

Avoidance of the singular set along the thick return time. Let $U_0 = \{u_t\}$ be a one parameter subgroup of U. We denote by $\mathscr{S}(U_0)$ the union of all closed orbits xL where $x \in \mathrm{RF}_+ \mathcal{M}$ and $L \in \mathcal{Q}_{U_0}$ is a *proper* subgroup of G. This set is called the *singular set* for U_0. Its complement in $\mathrm{RF}_+ \mathcal{M}$ is denoted by $\mathscr{G}(U_0)$, and called the set of *generic* elements of U_0. We have

$$\mathscr{S}(U_0) = \bigcup_{H \in \mathscr{H}} \Gamma\backslash\Gamma X(H, U_0)$$

where $\mathscr{H}$ is the countable collection of all proper connected closed subgroups H of G containing a unipotent element such that $\Gamma\backslash\Gamma H$ is closed and $H \cap \Gamma$ is Zariski dense in H, and $X(H, U_0) := \{g \in G : gU_0 g^{-1} \subset H\}$ (Proposition 5.10). We define $\mathcal{E} = \mathcal{E}_{U_0}$ to be the collection of all subsets of $\mathscr{S}(U_0)$ which are of the form

$$\bigcup \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\,\mathcal{M}$$

where $H_i \in \mathscr{H}$ is a finite collection, and D_i is a compact subset of $X(H_i, U_0)$. The following avoidance theorem is one of the main ingredients of our proof: let k be given by (2.1) for $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$:

Theorem 2.1 (Avoidance theorem)**.** *There exists a sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}$ with*

$$\mathscr{S}(U_0) \cap \mathrm{RF}\,\mathcal{M} = \bigcup_{j=1}^{\infty} E_j$$

satisfying the following: for each $j \in \mathbb{N}$ and for any compact subset $F \subset \mathrm{RF}\,\mathcal{M} - E_{j+1}$, there exists an open neighborhood $\mathcal{O}_j = \mathcal{O}_j(F)$ of E_j such that for any $x \in F$, the following set

$$\{t \in \mathbb{R} : xu_t \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}_j\} \tag{2.4}$$

is $2k$-thick.

It is crucial that the thickness size of the set (2.4), which is given by $2k$ here, can be controlled independently of the compact subsets E_j for applications in the orbit closure theorem. If E_j does not intersect any closed orbit of a proper subgroup of G, then obtaining E_{j+1} and $\mathcal{O}_j$ is much simpler. In general, E_j may intersect infinitely many intermediate closed orbits, and our proof is based on a careful analysis on the graded intersections of those closed orbits and a combinatorial argument, which we call an *inductive search argument.* This process is quite delicate, compared to the finite volume case treated in ([12], [41]) in which case the set $\{t : xu_t \in \mathrm{RF}\,\mathcal{M}\}$, being equal to $\mathbb{R}$, possesses the Lebesgue measure which can be used to measure the time outside of a neighborhood of E_j's.

We deduce the following from Theorem 2.1:

Theorem 2.2 (Accumulation on a generic point)**.** *Suppose that $(2)_m$ and $(3)_m$ hold in Theorem 1.2. Then the following holds for any connected closed subgroup $U < N$ with* $\text{co-dim}_N(U) = m+1$*: Let $U_0 = \{u_t : t \in \mathbb{R}\}$ be a* *one-parameter subgroup of U, and let $x_i \in \mathrm{RF}\,\mathcal{M}$ be a sequence converging to $x_0 \in \mathscr{G}(U_0)$ as $i \to \infty$. Then for any given sequence $T_i \to \infty$,*

$$\limsup_{i\to\infty}\{x_i u_{t_i} \in \mathrm{RF}\,\mathcal{M} : T_i \le |t_i| \le 2kT_i\} \tag{2.5}$$

contains a sequence $\{y_j : j = 1, 2, \cdots\}$ such that $\limsup_{j\to\infty} y_j U$ contains a point in $\mathscr{G}(U_0)$.[6]

Again, it is important that $2k$ is independent of x_i here. We prove two independent but related versions of Theorem 2.2 in section 15 depending on the relative location of x_i for the set $\mathrm{RF}\,\mathcal{M}$; we use Proposition 15.1 for the proof of $(1)_{m+1}$ and Proposition 15.2 for the proofs of $(2)_{m+1}$ and $(3)_{m+1}$.

Comparison with the finite volume case. If $\Gamma\backslash G$ is compact, the approach of Dani-Margulis [12] shows that if x_i converges to $x \in \mathscr{G}(U_0)$, then for any $\varepsilon > 0$, we can find a sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}$, and neighborhoods $\mathcal{O}_j$ of E_j such that $\mathscr{S}(U_0) = \bigcup E_j$, $x_i \notin \bigcup_{j\le i+1} \mathcal{O}_j$ and for all $i \ge j$ and $T > 0$,

$$\ell\{t \in [0, T] : x_i u_t \in \mathcal{O}_j\} \le \tfrac{\varepsilon}{2^i} T.$$

This implies that for all $i > 1$,

$$\ell\{t \in [0, T] : x_i u_t \in \bigcup_{j\le i} \mathcal{O}_j\} \le \varepsilon T. \tag{2.6}$$

In particular, the limsup set in (2.5) always contains an element of $\mathscr{G}(U_0)$, without using induction hypothesis. This is the reason why $(3)_m$ is not needed in obtaining $(1)_{m+1}$ and $(2)_{m+1}$ in Theorem 19.1 for the finite volume case.[7]

In comparison, we are able to get a generic point in Theorem 2.2 only with the help of the induction hypothesis $(2)_m$ and $(3)_m$ and after taking the limsup of the U-orbits of all accumulating points from the $2k$-thick sets obtained in Theorem 2.1.

Generic points in F^* as limits of $\mathrm{RF}\,\mathcal{M}$-points. In the inductive argument, it is important to find a closed orbit $x_0 L$ based at a point $x_0 \in F^*$ in order to promote a semi-group V^+ to a group V as described following (2.3). Another reason why this is critical is the following: implementing Theorem 2.2 (more precisely, its versions Theorems 15.1 and 15.2) requires having a sequence of $\mathrm{RF}\,\mathcal{M}$*-points of X* accumulating on a generic point of $x_0 L$ with respect to U_0.

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[6] Here we allow a constant sequence $y_j = y$ in which case $\limsup_{j\to\infty} y_j U$ is understood as $\overline{yU}$ and hence $y \in \mathscr{G}(U_0)$.

[7] We give a summary of our proof for the case when $\Gamma\backslash G$ is compact and has at least one $\mathrm{SO}^\circ(d-1,1)$ closed orbit in the appendix to help readers understand the whole scheme of the proof.

The advantage of having a closed orbit x_0L with $x_0 \in F^* \cap \mathrm{RF}_+ \mathcal{M} \cap \mathscr{G}(U_0)$ is that x_0 can be approximated by a sequence of $\mathrm{RF}\,\mathcal{M}$-points in $F^* \cap X$ (Lemmas 8.3 and 8.7).

We also point out that we use the ergodicity theorem obtained in [30] and [29] to guarantee that there are many U_0-generic points in any closed orbit x_0L as above.

Existence of a compact orbit in any noncompact closed orbit. In our setting, $\Gamma\backslash G$ always contains a closed orbit xL for some $x \in \mathrm{RF}\,\mathcal{M}$ and a proper subgroup $L \in \mathcal{L}_U$; namely those compact orbits of $\mathrm{SO}^\circ(d-1,1)$ over the boundary of core $\mathcal{M}$. Moreover, if $x_0\widehat{L}$ is a noncompact closed orbit for some $x_0 \in \mathrm{RF}\,\mathcal{M}$ and $\dim(\widehat{L} \cap N) \geq 2$, then $x_0\widehat{L}$ contains a compact orbit xL of some $L \in \mathcal{L}_U$ (Proposition 5.16). This fact was crucially used in deducing $(2)_{m+1}$ from $(1)_{m+1}, (2)_m$ and $(3)_m$ in Theorem 1.2 (more precisely, in Theorem 14.1).

Organization of the paper.

- In section 3, we set up notations for certain Lie subgroups of G, review some basic facts and gather preliminaries about them and geodesic planes of $\mathcal{M}$.
- In section 4, for each unipotent subgroup U of G, we define the minimal $H(U)$-invariant closed subset $F_{H(U)} \subset \Gamma\backslash G$ containing $\mathrm{RF}_+ \mathcal{M}$ and study its properties for a convex cocompact hyperbolic manifold of non-empty Fuchsian ends.
- In section 5, we define the singular set $\mathscr{S}(U, x_0L)$ for a closed orbit $x_0L \subset \Gamma\backslash G$, and prove a structure theorem and a countability theorem for a general convex cocompact manifold.
- In section 6, we prove Proposition 6.3, based on a combinatorial lemma 6.4, called an *inductive search lemma.* This proposition is used in the proof of Theorem 7.13 (Avoidance theorem).
- In section 7, we construct families of triples of intervals which satisfy the hypothesis of Proposition 6.3, by making a careful analysis of the graded intersections of the singular set and the linearization, and prove Theorem 7.13 from which Theorem 2.1 is deduced.
- In section 8, we prove several geometric lemmas which are needed to modify a sequence limiting on a generic point to a sequence of $\mathrm{RF}\,\mathcal{M}$-points which still converges to a generic point.
- In section 9, we study the unipotent blowup lemmas using quasi-regular maps and properties of thick subsets.
- In section 10, we study the translates of relative U-minimal sets Y into the orbit closure of an $\mathrm{RF}\,\mathcal{M}$ point; the results in this section are used in the step of finding a closed orbit in a given $H(U)$-orbit closure.
- In section 11, we describe closures of orbits contained in the boundary of $F_{H(U)}$.

- In section 12, we review the ergodicity theorem of [30] and [29] and deduce the density of almost all orbits of a connected unipotent subgroup in $\mathrm{RF}_+\mathcal{M}$.
- In section 13, the minimality of a horospherical subgroup action is obtained in the presence of compact factors.
- In section 14, we begin to prove Theorem 1.2; the base case $m = 0$ is addressed and the orbit closure of a singular U-orbit is classified under the induction hypothesis.
- In section 15 we prove two propositions on how to get an additional invariance from Theorem 7.13; the results in this section are used in the step of enlarging a closed orbit to a larger one inside a given U-invariant orbit closure in the proof of Theorem 1.2.
- We prove $(1)_{m+1}$, $(2)_{m+1}$ and $(3)_{m+1}$ respectively in sections 16, 17 and 18.
- In the appendix, we give an outline of our proof in the case when $\Gamma\backslash G$ is compact with at least one $\mathrm{SO}^\circ(d-1,1)$-closed orbit.

3. Lie subgroups and geodesic planes

Let G denote the connected simple Lie group $\mathrm{SO}^\circ(d,1)$ for $d \geq 2$. In this section, we fix notation and recall some background about Lie subgroups of G and geodesic planes of a hyperbolic d-manifold.

As a Lie group, we have $G \simeq \mathrm{Isom}^+(\mathbb{H}^d)$. In order to present a family of subgroups of G explicitly, we fix a quadratic form $Q(x_1, \cdots, x_{d+1}) = 2x_1x_{d+1} + x_2^2 + x_3^2 + \cdots + x_d^2$, and identify $G = \mathrm{SO}^\circ(Q)$. The Lie algebra of G is then given as:

$$\mathfrak{so}(d,1) = \{X \in \mathfrak{sl}_{d+1}(\mathbb{R}) : X^tQ + QX = 0\}$$

where

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{Id}_{d-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

A subset $S \subset G$ is said to be Zariski closed if S is defined as the zero set $\{(x_{ij}) \in G : p_1(x_{ij}) = \cdots = p_\ell(x_{ij}) = 0\}$ for a finite collection of polynomials with real coefficients in variables $(x_{ij}) \in \mathrm{M}_{d+1}(\mathbb{R})$. The Zariski closure of a subset $S \subset G$ means the smallest Zariski closed subset of G containing S. A connected sugroup $L < G$ is algebraic if L is equal to the identity component of its Zariski closure.

Subgroups of G. Inside G, we have the following subgroups:

$$\begin{aligned}
K &= \{g \in G : g^t g = \mathrm{Id}_{d+1}\} \simeq \mathrm{SO}(d),\\
A &= \left\{ a_s = \begin{pmatrix} e^s & 0 & 0 \\ 0 & \mathrm{Id}_{d-1} & 0 \\ 0 & 0 & e^{-s} \end{pmatrix} : s \in \mathbb{R} \right\},\\
M &= \text{ the centralizer of } A \text{ in } K \simeq \mathrm{SO}(d-1),\\
N^- &= \{\exp u^-(x) : x \in \mathbb{R}^{d-1}\},\\
N^+ &= \{\exp u^+(x) : x \in \mathbb{R}^{d-1}\},
\end{aligned}$$

where

$$u^-(x) = \begin{pmatrix} 0 & x^t & 0 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad u^+(x) = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^t & 0 \end{pmatrix}.$$

The Lie algebra of M consists of matrices of the form

$$m(C) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $C \in \mathrm{M}_{d-1}(\mathbb{R})$ is a skew-symmetric matrix, i.e., $C^t = -C$.

The subgroups N^- and N^+ are respectively the contracting and expanding horospherical subgroups of G for the action of A. We have the Iwasawa decomposition $G = KAN^{\pm}$. As we will be using the subgroup N^- frequently, we simply write $N = N^-$. We often identify the subgroup $N^{\pm}$ with $\mathbb{R}^{d-1}$ via the map $\exp u^{\pm}(x) \mapsto x$. For a connected closed subgroup $U < N$, we use the notation $U^{\perp}$ for the orthogonal complement of U in N as a vector subgroup of N, and $U^t = U^+$ for the transpose of U. We use the notation $B_U(r)$ to denote the ball of radius r centered at 0 in U for the Euclidean metric on $N = \mathbb{R}^{d-1}$.

We consider the upper-half space model of $\mathbb{H}^d = \mathbb{R}^+ \times \mathbb{R}^{d-1}$, so that its boundary is given by $\mathbb{S}^{d-1} = \{\infty\} \cup (\{0\} \times \mathbb{R}^{d-1})$. Set $o = (1, 0, \cdots, 0)$, and fix a standard basis $e_0, e_1, \cdots, e_{d-1}$ at $\mathrm{T}_o(\mathbb{H}^d)$. The map

$$g \mapsto (ge_0, \cdots, ge_{d-1})_{g(o)} \tag{3.1}$$

gives an identification of G with the oriented frame bundle $\mathrm{F}\,\mathbb{H}^d$. The stabilizer of o and e_0 in G are equal to K and M respectively, and hence the map (3.1) induces the identifications of the hyperbolic space $\mathbb{H}^d$ and the unit tangent bundle $\mathrm{T}^1\,\mathbb{H}^d$ with G/K and G/M respectively. The action of G on the hyperbolic space $\mathbb{H}^d = G/K$ extends continuously to the compactification $\mathbb{S}^{d-1} \cup \mathbb{H}^d$.

If $g \in G$ corresponds to a frame $(v_0, \cdots, v_{d-1}) \in \mathrm{F}\,\mathbb{H}^d$, we define $g^+, g^- \in \mathbb{S}^{d-1}$ to be the forward and backward end points of the directed geodesic tangent to v_0 respectively. The right translation action of A on $G = \mathrm{F}\,\mathbb{H}^d$

defines the frame flow and we have

$$g^{\pm} = \lim_{t \to \pm\infty} \pi(g a_t)$$

where $\pi : G = \mathrm{F}\,\mathbb{H}^d \to \mathbb{H}^d$ is the basepoint projection.

For the identity element $e = \mathrm{Id}_{d+1} \in G$, note that $e^+ = \infty$, and $e^- = 0$, and hence $g^+ = g(\infty)$ and $g^- = g(0)$. The subgroup MA fixes both points 0 and ∞, and the horospherical subgroup N fixes ∞, and the restriction of the map $g \mapsto g(0)$ to N defines an isomorphism $N \to \mathbb{R}^{d-1}$ given by $u^-(x) \mapsto x$.

For each non-trivial connected subgroup $U < N$, we denote by

$$H(U)$$

the smallest simple closed Lie subgroup of G containing A and U. It is generated by U and the transpose of U.

For a subset $S \subset G$, we denote by $\mathrm{N}_G(S)$ and $\mathrm{C}_G(S)$ the normalizer of S and the centralizer of S respectively. We denote by $\mathrm{N}(S)$ and $\mathrm{C}(S)$ the identity components of $\mathrm{N}_G(S)$ and $\mathrm{C}_G(S)$ respectively.

Example 3.1. Fix the standard basis $e_1, \cdots, e_{d-1}$ of $\mathbb{R}^{d-1}$. For $1 \leq k \leq d-1$, define U_k to be the connected subgroup of N spanned by $e_1, \cdots, e_k$.

The following can be checked directly:

$$\begin{aligned} H(U_k) &= \langle U_k, U_k^t \rangle \simeq \mathrm{SO}^\circ(k+1,1); \\ \mathrm{C}(H(U_k)) &\simeq \mathrm{SO}(d-k-1); \\ \mathrm{N}_G(H(U_k)) &\simeq \mathrm{O}(k+1,1)\mathrm{O}(d-k-1) \cap G; \\ \mathrm{N}(H(U_k)) &\simeq \mathrm{SO}^\circ(k+1,1)\mathrm{SO}(d-k-1). \end{aligned}$$

We set

$$H'(U) := \mathrm{N}(H(U)) = H(U)\,\mathrm{C}(H(U)),$$

which is a connected reductive algebraic subgroup of G with compact center.

The adjoint action of M on N corresponds to the standard action of $\mathrm{SO}(d-1)$ on $\mathbb{R}^{d-1}$. It follows that any connected closed subgroup $U < N$ is conjugate to U_k and $H(U)$ is conjugate to $H(U_k)$ by an element of M, where $k = \dim(U)$.

We set

$$\mathrm{C}_1(U) := \mathrm{C}(H(U)) = M \cap \mathrm{C}(U), \text{ and } \mathrm{C}_2(U) := M \cap \mathrm{C}(U^\perp) \subset H(U). \tag{3.2}$$

Lemma 3.2. *We have*

$$\mathrm{N}(U) = NA\,\mathrm{C}_1(U)\,\mathrm{C}_2(U) \text{ and } \mathrm{C}(U) = N\,\mathrm{C}_1(U).$$

Proof. For the first claim, it suffices to show that for $U = U_k$, $\mathrm{N}(U) = NA\,\mathrm{SO}(k)\,\mathrm{SO}(d-1-k)$. It is easy to check that $Q := NA\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$ normalizes U. Let $g \in \mathrm{N}(U)$. We claim that $g \in Q$. Using the decomposition $G = KAN$, we may assume that $g \in K$. Then $Ug(\infty) = gU(\infty) = g(\infty)$ since $U(\infty) = \infty$. Since $\infty \in \mathbb{S}^{d-1}$ is the unique fixed point of U, it follows $g(\infty) = \infty$. As $M = \mathrm{Stab}_K(\infty)$, we get $g \in M$. Now $gU(0) = Ug(0) = U(0)$. As $U(0) = \mathbb{R}^k$, $g\mathbb{R}^k = \mathbb{R}^k$. Therefore, as $g \in M$, we also have

$g\mathbb{R}^{d-1-k} = \mathbb{R}^{d-1-k}$, and consequently $g \in \mathrm{O}(k)\,\mathrm{O}(d-1-k)$. This shows that $NA\,\mathrm{SO}(k)\,\mathrm{SO}(d-1-k) \subset \mathrm{N}(U) \subset NA\,\mathrm{O}(k)\,\mathrm{O}(d-1-k)$. As $\mathrm{N}(U)$ is connected, this implies the claim.

For the second claim, note first that $N\,\mathrm{C}_1(U) < \mathrm{C}(U)$. Now let $g \in \mathrm{C}(U)$. Since $\mathrm{C}(U) < \mathrm{N}(U) = AN\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, we can write $g = ac_2nc_1 \in A\,\mathrm{C}_2(U)N\,\mathrm{C}_1(U)$. Since nc_1 commutes with U, it follows $ac_2 \in \mathrm{C}(U)$. Now observe that the adjoint action of a on U is a dilation and the adjoint action of c_2 on U is a multiplication by an orthogonal matrix. Therefore we get $a = c_2 = e$, finishing the proof. ☐

Denote by $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of G. By a one-parameter subsemigroup of G, we mean a set of the form $\{\exp(t\xi) \in G : t \geq 0\}$ for some non-zero $\xi \in \mathfrak{g}$. Note that the product $AU^\perp\,\mathrm{C}_2(U)$ is a subgroup of G.

Lemma 3.3. *An unbounded one-parameter subsemigroup S of $AU^\perp\,\mathrm{C}_2(U)$ is one of the following form:*

$$\begin{aligned}&\{\exp(t\xi_A)\exp(t\xi_C)\ : t \geq 0\};\\ &\{\left(v\exp(t\xi_A)v^{-1}\right)\exp(t\xi_C) : t \geq 0\};\\ &\{\exp(t\xi_V)\exp(t\xi_C) : t \geq 0\}\end{aligned}$$

for some $\xi_A \in \mathrm{Lie}(A) - \{0\}, \xi_C \in \mathrm{Lie}(\mathrm{C}_2(U)), v \in U^\perp - \{e\}$, and $\xi_V \in \mathrm{Lie}(U^\perp) - \{0\}$.

Proof. Let $\xi \in \mathrm{Lie}(AU^\perp\,\mathrm{C}_2(U))$ be such that $S = \{\exp(t\xi) : t \geq 0\}$. Write $\xi = \xi_0 + \xi_C$ where $\xi_0 \in \mathrm{Lie}(AU^\perp)$ and $\xi_C \in \mathrm{Lie}(\mathrm{C}_2(U))$. Since $AU^\perp$ commutes with $\mathrm{C}_2(U)$, $\exp(t\xi) = \exp(t\xi_0)\exp(t\xi_C)$ for any $t \in \mathbb{R}$. Hence we only need to show that either $\xi_0 \in \mathrm{Lie}(U^\perp)$ or

$$\{\exp(t\xi_0) : t \geq 0\} = \{v\exp(t\xi_A)v^{-1} : t \geq 0\} \tag{3.3}$$

for some $v \in U^\perp$ and $\xi_A \in \mathrm{Lie}(A)$. Now if $\xi_0 \notin \mathrm{Lie}(U^\perp)$, then writing

$$\xi_0 = \begin{pmatrix} a & x^t & 0 \\ 0 & 0_{d-1} & -x \\ 0 & 0 & -a \end{pmatrix} \in \mathrm{Lie}(AU^\perp)$$

with $a \neq 0$, a direct computation shows that $\xi_0 = v\xi_A v^{-1}$ where

$$\log v = \begin{pmatrix} 0 & -x^t/a & 0 \\ 0 & 0_{d-1} & x/a \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \xi_A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0_{d-1} & 0 \\ 0 & 0 & -a \end{pmatrix},$$

proving (3.3). ☐

Lemma 3.4. *If $v_i \to \infty$ in $U^\perp$, then $\limsup_{i\to\infty} v_iAv_i^{-1}$ contains one-parameter subgroup of $U^\perp$.*

Proof. Writing $v_i = \exp u^-(x_i)$ for $x_i \in \mathbb{R}^{d-1}$, we have

$$v_ia_sv_i^{-1} = \begin{pmatrix} e^s & (1-e^s)x_i^t & -\|(e^{s/2}-e^{-s/2})x_i\|^2/2 \\ 0 & \mathrm{Id}_{d-1} & (1-e^{-s})x_i \\ 0 & 0 & e^{-s} \end{pmatrix}.$$

Passing to a subsequence, $x_i/\|x_i\|$ converges to some unit vector x_0 as $i \to \infty$. For any $r \in \mathbb{R}$, if we set $s_i := \log(1 - r\|x_i\|^{-1})$, then the sequence $v_i a_{s_i} v_i^{-1}$ converges to $\exp u^-(rx_0)$. Therefore the set $V := \{\exp u^-(rx_0) : r \in \mathbb{R}\} < U^\perp$ gives the desired subgroup. □

The complementary subspaces $\mathfrak{h}_U^\perp$ and $\mathfrak{h}^\perp$. If L is a reductive Lie subgroup of G with $\mathfrak{l} = \mathrm{Lie}(L)$, the restriction of the adjoint representation of G to L is completely reducible, and hence there exists an $\mathrm{Ad}(L)$-invariant complementary subspace $\mathfrak{l}^\perp$ so that

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}^\perp.$$

It follows from the inverse function theorem that the map $L \times \mathfrak{l}^\perp \to G$ given by $(g, X) \mapsto g \exp X$ is a local diffeomorphism onto an open neighborhood of e in G.

Let $U = U_k$. Denote by $\mathfrak{h}_U \subset \mathfrak{g}$ the Lie algebra of $H(U)$, by $\mathfrak{u}^\perp$ the subspace $\mathrm{Lie}(U^\perp)$, and by $(\mathfrak{u}^\perp)^t$ its transpose. Then $\mathfrak{h}_U^\perp$ can be given explicitly as follows:

$$\mathfrak{h}_U^\perp = \mathfrak{u}^\perp \oplus (\mathfrak{u}^\perp)^t \oplus \mathfrak{m}_0 \tag{3.4}$$

where $\mathfrak{m}_0$ is given by

$$\left\{ m(C) : C = \begin{pmatrix} 0 & Y \\ -Y^t & Z \end{pmatrix}, Z \in \mathrm{M}_{d-1-k}(\mathbb{R}), Z^t = -Z, Y \in \mathrm{M}_{k \times (d-1-k)}(\mathbb{R}) \right\};$$

to see this, it is enough to check that $\dim(\mathfrak{g}) = \dim(\mathfrak{h}_U) + \dim(\mathfrak{h}_U^\perp)$ and that $\mathfrak{h}_U^\perp$ is $\mathrm{Ad}(H(U))$-invariant, which can be done by direct computation.

Similarly, setting $\mathfrak{h} := \mathrm{Lie}(H'(U))$, $\mathfrak{h}^\perp$ is given by

$$\mathfrak{h}^\perp = \mathfrak{u}^\perp \oplus (\mathfrak{u}^\perp)^t \oplus \mathfrak{m}_0' \tag{3.5}$$

where

$$\mathfrak{m}_0' := \left\{ m(C) : C = \begin{pmatrix} 0 & Y \\ -Y^t & 0 \end{pmatrix} \right\}.$$

Lemma 3.5. *If $r_i \to e$ in $\exp \mathfrak{h}^\perp - \mathrm{C}(H(U))$, then either $r_i \notin \mathrm{N}(U)$ for all i, or $r_i \notin \mathrm{N}(U^+)$ for all i, by passing to a subsequence.*

Proof. By Lemma 3.2 and (3.5), there exists a neighborhood $\mathcal{O}$ of 0 in $\mathfrak{g}$ such that

$$\mathrm{N}(U) \cap \mathrm{N}(U^+) \cap \exp(\mathfrak{h}^\perp \cap \mathcal{O}) \subset \mathrm{C}(H(U)).$$

Hence the claim follows. □

Reductive subgroups of G.

Definition 3.6. For a connected reductive algebraic subgroup $L < G$, we denote by L_{nc} the maximal connected normal semisimple subgroup of L with no compact factors.

A connected reductive algebraic subgroup L of G is an almost direct product

$$L = L_{nc} C T \tag{3.6}$$

where C is a connected semisimple compact normal subgroup of L and T is the central torus of L. If L contains a unipotent element, then L_{nc} is non-trivial, and simple, containing a conjugate of A, and the center of L is compact.

Proposition 3.7. *If $L < G$ is a connected reductive algebraic subgroup normalized by A and containing a unipotent element, then*

$$L = H(U)C$$

where $U < N$ is a non-trivial connected subgroup and C is a closed subgroup of $\mathrm{C}(H(U))$. In particular, L_{nc} and $\mathrm{N}(L_{nc})$ are equal to $H(U)$ and $H'(U)$ respectively.

Proof. If L is normalized by A, then so is L_{nc}. Therefore it suffices to prove that a connected non-compact simple Lie subgroup $H < G$ normalized by A is of the form $H = H(U)$ where $U < N$ is a non-trivial connected subgroup.

First, consider the case when $A < H$. Let $\mathfrak{h}$ be the Lie algebra of H, and $\mathfrak{a}$ be the Lie algebra of A. Since $\mathfrak{h}$ is simple, its root space decomposition for the adjoint action of $\mathfrak{a}$ is of the form $\mathfrak{h} = \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{u}^+ \oplus \mathfrak{u}^-$ where $\mathfrak{u}^\pm$ are the sum of all positive and negative root subspaces respectively and $\mathfrak{z}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$. Since the sum of all negative root subspaces for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ is $\mathrm{Lie}(N^-)$, it follows that $U := \exp(\mathfrak{u}^-) < N^-$ and $H = H(U)$.

Now for the general case, H contains a conjugate gAg^{-1} for some $g \in G$. Hence $g^{-1}Hg = H(U)$. Since $H(U)$ contains both A and $g^{-1}Ag$, they must be conjugate within $H(U)$, so $A = h^{-1}g^{-1}Agh$ for some $h \in H(U)$. Hence $gh \in \mathrm{N}_G(A) = AM$. Therefore $H = gH(U)g^{-1}$ is equal to $mH(U)m^{-1}$ for some $m \in M$. Since m normalizes N and $mH(U)m^{-1} = H(mUm^{-1})$, the claim follows. □

Corollary 3.8. *Any connected closed subgroup L of G generated by unipotent elements is conjugate to either U or $H(U)$ for some non-trivial connected subgroup $U < N$.*

Proof. The subgroup S admits a Levi decomposition $L = SV$ where S is a connected semisimple subgroup with no compact factors and V is the unipotent radical of S [41, Lemma 2.9]. If S is trivial, the claim follows since any connected unipotent subgroup can be conjugate into N. Suppose that S is not-trivial. Then $S = H(U)$ for some non-trivial $U < N$ by Proposition 3.7. Unless V is trivial, the normalizer of V is contained in a conjugate of NAM, in particular, it cannot contain $H(U)$. Hence $V = \{e\}$. □

Totally geodesic immersed planes. Let Γ be a discrete, torsion free, non-elementary, subgroup of G, and consider the associated hyperbolic manifold

$$\mathcal{M} = \Gamma\backslash\mathbb{H}^d = \Gamma\backslash G/K.$$

We refer to [35] for basic properties of hyperbolic manifolds. As in the introduction, we denote by Λ the limit set of Γ and by $\operatorname{core}\mathcal{M}$ the convex core of $\mathcal{M}$. Note that $\operatorname{core}\mathcal{M}$ contains all bounded geodesics in $\mathcal{M}$.

We denote by $\mathrm{F}\,\mathcal{M} \simeq \Gamma\backslash G$ the bundle of all oriented orthonormal frames over $\mathcal{M}$. We denote by

$$\pi : \Gamma\backslash G \to \mathcal{M} = \Gamma\backslash G/K \tag{3.7}$$

the base-point projection. By abuse of notation, we also denote by

$$\pi : G \to \mathbb{H}^d = G/K \tag{3.8}$$

the base-point projection. For $g \in G$, $[g]$ denotes its image under the covering map $G \to \Gamma\backslash G$.

Fix $1 \le k \le d-2$ and let

$$H = H(U_k) \simeq \mathrm{SO}^\circ(k+1,1) \quad \text{and} \quad H' = H'(U_k) \simeq \mathrm{SO}^\circ(k+1,1)\cdot\mathrm{SO}(d-k-1). \tag{3.9}$$

Let $C_0 := \mathbb{R}^k \cup \{\infty\}$ denote the unique oriented k-sphere in $\mathbb{S}^{d-1}$ stabilized by H'. Then $\tilde{S}_0 := \operatorname{hull}(C_0)$ is the unique oriented totally geodesic subspace of $\mathbb{H}^d$ stabilized by H', and $\partial\tilde{S}_0 = C_0$. We note that H' (resp. H) consists of all oriented frames $(v_0, \cdots, v_{d-1}) \in G$ (resp. $(v_0, \cdots, v_k, e_{k+1}, \cdots, e_{d-1}) \in G$) such that the k-tuple $(v_0, \cdots, v_k)$ is tangent to $\tilde{S}_0$, compatible with the orientation of $\tilde{S}_0$. The group G acts transitively on the space of all oriented k spheres in $\mathbb{S}^{d-1}$ giving rise to the isomorphisms of G/H' with

$$\mathcal{C}^k = \text{ the space of all oriented } k\text{-spheres in } \mathbb{S}^{d-1}$$

and with

$$\text{the space of all oriented totally geodesic } (k+1)\text{-planes of } \mathbb{H}^d.$$

We discuss the fundamental group of an immersed geodesic k-plane $S \subset \mathcal{M}$. Choose a totally geodesic subspace $\tilde{S}$ of $\mathbb{H}^d$ which covers S. Then $\tilde{S} = g\tilde{S}_0$ for some $g \in G$, and the stabilizer of $\tilde{S}$ in G is equal to $gH'g^{-1}$. We have

$$\Gamma_{\tilde{S}} = \{\gamma \in \Gamma : \gamma\tilde{S} = \tilde{S}\} = \Gamma \cap gH'g^{-1}$$

and get an immersion $\tilde{f} : \Gamma_{\tilde{S}}\backslash\tilde{S} \to \mathcal{M}$ with image S. Consider the projection map

$$p : gH'g^{-1} \to gHg^{-1}. \tag{3.10}$$

Then p is injective on $\Gamma_{\tilde{S}}$ and

$$\Gamma_{\tilde{S}}\backslash\tilde{S} \simeq p(\Gamma_{\tilde{S}})\backslash\tilde{S}$$

is an isomorphism, since $g\,\mathrm{C}(H)g^{-1}$ acts trivially on $\tilde{S}$. Hence $\tilde{f}$ gives an immersion

$$f : p(\Gamma_{\tilde{S}})\backslash\tilde{S} \to \mathcal{M} \tag{3.11}$$

with image S. We say S properly immersed if f is a proper map.

Proposition 3.9. *Let $x \in \Gamma\backslash G$, and set $S := \pi(xH') \subset \mathcal{M}$. Then*

1. *xH' is closed in $\Gamma\backslash G$ if and only if S is properly immersed in $\mathcal{M}$.*
2. *If M is convex cocompact and S is properly immersed, then S is convex cocompact and*

$$\partial\tilde{S} \cap \Lambda = \Lambda(p(\Gamma_{\tilde{S}}))$$

for any geodesic subspace $\tilde{S} \subset \mathbb{H}^d$ which covers S.

Proof. Choose a representative $g \in G$ of x and consider the totally geodesic subspace $\tilde{S} := g\tilde{S}_0$. Then $S = \mathrm{Im}(f)$ as f given by (3.11). Now the closedness of xH' in $\Gamma\backslash G$ is equivalent to the properness of the map $(H' \cap g^{-1}\Gamma g)\backslash H' \to \Gamma\backslash G$ induced from map $h \mapsto xh$. This in turn is equivalent to the properness of the induced map $(H' \cap g^{-1}\Gamma g)\backslash H'/(H' \cap K) \to \Gamma\backslash G/K$. If Δ is the image of $H' \cap g^{-1}\Gamma g$ under the projection map $H' \to H$, then the natural injective map $\Delta\backslash H/H \cap K \to (H' \cap g^{-1}\Gamma g)\backslash H'/H' \cap K$ is an isomorphism. Since

$$p(\Gamma_{\tilde{S}})\backslash\tilde{S} = p(\Gamma_{\tilde{S}})\backslash gH/(H \cap K) \simeq \Delta\backslash H/(H \cap K),$$

the first claim follows. The second claim follows from [33, Theorem 4.7]. □

4. Hyperbolic manifolds with Fuchsian ends and thick return time

In this section, we study the closed $H(U)$-invariant subset $F_{H(U)} := \mathrm{RF}_+\mathcal{M} \cdot H(U)$ when $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ is a convex cocompact manifold with Fuchsian ends. At the end of the section, we address the global thickness of the return time of any one-parameter subgroup of N to $\mathrm{RF}\,\mathcal{M}$.

Definition 4.1. A convex cocompact hyperbolic manifold $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ is said to have non-empty *Fuchsian ends* if one of the following equivalent conditions holds:

1. its convex core has non-empty interior and non-empty totally geodesic boundary.
2. the domain of discontinuity of Γ

$$\Omega := \mathbb{S}^{d-1} - \Lambda = \bigcup_{i=1}^{\infty} B_i$$

is a dense union of infinitely many round balls with mutually disjoint closures.

In the whole section, let $\mathcal{M}$ be a convex cocompact hyperbolic manifold of non-empty Fuchsian ends.

Renormalized frame bundle. The renormalized frame bundle $\mathrm{RF}\,\mathcal{M} \subset \mathrm{F}\,\mathcal{M}$ is defined as the following AM-invariant subset

$$\mathrm{RF}\,\mathcal{M} = \{[g] \in \Gamma\backslash G : g^{\pm} \in \Lambda\} = \{x \in \Gamma\backslash G : xA \text{ is bounded}\}$$

i.e., the closed set consisting of all oriented frames $(v_0, \cdots, v_{d-1})$ such that the complete geodesic through v_0 is contained in $\mathrm{core}\,\mathcal{M}$.

Unless mentioned otherwise[8], we set $A^+ = \{a_t : t \geq 0\}$. We define

$$\mathrm{RF}_+\,\mathcal{M} = \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\} = \{x \in \Gamma\backslash G : xA^+ \text{ is bounded}\}$$

which is a closed NAM-invariant subset. As $\pi(xNA) = \pi(xG) = \mathcal{M}$ for any $x \in \Gamma\backslash G$, we have $\pi(\mathrm{RF}_+\,\mathcal{M}) = \mathcal{M}$.

Lemma 4.2. *For $x \in \mathrm{RF}_+\,\mathcal{M}$, $\overline{xA^+}$ meets $\mathrm{RF}\,\mathcal{M}$.*

Proof. Take any sequence $a_i \to \infty$ in A^+. Since xA^+ is bounded, xa_i converges to some $x_0 \in \overline{xA^+}$ by passing to a subsequence. On the other hand, as $A = \liminf a_i^{-1}A^+$, we have $x_0A \subset \limsup(xa_i)(a_i^{-1}A^+) \subset \overline{xA^+}$. Since $x \in \mathrm{RF}_+\,\mathcal{M}$, $\overline{xA^+}$ is bounded, so is x_0A. Hence $x_0 \in \mathrm{RF}\,\mathcal{M}$ as desired. □

$H(U)$-invariant subsets: $F_{H(U)}, F^*_{H(U)}$, $\partial F_{H(U)}$. Fix a non-trivial connected subgroup $U < N$, and consider the associated subgroups $H(U)$ and $H'(U)$ as defined in section 3.

We define

$$F_{H(U)} := \mathrm{RF}_+\,\mathcal{M} \cdot H(U). \tag{4.1}$$

We denote by $F^*_{H(U)}$ the interior of $F_{H(U)}$ and by $\partial F_{H(U)}$ the boundary of $F_{H(U)}$. When there is no room for confusion, we will omit the subscript $H(U)$ and simply write F, F^* and ∂F.

If $C \subset \mathbb{S}^{d-1}$ denotes the oriented k-sphere stabilized by $H(U)$, then $g \in F_{H(U)}$ if and only if $gC \cap \Lambda \neq \emptyset$. Therefore the closedness of $F_{H(U)}$ follows from the compactness of Λ. The set $F_{H(U)}$ is also $H'(U)$-invariant, since $\mathrm{RF}_+\,\mathcal{M}$ is M-invariant and $\mathrm{C}(H(U))$ is contained in M. For $g \in G$, we denote by $C_g = C_{gH(U)} \subset \mathbb{S}^{d-1}$ the sphere given by the boundary of the geodesic plane $\pi(gH(U))$. Then $\mathrm{hull}\,C_g = \pi(g(H(U))$, and $C_g = gH(U)^+ = gH(U)^-$ where $H(U)^{\pm} = \{h^{\pm} : h \in H(U)\}$. It follows that

$$F_{H(U)} = \{[g] \in \Gamma\backslash G : C_g \cap \Lambda \neq \emptyset\}. \tag{4.2}$$

Lemma 4.3. *Fix $x = [g] \in \Gamma\backslash G$. Let L be a closed subgroup of G such that the closure of $\pi(gL)$ in $\mathbb{H}^d \cup \mathbb{S}^{d-1}$ does not meet Λ. Then the map $L \to xL \subset \Gamma\backslash G$ given by $\ell \mapsto x\ell$ is a proper map, and hence xL is closed.*

[8]At certain places, we use notation A^+ for any subsemigroup of A

Proof. Suppose that $x\ell_i$ converges to some $[g_0] \in \Gamma\backslash G$ for some sequence $\ell_i \to \infty$ in L. Then there exists $\gamma_i \in \Gamma$ such that $d(\gamma_i\pi(g\ell_i), \pi(g_0)) = d(\pi(g\ell_i), \gamma_i\pi(g_0)) \to 0$ as $i \to \infty$. As $g\ell_i \to \infty$, $\gamma_i\pi(g_0)$ converges to a limit point $\xi \in \Lambda$, after passing to a subsequence. Hence $\overline{\pi(gL)} \cap \Lambda \neq \emptyset$. ☐

This lemma implies that if $x \notin \mathrm{RF}_+ \mathcal{M}$ (resp. $x \notin F_{H(U)}$) , then xU (resp. $xH(U)$) is closed for any closed subgroup $U < N$.

Lemma 4.4. *If $\mathcal{M}$ is a convex cocompact hyperbolic manifold of non-empty Fuchsian ends, then*

$$F_{H(U)} = \{x \in \Gamma\backslash G : \pi(\overline{xH(U)}) \cap \operatorname{core} \mathcal{M} \neq \emptyset\}.$$

Proof. Denote by Q the subset on the right-hand side of the above equality. To show $F_{H(U)} \subset Q$, let $x \in F_{H(U)}$. By modifying it using an element of $H(U)$, we may assume that $x \in \mathrm{RF}_+ \mathcal{M}$. By Lemma 4.2, $\overline{xA^+}$ contains $x_0 \in \mathrm{RF}\,\mathcal{M}$. Since x_0A is bounded, $\pi(x_0A)$ is a bounded geodesic, and hence

$$\pi(x_0A) \subset \pi(\overline{xH(U)}) \cap \operatorname{core} \mathcal{M}$$

because $\operatorname{core} \mathcal{M}$ contains all bounded geodesics. Therefore $x \in Q$. To show the other inclusion $Q \subset F_{H(U)}$, we use the hypothesis on $\mathcal{M}$. Suppose $x = [g] \notin F_{H(U)}$. Then $C_g \cap \Lambda = \emptyset$, and hence C_g must be contained in a connected component, say B_i, of Ω. Hence $\pi(gH(U)) = \mathrm{hull}(C_g)$ is contained in the interior of $\mathrm{hull}(B_i)$, which is disjoint from $\mathrm{hull}(\Lambda)$, by the convexity of B_i. Therefore the orbit $\Gamma\pi(gH(U))$ is a closed subset of $\mathbb{H}^d$, disjoint from $\mathrm{hull}(\Lambda)$. Hence $x \notin Q$, proving the claim. ☐

Note also that

$$\begin{aligned} \mathrm{RF}\,\mathcal{M} \cdot H(U) &= \{[g] \in \Gamma\backslash G : \#C_g \cap \Lambda \geq 2\} \\ &= \{x \in \Gamma\backslash G : \pi(xH(U)) \cap \operatorname{core} \mathcal{M} \neq \emptyset\}. \end{aligned} \tag{4.3}$$

This can be seen using the fact that for any two distinct points $\xi^+, \xi^- \in C_g$, there exists $h \in H(U)$ such that $gh(\infty) = \xi^+$ and $gh(0) = \xi^-$; this fact is clear if $H(U) = H(U_k)$ for some k, and a general case follows since $H(U) = mH(U_k)m^{-1}$ for some $m \in M$, and M fixes both 0 and ∞.

Denote by $\mathcal{M}^*$ the interior of the core of $\mathcal{M}$ and by $F^*_{H(U)}$ the interior of $F_{H(U)}$. Then

$$F^*_{H(U)} = \{x \in \Gamma\backslash G : \pi(xH(U)) \cap \mathcal{M}^* \neq \emptyset\}.$$

To see this, note that the right-hand side is equal to

$$\begin{aligned} &\{[g] \in F_{H(U)} : \mathrm{hull}\, C_g \cap \mathrm{Interior}(\mathrm{hull}(\Lambda)) \neq \emptyset\} \\ &= \{[g] \in F_{H(U)} : C_g \not\subset \overline{B}_i \text{ for any } i\} \end{aligned} \tag{4.4}$$

which can then be seen to be equal to $F^*_{H(U)}$ in view of (4.2). Note that (4.4) implies that for $[g] \in F^*_{H(U)}$, $\#C_g \cap \Lambda \geq 2$ and hence

$$F^*_{H(U)} \subset \mathrm{RF}\,\mathcal{M} \cdot H(U). \tag{4.5}$$

In particular, $\mathrm{RF}\,\mathcal{M}\cdot H(U)$ is dense in $F_{H(U)}$.

Lemma 4.5. *We have*

$$\mathrm{RF}_+\,\mathcal{M}\cap F^*_{H(U)}\subset \mathrm{RF}\,\mathcal{M}\cdot U.$$

Proof. Let $y\in \mathrm{RF}_+\,\mathcal{M}\cap F^*_{H(U)}$. We need to show that $yU\cap \mathrm{RF}\,\mathcal{M}\neq\emptyset$. Choose $g\in G$ so that $[g]=y$. As $y\in \mathrm{RF}_+\,\mathcal{M}$, $g^+=g(\infty)\in\Lambda$, and hence $C_g\cap\Lambda\neq\emptyset$. If $\#C_g\cap\Lambda=1$, then C_g must be contained in $\overline{B_i}$ for some i, which implies $[g]\notin F^*_{H(U)}$. Therefore $\#C_g\cap\Lambda\geq 2$. We note that $gU(0)\cup\{g(\infty)\}=C_g$; this is clear when $U=U_k$ for some $k\geq 1$ and $g=e$, to which a general case is reduced. Hence there exists $u\in U$ such that $gu(0)\in\Lambda$. Since $gu(\infty)=g(\infty)\in\Lambda$, we have $yu=[g]u\in \mathrm{RF}\,\mathcal{M}$. □

We denote by $\partial F_{H(U)}$ the boundary of $F_{H(U)}$, that is,

$$\partial F_{H(U)}=F_{H(U)}-F^*_{H(U)}=\{[g]\in F_{H(U)}:C_g\subset\overline{B_i}\text{ for some }i\}.$$

When there is no room for confusion, we will omit the subscript $H(U)$ and simply write F,F^* and ∂F.

We call an oriented frame $g=(v_0,\cdots,v_{d-1})\in \mathrm{F}\,\mathcal{M}=G$ a boundary frame if the first $(d-1)$ vectors $v_0,\cdots,v_{d-2}$ are tangent to the boundary of $\operatorname{core}\mathcal{M}$. Set

$$\check{H}:=H(U_{d-2})=\mathrm{SO}^\circ(d-1,1),$$

and denote by $\check{V}$ the one-dimensional subgroup $\mathbb{R}e_{d-1}$ of $N=\mathbb{R}^{d-1}$; note that $\check{V}=(\check{H}\cap N)^\perp$.

We denote by $\mathrm{BF}\,\mathcal{M}$ the set of all boundary frames of $\mathcal{M}$; it is a union of compact $\check{H}$-orbits:

$$\mathrm{BF}\,\mathcal{M}=\bigcup_{i=1}^k z_i\check{H}$$

such that $\pi(z_i\check{H})=\Gamma\backslash\Gamma\,\mathrm{hull}(B_i)$ for some component B_i of Ω.

The boundary $\partial F_{H(U)}$ for $U<\check{H}\cap N$. Suppose that U is contained in $\check{H}\cap N=\mathbb{R}^{d-2}$. Then there exists a one-parameter semigroup $\check{V}^+$ of $\check{V}$ such that

$$\partial F=\mathrm{BF}\,\mathcal{M}\cdot\check{V}^+\cdot H'(U).$$

We use the notation $\check{V}^-=\{v^{-1}:v\in\check{V}^+\}$. Note that

(4.6)
$$\partial F\cap\mathrm{RF}\,\mathcal{M}=\mathrm{BF}\,\mathcal{M}\cdot\mathrm{C}(H(U))\ \text{ and }\ \partial F\cap\mathrm{RF}_+\,\mathcal{M}=\mathrm{BF}\,\mathcal{M}\cdot\check{V}^+\cdot\mathrm{C}(H(U)).$$

For a general proper connected closed subgroup $U<N$, $mUm^{-1}\subset\check{H}\cap N$ for some $m\in M$, and

$$\partial F\cap\mathrm{RF}\,\mathcal{M}=\mathrm{BF}\,\mathcal{M}\cdot m\cdot\mathrm{C}(H(U))$$

where $\mathrm{BF}\,\mathcal{M}\cdot m$ is now a union of finitely many $m^{-1}\check{H}m$-compact orbits.

Lemma 4.6. *Let $U<\check{H}\cap N$, $z\in\mathrm{BF}\,\mathcal{M}$ and $v\in\check{V}-\{e\}$. If $zv\in\mathrm{RF}\,\mathcal{M}$, then $zv\in F^*$.*

Proof. Let $z = [g] \in \mathrm{BF}\,\mathcal{M}$. Then $\partial(\pi(g\check{H})) = \partial B_j$ for some j. Let $v \in \check{V} - \{e\}$ be such that $zv \in \mathrm{RF}\,\mathcal{M}$. Suppose $zv \in \partial F_{H(U)}$. Then $C_{gv} \subset \overline{B_i}$ for some i. Since the sphere $C_{gv} = \{gvh(\infty) : h \in H(U)\}$ contains $g(\infty)$ which belongs to ∂B_j, we have $i = j$, as $\overline{B}_i$'s are mutually disjoint. As $zv \in \mathrm{RF}\,\mathcal{M}$, $C_{gv} \subset \partial B_j$. Hence $gvH(U)^+ \subset g\check{H}^+$. It follows that $gvH(U) \subset g\check{H}$, and hence $vH(U) \cap \check{H} \neq \emptyset$, which is a contradiction since $v \notin \check{H}$, and $H(U) \subset \check{H}$. □

Properly immersed geodesic planes. Let $H = H(U_k)$ and $H' = H'(U_k)$ be as in (3.9), and p be the map in (3.10). In (3.11), if $p(\Gamma_{\tilde{S}})\backslash\tilde{S}$ is a convex cocompact hyperbolic k-manifold with Fuchsian ends and f is proper, then the image $S = \mathrm{Im}(f)$ is referred to as a properly immersed convex cocompact geodesic k-plane of Fuchsian ends.

Proposition 4.7. *If xH' is closed for $x \in \mathrm{RF}\,\mathcal{M}$, then $S = \pi(xH')$ is a properly immersed convex cocompact geodesic plane with (possibly empty) Fuchsian ends.*

Proof. Choose $g \in G$ so that $x = [g]$. Let $\tilde{S}$ and $\Gamma_{\tilde{S}}$ be as in Proposition 3.9. Set $C = \partial\tilde{S}$. By loc. cit., S is properly immersed, and $C \cap \Lambda = \Lambda(p(\Gamma_{\tilde{S}}))$. Write

$$C - (C \cap \Lambda) = \bigcup_{i \in I}(C \cap B_i) \tag{4.7}$$

where I is the collection of all i such that $C \cap B_i \neq \emptyset$. If $C \cap \Lambda$ contains a non-empty open subset of C, then the limit set of $p(\Gamma_{\tilde{S}})$ has Hausdorff dimension equal to the dimension of C. So $p(\Gamma_{\tilde{S}})$ is a uniform lattice in gHg^{-1}, and hence S is compact. In the other case, I is an infinite set and $\bigcup_{i\in I}(C \cap B_i)$ is dense in C; so S is a convex cocompact hyperbolic submanifold with Fuchsian ends by Definition 4.1(2). □

Lemma 4.8. *For any sphere C in $\mathbb{S}^{d-1}$ with $\#C \cap \Lambda \geq 2$, the intersection $C \cap \Lambda$ is Zariski dense in C.*

Proof. The claim is clear if $C \cap \Lambda$ contains a non-empty open subset of C. If not, $C \cap \Lambda$ contains infinitely many $C \cap \partial B_i$'s, each of which is an irreducible co-dimension one real subvariety of C. It follows that the Zariski closure of $C \cap \Lambda$ has dimension strictly greater than $\dim C - 1$, hence is equal to C. □

We let

$$\pi_1 : H' \to H, \quad \text{and} \quad \pi_2 : H' \to \mathrm{C}(H) \tag{4.8}$$

denote the canonical projections.

Proposition 4.9. *Suppose that xH' is closed for $x = [g] \in \mathrm{RF}\,\mathcal{M}$, and set $\Gamma' := g^{-1}\Gamma g \cap H'$. Then*

$$\overline{xH} = xHC \tag{4.9}$$

where $C = \overline{\pi_2(\Gamma')}$ *and* HC *is equal to the identity component of the Zariski closure of* Γ'.

Proof. Without loss of generality, we may assume $g = e$. As H' is a direct product $H \times \mathrm{C}(H)$, we write an element of H' as (h, c) with $h \in H$ and $c \in \mathrm{C}(H)$. For all $\gamma \in \Gamma'$,

$$xH = [(e, e)]H = [(e, \pi_2(\gamma))]H = [(e, e)]H\pi_2(\gamma)$$

and hence $xH = xH\pi_2(\Gamma')$. It follows that $xHC \subset \overline{xH}$.

To show the other inclusion, let $(h_0, c_0) \in H\,\mathrm{C}(H)$ be arbitrary. If $[(h_0, c_0)] \in \overline{xH} = \overline{[(e, e)]H}$, then there exist sequences $\gamma_i \in \Gamma'$ and $h_i \in H$ such that $\gamma_i(h_i, e) \to (h_0, c_0)$ in H' as $i \to \infty$. In particular, $\pi_2(\gamma_i) \to c_0$ in $\mathrm{C}(H)$ as $i \to \infty$ and hence $c_0 \in C = \overline{\pi_2(\Gamma')}$. This finishes the proof of (4.9). Let W denote the identity component of the Zariski closure of Γ' in H'. Since any proper algebraic subgroup of G stabilizes either a point, or a proper sphere in $\mathbb{S}^{d-1}$, it follows from Proposition 3.9 and Lemma 4.8 that $\pi_1(\Gamma')$ is Zariski dense in H; so $\pi_1(W) = H$. So the quotient $W\backslash H'$ is compact. This implies that W contains a maximal real-split connected solvable subgroup, say, P of H'. Now $H \cap W$ is a normal subgroup of H, as $\pi_1(W) = H$. Since $P < H \cap W$ and H is simple, we conclude that $H \cap W = H$, i.e., $H < W$. Hence $W = H\pi_2(W)$. As any compact linear group is algebraic, C is algebraic and hence $C = \pi_2(W) = \overline{\pi_2(\Gamma')}$. Therefore $W = HC$, finishing the proof. □

Global thickness of the return time to $\mathrm{RF}\,\mathcal{M}$**.** We recall the various notions of thick subsets of $\mathbb{R}$, following [26] and [28].

Definition 4.10. Fix $k > 1$.

- A closed subset $\mathsf{T} \subset \mathbb{R}$ is locally k-thick at t if for any $\lambda > 0$,
$$\mathsf{T} \cap (t \pm [\lambda, k\lambda]) \neq \emptyset.$$
- A closed subset $\mathsf{T} \subset \mathbb{R}$ is k-thick if T is locally k-thick at 0.
- A closed subset $\mathsf{T} \subset \mathbb{R}$ is k-thick at ∞ if
$$\mathsf{T} \cap (\pm[\lambda, k\lambda]) \neq \emptyset$$
for all sufficiently large $\lambda \gg 1$.
- A closed subset $\mathsf{T} \subset \mathbb{R}$ is globally k-thick if $\mathsf{T} \neq \emptyset$ and T is locally k-thick at every $t \in \mathsf{T}$.

We will frequently use the fact that if T_i is a sequence of k-thick subsets, then $\limsup \mathsf{T}_i$ is also k-thick, and that if T is k-thick, so is $-\mathsf{T}$.

The following proposition shows that $\mathrm{RF}\,\mathcal{M}$ has a thick return property under the action of any one-dimensional subgroup U of N.

Proposition 4.11. *There exists a constant* $k > 1$ *depending only on the systole of the double of* $\mathrm{core}\,\mathcal{M}$ *such that for any one-parameter subgroup* $U = \{u_t : t \in \mathbb{R}\}$ *of* $N^{\pm}$, *and any* $y \in \mathrm{RF}\,\mathcal{M}$,

$$\mathsf{T}(y) := \{t \in \mathbb{R} : yu_t \in \mathrm{RF}\,\mathcal{M}\}$$

is globally k-thick.

Proof. Let $\eta > 0$ be the systole of the hyperbolic double of core $\mathcal{M}$, which is a closed hyperbolic manifold. Let $k > 1$ be given by the equation

$$d\big(\text{hull}([-k,-1]),\ \text{hull}([1,k])\big) = \eta/4 \tag{4.10}$$

where d is the hyperbolic distance in the upper half plane $\mathbb{H}^2$.

Note that

$$\inf_{i \neq j} d(\text{hull}B_i,\ \text{hull}B_j) \geq \eta/2 \tag{4.11}$$

as the geodesic realizing this distance is either a closed geodesic or half of a closed geodesic in the double of core $\mathcal{M}$.

We first prove the case when $U < N$. Let $s \in \mathsf{T}(y)$ be arbitrary. To show that $\mathsf{T}(y)$ is locally k-thick at s, we may assume that $s = 0$, by replacing y with $yu_s \in \text{RF}\,\mathcal{M}$. We may also assume that $y = [g]$ where $g(\infty) = \infty$ and $g(0) = 0$. As $y \in \text{RF}\,\mathcal{M}$, this implies that $0, \infty \in \Lambda$. Since $gu_t(\infty) = g(\infty) \in \Lambda$, we have

$$\mathsf{T}(y) = \{t \in \mathbb{R} : gu_t(0) \in \Lambda\}.$$

Suppose that $\mathsf{T}(y)$ is not locally k-thick at 0. Then there exist $w \in U$ and $t > 0$ such that

$$([-kt,-t] \cdot w \cup [t,kt] \cdot w) \cap \Lambda = \emptyset.$$

Since each component of Ω is convex and $0 \notin \Omega$, it follows that $[-kt,-t] \cdot w$ and $[t,kt] \cdot w$ lie in distinct components of Ω, say B_i and B_j, $(i \neq j)$. But this yields

$$d_w(\text{hull}([-kt,-t] \cdot w),\ \text{hull}([t,kt] \cdot w)) \geq d(\text{hull}B_i,\ \text{hull}B_j) \geq \eta/2. \tag{4.12}$$

where d_w denotes the hyperbolic distance of the plane above the line $\mathbb{R}w$. Observe that the distance in (4.12) is independent of $w \in \mathbb{R}^{d-1}$ and $t > 0$, because both the dilation centered at 0 and the $(d-2)$-dimensional rotation with respect to the vertical axis above 0 are hyperbolic isometries. Therefore, we get a contradiction to (4.10). The case of $U < N^+$ is proved similarly, just replacing the role of g^+ and g^- in the above arguments. □

Remark 4.12. It follows from the proof that k is explicitly given by (4.10), equivalently, $k + k^{-1} = e^{\eta/4} + 2e^{\eta/8} - 1$ where $\eta > 0$ is the systole of the double of core $\mathcal{M}$.

5. Structure of singular sets

Let $\Gamma < G = \text{SO}^\circ(d,1)$ be a convex cocompact torsion-free Zariski-dense subgroup. Let $U < G$ be a connected closed subgroup of G generated by unipotent elements in it. In this section, we define the singular set $\mathscr{S}(U)$ associated to U and study its structural property. The singular set $\mathscr{S}(U)$ is defined so that it contains all closed orbits of intermediate subgroups between U and G.

Definition 5.1 (Singular set)**.** We set

$$\mathscr{S}(U) = \left\{ x \in \Gamma\backslash G : \begin{array}{c} \text{there exists a proper connected} \\ \text{closed subgroup } W \supset U \text{ such that } xW \\ \text{is closed and } \mathrm{Stab}_W(x) \text{ is Zariski dense in } W. \end{array} \right\}.$$

Definition 5.2 (Definition of $\mathscr{H}$)**.** We denote by $\mathscr{H}$ the collection of all *proper* connected closed subgroups $H < G$ containing a unipotent element such that

- $\Gamma\backslash\Gamma H$ is closed, and
- $H \cap \Gamma$ is Zariski dense in H.

Proposition 5.3. *If $H \in \mathscr{H}$, then H is a reductive subgroup of G, and hence is of the form $gH(U)Cg^{-1}$ for some connected subgroup $U < N$, a closed subgroup $C < \mathrm{C}(H(U))$ and $g \in G$ such that $[g] \in \mathrm{RF}\,\mathcal{M}$.*

Proof. In order to prove that H is reductive, suppose not. Then its unipotent radical is non-trivial, which we can assume to be a subgroup U of N, up to a conjugation. Now we write $H = H_{nc}CTU$ where C is a connected semisimple compact subgroup and T is a torus centralizing $H_{nc}C$. As H is contained in $\mathrm{N}(U) = NA\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, which does not contain any non-compact simple Lie subgroup, it follows that H_{nc} is trivial. Now if T were compact, then $H \cap \Gamma$ would consist of parabolic elements, which is a contradiction as Γ is convex cocompact. Hence T is non-compact. Write $T = T_0S$ where S is a split torus and T_0 is compact. Then T_0 is equal to a conjugate of A, say, $g^{-1}Ag$ for some $g \in G$. As T_0 normalizes U, and $\mathrm{N}(U)$ fixes ∞, we deduce that $g(\infty)$ is either ∞ or 0. Since $\mathrm{Stab}_G(\infty) = NAM$, $g(\infty) = \infty$ implies $g \in NAM$, and $g(\infty) = 0$ implies $jg \in NAM$ where $j \in G$ is an element of order 2 such that $j(0) = \infty$. In either case, $T_0 = v^{-1}Av$ for some $v \in N$. By replacing H with vHv^{-1}, we may assume that $T_0 = A$. Since CS is a compact subgroup commuting with A, $CS \subset M$. Therefore H is of the form M_0AU where M_0 is a closed subgroup of $M \cap \mathrm{N}(U)$; note that we used the fact that v commutes with U. Now the commutator subgroup $[H, H]$ is equal to $[M_0, M_0]U$. Since $[H \cap \Gamma, H \cap \Gamma]$ must be Zariski dense in $[H, H]$, we deduce that Γ contains an element $m_0u \in M_0U$ with u non-trivial. Since m_0u is a parabolic element of Γ, this is a contradiction to the assumption that Γ is convex cocompact. This proves that H is reductive.

By Proposition 3.7, H is of the form $gH(U)Cg^{-1}$ for some $g \in G$ and $C < \mathrm{C}(H(U))$. For some $m \in M$ and $1 \le k \le d-2$, $H(U) = mH(U_k)m^{-1}$. Hence $\Gamma\backslash\Gamma gmH(U_k)C_0$ is closed where $C_0 = m^{-1}Cm$. By Proposition 3.9, the boundary of the geodesic plane $\pi(gmH(U_k))$ contains uncountably many points of Λ, since $(gm)H(U_k)C_0(gm)^{-1} \cap \Gamma$ is Zariski dense in $(gm)H(U_k)C_0(gm)^{-1}$. Using two such limit points, we can find an element $h \in H(U_k)$ such that $(gmh)^{\pm} \in \Lambda$. Since $(gmhm^{-1})^{\pm} = (gmh)^{\pm}$ and $mhm^{-1} \in H(U)$, it follows that $[g]H(U) \cap \mathrm{RF}\,\mathcal{M} \neq \emptyset$, and hence we can take $[g] \in \mathrm{RF}\,\mathcal{M}$ by modifying it with an element of $H(U)$ if necessary. This finishes the proof. ∎

Therefore, for each $H \in \mathscr{H}$, the non-compact semisimple part H_{nc} of H is well defined.

Proposition 5.4. *If $H \in \mathscr{H}$, then*
- *$H \cap \Gamma$ is finitely generated;*
- *$[\mathrm{N}_G(H_{nc}) \cap \Gamma; H \cap \Gamma] < \infty$.*

Proof. Let p denote the projection map $\mathrm{N}_G(H_{nc}) \to H_{nc}$. Note that p is an injective map on $\mathrm{N}_G(H_{nc}) \cap \Gamma$, as Γ is torsion free and the kernel of p is a compact subgroup. It follows from Proposition 5.3 that H_{nc} is co-compact in $\mathrm{N}_G(H_{nc})$. Since $H \in \mathscr{H}$, the orbit $[e]H$ is closed and hence $[e]\,\mathrm{N}_G(H_{nc})$ is closed. It follows that both $p(H \cap \Gamma)$ and $p(\mathrm{N}_G(H_{nc}) \cap \Gamma)$ are convex cocompact Zariski dense subgroups of H_{nc} by Proposition 3.9. As any convex cocompact subgroup is finitely generated [3], $p(H \cap \Gamma)$ is finitely generated. Hence $H \cap \Gamma$ is finitely generated by the injectivity of $p|_{H \cap \Gamma}$.

Since $p(H \cap \Gamma)$ is a normal subgroup of $p(\mathrm{N}_G(H_{nc}) \cap \Gamma)$, it follows that $p(H \cap \Gamma)$ has finite index in $p(\mathrm{N}_G(H_{nc}) \cap \Gamma)$ by Lemma 5.5 below. Since $p|_{\mathrm{N}_G(H_{nc}) \cap \Gamma}$ is injective, it follows that $H \cap \Gamma$ has finite index in $\mathrm{N}_G(H_{nc}) \cap \Gamma$. ☐

Lemma 5.5. *Let Γ_1 and Γ_2 be non-elementary convex cocompact subgroups of G. If Γ_2 is a normal subgroup of Γ_1, then $[\Gamma_1 : \Gamma_2] < \infty$.*

Proof. Let Λ_i be the limit set of Γ_i for $i = 1, 2$. Since $\Gamma_2 < \Gamma_1$, $\Lambda_2 \subset \Lambda_1$. As Γ_2 is normalized by Γ_1, Λ_2 is Γ_1-invariant. Since Γ_1 is non-elementary, Λ_1 is a minimal Γ_1-invariant closed subset. Hence $\Lambda_1 = \Lambda_2$. Let $\mathcal{M}_i := \Gamma_i \backslash \mathbb{H}^d$. Then the convex core of $\mathcal{M}_1$ is equal to $\Gamma_1 \backslash \mathrm{hull}(\Lambda_2)$ and covered by $\mathrm{core}\, \mathcal{M}_2 = \Gamma_2 \backslash \mathrm{hull}(\Lambda_2)$. Since $\mathrm{core}\, \mathcal{M}_2$ is compact, it follows that $[\Gamma_1 : \Gamma_2] < \infty$. ☐

Definition 5.6 (Definition of $\mathscr{H}^\star$)**.**

$$\mathscr{H}^\star := \{\mathrm{N}_G(H_{nc}) : H \in \mathscr{H}\}. \tag{5.1}$$

Corollary 5.7 (Countability)**.** *The collection $\mathscr{H}$ is countable, and the map $H \to \mathrm{N}_G(H_{nc})$ defines a bijection between $\mathscr{H}$ and $\mathscr{H}^\star$.*

Proof. As Γ is convex cocompact, it is finitely generated. Therefore there are only countably many finitely generated subgroups of Γ. By Proposition 5.4, there are only countably many possible $H \cap \Gamma$ for $H \in \mathscr{H}$. Since H is determined by $H \cap \Gamma$, being its Zariski closure, the first claim follows.

Since $H \cap \Gamma$ has finite index in $\mathrm{N}_G(H_{nc}) \cap \Gamma$ by Proposition 5.4, H is determined as the identity component of the Zariski closure of $\mathrm{N}_G(H_{nc}) \cap \Gamma$. This proves the second claim. ☐

In the case of a convex cocompact hyperbolic manifold of Fuchsian ends, there is a one to one correspondence between $\mathscr{H}$ and the collection of all closed $H'(U)$-orbits of points in $\mathrm{RF}\,\mathcal{M}$ for $U < N$: if $H \in \mathscr{H}$, then $H = gH(U)Cg^{-1}$ for some $U < N$ and $g \in G$ with $[g] \in \mathrm{RF}\,\mathcal{M}$ and $[g]H'(U)$ is closed. Conversely, if $[g]H'(U)$ is closed for some $[g] \in \mathrm{RF}\,\mathcal{M}$, then the identity component of the Zariski closure of $\Gamma \cap gH'(U)g^{-1}$ is given by $gH(U)Cg^{-1}$ for some closed subgroup $C < \mathrm{C}(H(U))$ by Proposition 4.9, and hence $gH(U)Cg^{-1} \in \mathscr{H}$. Moreover, since the normalizer of $H(U)C$ is contained in $H'(U)$, if $g_1H(U)Cg_1^{-1} = g_2H(U)Cg_2^{-1}$, then $g_2^{-1}g_1 \in H'(U)$, so $[g_1]H'(U) = [g_2]H'(U)$. Therefore Corollary 5.7 implies the following corollary by Propositions 3.9 and 4.9.

Corollary 5.8. *Let $\mathcal{M}$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Then*

1. *there are only countably many properly immersed geodesic planes of dimension at least 2 intersecting* $\operatorname{core}\mathcal{M}$*.*
2. *For each $1 \leq m \leq d-2$, there are only countably many spheres $S \subset \mathbb{S}^{d-1}$ of dimension m, such that $\#S \cap \Lambda \geq 2$ and ΓS is closed in the space $\mathcal{C}^m$.*

Remark 5.9. In (2), we may replace the condition $\#S\cap\Lambda \geq 2$ with $\#S\cap\Lambda \geq 1$, because if $\#S \cap \Lambda = 1$, then ΓS is not closed (see Remark 11.6).

For a subgroup $H < G$, define

$$X(H,U) := \{g \in G : gUg^{-1} \subset H\}. \tag{5.2}$$

Note that $X(H,U)$ is left-$\mathrm{N}_G(H)$ and right-$\mathrm{N}_G(U)$-invariant, and for any $g \in G$,

$$X(gHg^{-1}, U) = gX(H,U). \tag{5.3}$$

For $H \in \mathscr{H}$ and any connected unipotent subgroup $U < G$, observe that

$$X(H,U) = X(H_{nc},U) = X(\mathrm{N}_G(H_{nc}),U); \tag{5.4}$$

this follows since any unipotent element of $\mathrm{N}_G(H_{nc})$ is contained in H_{nc}.

Proposition 5.10. *We have*

$$\mathscr{S}(U) = \bigcup_{H \in \mathscr{H}^\star} \Gamma\backslash\Gamma X(H,U).$$

Proof. If $x = [g] \in \mathscr{S}(U)$, then there exists a proper connected closed subgroup W of G containing U such that $[g]W$ is closed and $\mathrm{Stab}_W(x)$ is Zariski dense in W. This means $H := gWg^{-1} \in \mathscr{H}$ and $g \in X(H,U)$. Since $X(H,U) = X(\mathrm{N}_G(H_{nc}),U)$, and $\mathrm{N}_G(H_{nc}) \in \mathscr{H}^\star$, this proves the inclusion $\subset$. Conversely, let $g \in X(\mathrm{N}_G(H_{nc}),U)$ for some $H \in \mathscr{H}$. Set $W := g^{-1}Hg$. Then $U \subset W$, $[g]W = \Gamma Hg$ is closed and $\mathrm{Stab}_W([g]) = g^{-1}(\Gamma \cap H)g$ is Zariski dense in W. Hence $[g] \in \mathscr{S}(U)$. $\square$

Singular subset of a closed orbit. Let $L < G$ be a connected reductive subgroup of G containing unipotent elements. For a closed orbit x_0L of $x_0 \in \mathrm{RF}\,\mathcal{M}$, and a connected subgroup $U_0 < L \cap N$, we define the singular

set $\mathscr{S}(U_0, x_0L)$ by
(5.5)

$$\mathscr{S}(U_0, x_0L) = \left\{ x \in x_0L : \begin{array}{c} \text{there exists a connected closed subgroup } W < L, \\ \text{containing } U_0 \text{ such that } \dim W_{nc} < \dim L_{nc}, \\ xW \text{ is closed and } \mathrm{Stab}_W(x) \text{ is Zariski dense in } W \end{array} \right\}.$$

It follows from Proposition 5.10 and Proposition 5.3 that the subgroup W in the definition 5.1 is conjugate to $H(\widehat{U})C$ for some $\widehat{U} < N$. Hence W being a proper subgroup of G is same as requiring $\dim W_{nc} < \dim G$. Therefore $\mathscr{S}(U_0) = \mathscr{S}(U_0, \Gamma\backslash G)$ and

$$\mathscr{S}(U_0, x_0L) = x_0L \cap \bigcup \Gamma\backslash\Gamma X(H, U_0)$$

where the union is taken over all subgroups $H \in \mathscr{H}^\star$ such that H is a subgroup of $g_0Lg_0^{-1}$ with $\dim H_{nc} < \dim L_{nc}$ and $x_0 = [g_0]$. Equivalently,

$$\mathscr{S}(U_0, x_0L) = \bigcup_{W \in \mathscr{H}^\star_{x_0L}} x_0(L \cap X(W, U_0)) \tag{5.6}$$

where $\mathscr{H}^\star_{x_0L}$ consists of all subgroups of the form $W = g_0^{-1}Hg_0 \cap L$ for some $H \in \mathscr{H}^\star$ and $\dim W_{nc} < \dim L_{nc}$. Then the generic set $\mathscr{G}(U_0, x_0L)$ is defined by

$$\mathscr{G}(U_0, x_0L) := (x_0L \cap \mathrm{RF}_+\mathcal{M}) - \mathscr{S}(U_0, x_0L). \tag{5.7}$$

Definition of $\mathcal{L}_U$ and $\mathcal{Q}_U$. Fix a non-trivial connected closed subgroup $U < N$. We define the collection $\mathcal{L}_U$ of all subgroups of the form $H(\widehat{U})C$ where $U < \widehat{U} < N$ and C is a closed subgroup of $\mathrm{C}(H(\widehat{U}))$ satisfying the following:
(5.8)

$$\mathcal{L}_U := \left\{ L = H(\widehat{U})C : \begin{array}{c} \text{for some } [g] \in \mathrm{RF}_+\mathcal{M},\ [g]L \text{ is closed in } \Gamma\backslash G \\ \text{and } L \cap g^{-1}\Gamma g \text{ is Zariski dense in } L \end{array} \right\}.$$

Observe that for $L = H(\widehat{U})C \neq G$, the condition $L \in \mathcal{L}_U$ with $[g]L$ closed is equivalent to the condition that

$$gLg^{-1} \in \mathscr{H}.$$

Lemma 5.11. *Let L_1 and L_2 be members of $\mathcal{L}_U$ such that xL_1 and xL_2 are closed for some $x \in \mathrm{RF}\,\mathcal{M}$. If $(L_1)_{nc} = (L_2)_{nc}$, then $L_1 = L_2$.*

Proof. If L_1 or L_2 is equal to G, then the claim is trivial. Suppose that both L_1 and L_2 are proper subgroups of G. If $x = [g]$, then both subgroups $H_1 := gL_1g^{-1}$ and $H_2 := gL_2g^{-1}$ belong to $\mathscr{H}$. Since $(H_1)_{nc} = (H_2)_{nc}$, we have $H_1 = H_2$ by Corollary 5.7. Hence $L_1 = L_2$. □

We also define

$$\mathcal{Q}_U := \{vLv^{-1} : L \in \mathcal{L}_U, v \in \mathrm{N}(U)\}. \tag{5.9}$$

Since $\mathrm{N}(U) = AN\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$ by Lemma 3.2, and the collection $\mathcal{L}_U$ is invariant under a conjugation by an element of $AU\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, we have

$$\mathcal{Q}_U = \{vLv^{-1} : L \in \mathcal{L}_U, v \in U^\perp\}. \tag{5.10}$$

Lemma 5.12. *For $U_0 < U < N$, we have*

$$X(H(U), U_0) = \mathrm{N}_G(H(U))\,\mathrm{N}_G(U_0).$$

Proof. Without loss of generality, we may assume $U = U_m$ and $U_0 = U_\ell$ with $1 \le \ell \le m \le d-1$. Set $H = H(U_m)$. If $m = d-1$, then $H = G$, and the statement is trivial. Assume $m \le d-2$ below. We will prove the inclusion $X(H, U_0) \subset \mathrm{N}_G(H)\,\mathrm{N}_G(U_0)$, as the other one is clear. Let $g \in X(H, U_0)$ be arbitrary. By multiplying g by an element of $\mathrm{N}_G(H)$ on the left as well as by an element of $\mathrm{N}_G(U_0)$ on the right, we will reduce g to an element of $\mathrm{N}_G(U_0)$, which implies the claim. In view of the Iwasawa decomposition $G = KAN$, since $AN < \mathrm{N}_G(U_0)$, we may assume that $g = k \in K$. As $k \in X(H, U_0)$, we have $kU_0k^{-1} \subset H$. Since $K \cap H$ is a maximal compact subgroup of H, any maximal horospherical subgroups of H are conjugate to each other by an element of $K \cap H$. Hence there exists $w \in K \cap H$ such that $kU_0k^{-1} = wU_0w^{-1}$.

Since $w^{-1}kU_0 = U_0w^{-1}k$, we deduce $w^{-1}k(\infty) = U_0(w^{-1}k(\infty))$. Since $\infty \in \mathbb{S}^{d-1}$ is the unique fixed point of U_0, $w^{-1}k(\infty) = \infty$. Hence $w^{-1}k \in K \cap (MAN) = M$. Since $w \in H$, we may now assume that $k \in M$. From $kU_0 \subset Hk$, we get $kU_0(0) \subset Hk(0) = H(0)$ and hence $\langle ke_1, \cdots, ke_\ell\rangle \subset \langle e_1, \cdots, e_m\rangle$. By considering the action of $H \cap K$ on space of ℓ-tuples of orthonormal vectors in the subspace $\langle e_1, \cdots, e_m\rangle$, we may assume $ke_1 = e_1$, $\cdots$, $ke_{\ell-1} = e_{\ell-1}$, and $ke_\ell = \pm e_\ell$. This implies that $k \in \mathrm{C}_1(U_0)$, or $k\omega \in \mathrm{C}_1(U_0)$ where $\omega \in M$ is an involution which fixes all e_i, $i \ne \ell, \ell+1$ and $\omega(e_i) = -e_i$ for $i = \ell, \ell+1$. As $\mathrm{N}_G(U_0)$ contains $\mathrm{C}_1(U_0)$ and ω, the proof is complete. $\square$

Proposition 5.13. *Consider a closed orbit x_0L for $L \in \mathcal{Q}_U$ and $x_0 \in \mathrm{RF}\,\mathcal{M}$. If $x \in \mathscr{S}(U_0, x_0L)$ for a connected closed subgroup $U_0 < U$, then there exists a subgroup $Q \in \mathcal{Q}_{U_0}$ such that*

- $\dim Q_{nc} < \dim L_{nc}$*;*
- *xQ is closed;*
- $\overline{xU_0} \subset xQ$*.*

Proof. If $x = [g] \in \mathscr{S}(U_0, x_0L)$, then $g \in X(H, U_0)$ for some $H \in \mathscr{H}$ such that $\dim H_{nc} < \dim L_{nc}$. Then $\overline{xU_0} \subset x(g^{-1}Hg)$. By Proposition 5.3, $H = qH(\widehat{U})Cq^{-1}$ for some $U_0 < \widehat{U} < L \cap N$ and some $[q] \in \mathrm{RF}\,\mathcal{M}$. Note that $q^{-1}g \in X(H(\widehat{U}), U_0)$. By Lemma 5.12, we have

$$q^{-1}g \in \mathrm{N}_G(H(\widehat{U}))\,\mathrm{N}_G(U_0).$$

Hence $g^{-1}Hg = vH(\widehat{U})Cv^{-1}$ for some $v \in \mathrm{N}_G(U_0)$, and $\overline{xU_0} \subset xvH(\widehat{U})Cv^{-1}$. It suffices to set $Q := vH(\widehat{U})Cv^{-1}$. $\square$

Lemma 5.14. *Let* $L = H(\widehat{U})C$ *for a connected closed subgroup* $\widehat{U} < N$ *and closed subgroup* $C < \mathrm{C}(H(\widehat{U}))$. *Let* $W = g^{-1}H(\tilde{U})C_0 g$ *be a subgroup of* L *where* $g \in L$, $\tilde{U}$ *is a proper connected closed subgroup of* $\widehat{U}$ *and* C_0 *is a closed subgroup of* $H(\tilde{U})$. *Then for any non-trivial closed connected subgroup* $U < \widehat{U}$, $(L \cap X(W,U))H(U)$ *is a nowhere dense subset of* L.

Proof. Write $g = hc \in H(\widehat{U})C$. Note that

$$\begin{aligned} L \cap X(W,U) &= L \cap X(g^{-1}H(\tilde{U})g, U) \\ &= L \cap X(h^{-1}H(\tilde{U})h, U) \\ &= h(L \cap X(H(\tilde{U}), U)) \\ &= h(H(\widehat{U}) \cap X(H(\tilde{U}), U))C. \end{aligned}$$

Hence it suffices to show that $(H(\widehat{U}) \cap X(H(\tilde{U}),U))H(U)$ is a nowhere dense subset of $H(\widehat{U})$. Without loss of generality, we may now assume $H(\widehat{U}) = G$. We observe that using Lemma 5.12,

$$\begin{aligned} &X(H(\tilde{U}),U)H(U) = \mathrm{N}_G(H(\tilde{U}))\,\mathrm{N}_G(U)H(U) \\ &= H(\tilde{U})\,\mathrm{C}_1(\tilde{U})AN\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)H(U) \\ &= (K \cap H(\tilde{U}))U^{\perp}H'(U). \end{aligned}$$

Let $\dim \tilde{U} = m$ and $\dim U = k$. If $k \geq m$, then $X(W,U) = \emptyset$. Hence we may assume that $1 \leq k \leq m < d-1 = \dim N$. Now, if we view the subset $(K \cap H(\tilde{U}))U^{\perp}H'(U)/H'(U)$ in the space $\mathcal{C}^k = G/H'(U)$, this set is contained in the set of all spheres $C \in \mathcal{C}^k$ which are tangent to the m-sphere given by $S_0 := (K \cap H(\tilde{U}))(\infty)$. Since $m < d-1$, it follows that $X(H(\tilde{U}),U)H(U)/H'(U)$ is a nowhere dense subset of $\mathcal{C}^k$, and hence $X(H(\tilde{U}),U)H(U)$ is a nowhere dense subset of G. □

Recall from (4.1) that $F = \mathrm{RF}_+\mathcal{M} \cdot H(U)$.

Lemma 5.15. *Let* $x_0\widehat{L}$ *be a closed orbit of* $\widehat{L} \in \mathcal{L}_U$ *with* $x_0 \in \mathrm{RF}\,\mathcal{M}$. *If* U *is a proper subgroup of* $\widehat{L} \cap N$, *then* $\mathscr{S}(U, x_0\widehat{L}) \cdot H(U) \cap F_{H(U)}$ *is a proper subset of* $x_0\widehat{L} \cap F_{H(U)}$.

Proof. Choose $g_0 \in G$ so that $x_0 = [g_0]$. Let $p : G \to \Gamma\backslash G$ be the canonical projection map. Then $p^{-1}(\mathscr{S}(U, x_0\widehat{L}) \cdot H(U))$ is a countable union $\gamma g_0(\widehat{L} \cap X(W,U))H(U)$ where $\gamma \in \Gamma$ and $W \in \mathscr{H}^{\star}_{x_0\widehat{L}}$ by (5.6). Hence by Lemma 5.14, $\mathscr{S}(U, x_0\widehat{L}) \cdot H(U)$ is a countable union of nowhere dense subsets of $x_0 L$. Since $F^*_{H(U)} \cap x_0\widehat{L}$ is an open subset of $x_0\widehat{L}$, it follows from the Baire category theorem that $F^*_{H(U)} \cap x_0\widehat{L} \not\subset \mathscr{S}(U, x_0\widehat{L}) \cdot H(U)$. This proves the claim. □

The following geometric property of a convex cocompact hyperbolic manifold with Fuchsian ends is one of its key features which is needed in the proof of our main theorems stated in the introduction.

Proposition 5.16. *Let $\mathcal{M}$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Let $x_0\widehat{L}$ be a closed orbit of $\widehat{L} \in \mathcal{L}_U$ with $x_0 \in \mathrm{RF}\,\mathcal{M}$ and with $\dim(\widehat{L} \cap N) \geq 2$. Either $x_0\widehat{L}$ is compact or $\mathscr{S}(U, x_0\widehat{L})$ contains a compact orbit zL_0 with $L_0 \in \mathcal{L}_U$.*

Proof. Write $\widehat{L} = H(\widehat{U})C$ for a connected closed subgroup $U < \widehat{U} < N$. Since $x_0\widehat{L}$ is closed, $\pi(x_0\widehat{L}) = \pi(x_0H'(\widehat{U}))$ is a properly immersed convex cocompact geodesic plane of dimension at least 3 with Fuchsian ends by Proposition 4.7. Suppose that x_0L is not compact. Then $\pi(x_0L)$ has non-empty Fuchian ends. This means that there exist a co-dimension one subgroup U_0 of $\widehat{U}$ and $z \in \widehat{L}$ such that $zH'(U_0)$ is compact and $\pi(zH'(U_0))$ is a component of the core of $\pi(x_0\widehat{L})$. By Proposition 4.9, there exists a closed subgroup $C_0 < \mathrm{C}(H(U_0)) \cap \widehat{L}$ such that $H(U_0)C_0 \in \mathcal{L}_{U_0}$ and $zH(U_0)C_0$ is compact. Let $m \in M \cap \widehat{L}$ be an element such that $U \subset m^{-1}U_0m$. Then $zm(m^{-1}H(U_0)C_0m)$ is a compact orbit contained in $\mathscr{S}(U, x_0\widehat{L})$ and $m^{-1}H(U_0)C_0m \in \mathcal{L}_U$, finishing the proof. □

6. Inductive search lemma

In this section, we prove a combinatorial lemma 6.4, which we call an *inductive search lemma*, and use it to prove Proposition 6.3 on the thickness of a certain subset of $\mathbb{R}$, constructed by the intersection of a global thick subset T and finite families of triples of subsets of $\mathbb{R}$ with controlled regularity, degree and the multiplicity with respect to T. This proposition will be used in the proof of the avoidance theorem 7.13 in the next section.

Definition 6.1. Let $J^* \subset I$ be a pair of open subsets of $\mathbb{R}$.

- The degree of (I, J^*) is defined to be the minimal $\delta \in \mathbb{N} \cup \{\infty\}$ such that for each connected component I° of I, the number of connected components of J^* contained in I° is bounded by δ.
- For $\beta > 0$, the pair (I, J^*) is said to be *β-regular* if for any connected component I° of I, and any component J° of $J^* \cap I^\circ$,

$$J^\circ \pm \beta \cdot |J^\circ| \subset I^\circ$$

  where $|J^\circ|$ denotes the length of J°.

Definition 6.2. Let $\mathcal{X}$ be a family of countably many triples (I, J^*, J') of open subsets of $\mathbb{R}$ such that $I \supset J^* \supset J'$.

- Given $\beta > 0$ and $\delta \in \mathbb{N}$, we say that $\mathcal{X}$ is *β-regular of degree δ* if for every triple $(I, J^*, J') \in \mathcal{X}$, the pair (I, J^*) is β-regular with degree at most δ.

- Given a subset $\mathsf{T} \subset \mathbb{R}$, we say that $\mathcal{X}$ is of T-*multiplicity free* if for any two distinct triples (I_1, J_1^*, J_1') and (I_2, J_2^*, J_2') of $\mathcal{X}$, we have
$$I_1 \cap J_2' \cap \mathsf{T} = \emptyset.$$

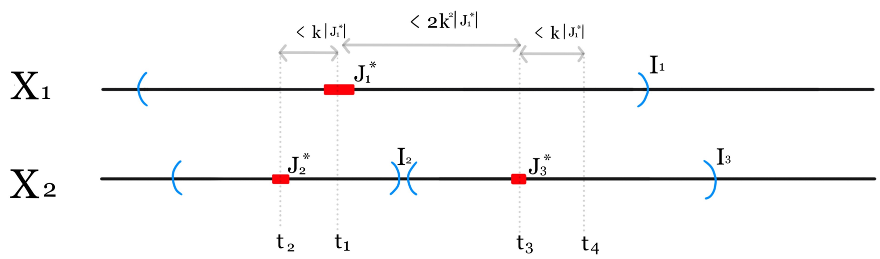
For a family $\mathcal{X} = \{(I_\lambda, J_\lambda^*, J_\lambda') : \lambda \in \Lambda\}$, we will use the notation
$$I(\mathcal{X}) := \bigcup_{\lambda\in\Lambda} I_\lambda, \quad J^*(\mathcal{X}) := \bigcup_{\lambda\in\Lambda} J_\lambda^* \quad \text{and} \quad J'(\mathcal{X}) := \bigcup_{\lambda\in\Lambda} J_\lambda'.$$

The goal of this section is to prove:

Proposition 6.3 (Thickness of $\mathsf{T} - J'(\mathcal{X})$)**.** *Given $n, k, \delta \in \mathbb{N}$, there exists a positive number $\beta_0 = \beta_0(n, k, \delta)$ for which the following holds: let $\mathsf{T} \subset \mathbb{R}$ be a globally k-thick set, and let $\mathcal{X}_1, \cdots, \mathcal{X}_\ell$, $\ell \le n$, be β_0-regular families of degree δ and of T-multiplicity free. Let $\mathcal{X} = \bigcup_{i=1}^{\ell} \mathcal{X}_i$. If $0 \in \mathsf{T} - I(\mathcal{X})$, then*
$$\mathsf{T} - J'(\mathcal{X})$$
is a $2k$-thick set.

We prove Proposition 6.3 using the inductive search lemma 6.4. The case of $n = 1$ and $\delta = 1$ is easy. As the formulation of the lemma is rather complicated in a general case, we first explain a simpler case of $n = 2$ and $\delta = 1$ in order to motivate the statement.

For simplicity, let us show that $\mathsf{T} - (J'(\mathcal{X}_1) \cup J'(\mathcal{X}_2))$ is $4k$-thick instead of $2k$-thick, given that $\mathcal{X}_1$ and $\mathcal{X}_2$ are $8k^2$-regular families of degree 1, and of T-multiplicity free. For any $r > 0$, we need to find a point
$$t \in \pm(r, 4kr) \cap \Big(\mathsf{T} - J'(\mathcal{X})\Big)$$
where $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$.



First, we know that there exists $t_1 \in \pm(2r, 2kr) \cap \mathsf{T}$, as T is locally k-thick at 0. If $t_1 \notin J'(\mathcal{X}_1) \cup J'(\mathcal{X}_2)$, then we are done. So we assume that $t_1 \in J'(\mathcal{X}_1)$. Our strategy is then to search for a sequence in T of length at most 4, starting with t_1, say (t_1, t_2, t_3, t_4) such that
$$\frac{|t_{i-1}|}{\sqrt[3]{2}} \le |t_i| \le \sqrt[3]{2}|t_{i-1}| \quad \text{for each } i = 2, 3, 4,$$

and the last element t_4 does not belong to $J'(\mathcal{X})$. This will imply $|t_1|/2 \leq |t_4| \leq 2|t_1|$ and hence

$$t := t_4 \in \pm(r, 4kr) \cap \Big(\mathsf{T} - J'(\mathcal{X})\Big)$$

as desired, because $2r \leq |t_1| \leq 2kr$.

We next sketch how we find t_2 from t_1 and so on. Let $t_1 \in J_1'$ where $(I_1, J_1^*, J_1') \in \mathcal{X}_1$. Since T is locally k-thick at t_1, there exists

$$t_2 \in (t_1 \pm (|J_1^*|, k|J_1^*|)) \cap \mathsf{T}. \tag{6.1}$$

We will refer to t_1 as a pivot for searching t_2 in (6.1), as t_2 was found in a symmetric interval around t_1. Note that $t_2 \in I_1 - J_1^*$ as (I_1, J_1^*) is k-regular. This implies that $t_2 \notin J'(\mathcal{X}_1)$ as the family $\mathcal{X}_1$ is of T-multiplicity free. Now we will assume $t_2 \in J_2'$ for some triple $(I_2, J_2^*, J_2') \in \mathcal{X}_2$, since otherwise, $t_2 \notin J'(\mathcal{X})$ and we are done.

To search for the next point $t_3 \in \mathsf{T}$, we choose our pivot between two candidates t_1 and t_2 as follows: we will choose t_1 if $|J_1^*| \geq |J_2^*|$, and t_2 otherwise. Without loss of generality, we will assume $|J_1^*| \geq |J_2^*|$. Since T is locally k-thick at t_1, we can find

$$t_3 \in (t_1 \pm 2k(|J_1^*|, k|J_1^*|)) \cap \mathsf{T}.$$

Note that $t_3 \in I_1 - J_1^*$ as the pair (I_1, J_1^*) is $2k^2$-regular. This implies $t_3 \notin J'(\mathcal{X}_1)$ as $\mathcal{X}_1$ is of T-multiplicity free. Now we can assume that $t_3 \in J_3'$ for some $(I_3, J_3^*, J_3') \in \mathcal{X}_2$, otherwise we are done. One can check that J_3^* cannot coincide with J_2^*. We claim that $|J_1^*| \geq |J_3^*|$. Suppose not, i.e. $|J_3^*| > |J_1^*|$. Then we would have $|t_2 - t_1| < k|J_3^*|$ and $|t_1 - t_3| < 2k^2|J_3^*|$, which implies that $t_2 \in I_3$, as the pair (I_3, J_3^*) is $(2k^2 + k)$-regular. This is a contradiction as $\mathcal{X}_2$ is T-multiplicity free and hence $J_2' \cap I_3 \cap \mathsf{T} = \emptyset$.

Finally, we will choose t_3 as a pivot and search for t_4. By the local k-thickness of T at t_3, we can find

$$t_4 \in (t_3 \pm (|J_3^*|, k|J_3^*|)) \cap \mathsf{T}.$$

Since the pair (I_3, J_3^*) is k-regular, we have $t_4 \in I_3 - J_3^*$. From the fact that the pair (I_1, J_1^*) is $(2k^2 + k)$-regular, one can check that $t_4 \in I_1 - J_1^*$. As a result, $t_4 \in (I_1 - J_1^*) \cup (I_3 - J_3^*)$ and hence $t_4 \notin J'(\mathcal{X})$.

It remains to check that $|t_{i-1}|/\sqrt[3]{2} \leq |t_i| \leq \sqrt[3]{2}|t_{i-1}|$ for each $i = 2, 3, 4$. This does not necessarily hold for the current sequence, but will hold after passing to a subsequence where t_{i-1} becomes a pivot for searching t_i for all i. In the previous case, (t_1, t_3, t_4) will be such a subsequence, as t_2 was not a pivot for searching t_3.

It follows from the $\beta := 8k^2$-regularity of (I_{i-1}, J_{i-1}^*) that $|t_{i-1}| - 8k^2|J_{i-1}^*| > 0$, as $t_{i-1} \in J_{i-1}^*$ and $0 \notin I_{i-1}$. On the other hand, observe that

$$t_i \in t_{i-1} \pm C_i(|J_{i-1}^*|, k|J_{i-1}^*|) \cap \mathsf{T}$$

for some $C_i \leq 2k^2$. This gives us the desired upper bound for $|t_i/t_{i-1}|$, as

$$|t_i| < |t_{i-1}| + C_i|J_{i-1}^*| \leq (1 + C_i(8k^2)^{-1})|t_{i-1}|$$

and $1+C_i(8k^2)^{-1} \le \sqrt[3]{2}$. The lower bound is obtained similarly, completing the proof for $n=2$ and $\delta=1$.

The general case reduces to the case of $\delta = 1$, by replacing n by $n\delta$. Roughly speaking, the following lemma gives an inductive argument for the search of a sequence of t_i's which is almost geometric in a sense that the ratio $|t_i|/|t_{i-1}|$ is coarsely a constant and which lands on $\mathsf{T}-J'(\mathcal{X})$ in a time controlled by n.

Lemma 6.4 (Inductive search lemma)**.** *Let $k>1$, $n\in\mathbb{N}$ and $0<\varepsilon<1$ be fixed. There exists $\beta=\beta(n,k,\varepsilon)>0$ for which the following holds: Let $\mathsf{T}\subset\mathbb{R}$ be a globally k-thick set, and let $\mathcal{X}_1,\cdots,\mathcal{X}_n$ be β-regular families of countably many triples $(I_\lambda, J^*_\lambda, J'_\lambda)$ with degree 1, and of T-multiplicity free. Set $\mathcal{X}=\mathcal{X}_1\cup\cdots\cup\mathcal{X}_n$, and assume $0\notin I(\mathcal{X})$. For any $t\in\mathsf{T}\cap J'(\mathcal{X})$ and any $1\le r\le n$, we can find distinct triples $(I_1,J^*_1,J'_1),\cdots,(I_{m-1},J^*_{m-1},J'_{m-1})\in\mathcal{X}$ with $2\le m\le 2^r$, and a sequence of pivots*

$$t=t_1\in\mathsf{T}\cap J'_1,\ t_2\in\mathsf{T}\cap J'_2,\cdots,t_{m-1}\in\mathsf{T}\cap J'_{m-1},\ t_m\in\mathsf{T}$$

which satisfy the following conditions:

(1) *either $t_m\notin J'(\mathcal{X})$, or $t_m\in J'_m$ for some $(I_m,J^*_m,J'_m)\in\mathcal{X}$, which is distinct from (I_i,J^*_i,J'_i) for all $1\le i\le m-1$, and the collection $\{(I_i,J^*_i,J'_i):1\le i\le m\}$ intersects at least $(r+1)$ number of $\mathcal{X}_i$'s;*

(2) *for all $1\le i\le j\le m$,*

$$|t_i-t_j|\le 2((4k)^r-1)k\max_{1\le p\le j-1}|J^*_p|;$$

(3) *for each $1\le i\le m$,*

$$(1-\varepsilon)^{i-1}|t_1|\le|t_i|\le(1+\varepsilon)^{i-1}|t_1|.$$

In particular, for any $t\in\mathsf{T}\cap J'(\mathcal{X})$, there exists $t'\in\mathsf{T}-J'(\mathcal{X})$ such that

$$(1-\varepsilon)^{2^n-1}|t|\le|t'|\le(1+\varepsilon)^{2^n-1}|t|.$$

Proof. We set

$$\beta=\beta(n,k,\varepsilon)=(4k)^{n+1}\varepsilon^{-1}. \tag{6.2}$$

Consider the increasing sequence $Q(r):=(4k)^r-1$ for $r\in\mathbb{N}$. Note that

$$Q(1)\ge 2\quad\text{and}\quad Q(r+1)\ge 4Q(r)k+1.$$

Moreover we check that

$$\beta>\max((Q(n)+4Q(n-1))k, Q(n)k\varepsilon^{-1}).$$

We proceed by induction on r. First consider the case when $r=1$. There exists $(I_1,J^*_1,J'_1)\in\mathcal{X}$ such that $t_1:=t\in J'_1\cap\mathsf{T}$. As T is globally k-thick, we can choose

$$t_2\in\big(t_1\pm Q(1)(|J^*_1|,k|J^*_1|)\big)\cap\mathsf{T}. \tag{6.3}$$

We claim that t_1,t_2 is our desired sequence with $m=2$. In the case when $t_2\in J'(\mathcal{X})$, there exists $(I_2,J^*_2,J'_2)\in\mathcal{X}$ such that $t_2\in J'_2$. We check:

(1): If $t_2 \in J'(\mathcal{X})$, then $t_2 \in J_2' - J_1^*$ implies that J_1^* and J_2^* are distinct. Hence (I_1, J_1^*, J_1') and (I_2, J_2^*, J_2') are distinct as well. Since $\beta > Q(1)k$, by the β-regularity of (I_1, J_1^*), we have $t_2 \in I_1$. By the T-multiplicity free condition, (I_1, J_1^*, J_1') and (I_2, J_2^*, J_2') don't belong to the same family, that is, $\{(I_1, J_1^*, J_1'), (I_2, J_2^*, J_2')\}$ intersects two of $\mathcal{X}_i$'s.

(2): By (6.3), $|t_1 - t_2| < Q(1)k|J_1^*| = (4k-1)k|J_1^*|$.

(3): Note that $0 \notin I_1$, since $0 \notin I(\mathcal{X})$. By the β-regularity of (I_1, J_1^*), we have $t_1 \pm \beta|J_1^*| \subset I_1$. Since $0 \notin I_1$ and $\beta > \varepsilon^{-1}Q(1)k$, we have

$$|t_1| - \varepsilon^{-1}Q(1)k|J_1^*| > 0.$$

On the other hand, by (6.3),

$$|t_2 - t_1| \leq Q(1)k|J_1^*| \leq \varepsilon|t_1|.$$

In particular,

$$\begin{aligned} |t_2| &\leq |t_1| + |t_2 - t_1| < |t_1| + Q(1)k|J_1^*| < (1+\varepsilon)\,|t_1| \text{ and} \\ |t_2| &\geq |t_1| - |t_2 - t_1| > |t_1| - Q(1)k|J_1^*| > (1-\varepsilon)\,|t_1|. \end{aligned}$$

This proves the base case of $r = 1$.

Next, assume the induction hypothesis for r. Hence we have a sequence

$$t_1(=t) \in J_1',\ t_2 \in J_2',\ \cdots,\ t_{m-1} \in J_{m-1}',\ \text{and } t_m$$

in T with $m \leq 2^r$ together with $\{(I_i, J_i^*, J_i') : 1 \leq i \leq m-1\}$ satisfying the three conditions listed in the lemma. If $t_m \notin J'(\mathcal{X})$, the same sequence would satisfy the hypothesis for $r+1$ and we are done. Now we assume that $t_m \in J_m'$ for some $(I_m, J_m^*, J_m') \in \mathcal{X}$, and that $\{(I_i, J_i^*, J_i') : 1 \leq i \leq m\}$ intersect at least $(r+1)$ numbers of $\mathcal{X}_i$'s. We may assume that they intersect exactly $(r+1)$-number of $\mathcal{X}_i$'s, which we may label as $\mathcal{X}_1, \cdots, \mathcal{X}_{r+1}$, since if they intersect more than $(r+1)$ of them, we are already done. Choose a largest interval J_ℓ^* among $J_1^*, \cdots, J_m^*$. Again using the global k-thickness of T, we can choose

$$s_1 \in \big(t_\ell \pm Q(r+1)(|J_\ell^*|, k|J_\ell^*|)\big) \cap \mathsf{T}. \tag{6.4}$$

First, consider the case when $s_1 \notin J'(\mathcal{X})$. We will show that the points $t_1, \cdots, t_m, s_1$ give the desired sequence. Indeed, the condition (1) is immediate. For (2), observe that by the induction hypothesis for r, we have

$$|s_1 - t_i| \leq |s_1 - t_\ell| + |t_\ell - t_i| \leq (Q(r+1)k + 2Q(r)k)|J_\ell^*|$$

for all $1 \leq i \leq m$. The conclusion follows as $Q(r+1) > 2Q(r)$. To show (3), since $\beta > \varepsilon^{-1}Q(r+1)k$ and $0 \notin I_\ell$, by applying the β-regularity to the pair (I_ℓ, J_ℓ^*), we have

$$|t_\ell| - \varepsilon^{-1}Q(r+1)k|J_\ell^*| > 0.$$

It follows that

$$\begin{aligned} |s_1| &\leq |t_\ell| + |s_1 - t_\ell| < |t_\ell| + Q(r+1)k|J_\ell^*| < (1+\varepsilon)\,|t_\ell| \leq (1+\varepsilon)^m\,|t_1|; \\ |s_1| &\geq |t_\ell| - |s_1 - t_\ell| > |t_\ell| - Q(r+1)k|J_\ell^*| > (1-\varepsilon)\,|t_\ell| \geq (1-\varepsilon)^m\,|t_1|. \end{aligned}$$

This proves (3).

For the rest of the proof, we now assume that $s_1 \in J'(\mathcal{X})$. Apply the induction hypothesis for r to $s_1 \in \mathsf{T} \cap J'(\mathcal{X})$ to obtain a sequence $\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') \in \mathcal{X} : 1 \le j \le m'-1\}$ with $m' \le 2^r$ and

$$s_1 \in \tilde{J}_1' \cap \mathsf{T},\ s_2 \in \tilde{J}_2' \cap \mathsf{T},\ \cdots,\ s_{m'-1} \in \tilde{J}_{m'-1}' \cap \mathsf{T},\ \text{and } s_{m'} \in \mathsf{T}.$$

Set q_0 to be the smallest $1 \le q \le m'-1$ satisfying

$$\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le q\} \not\subset \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_{r+1} \tag{6.5}$$

if it exists, and $q_0 := m'$ otherwise. We claim that the sequence

$$t_1, \cdots, t_m, s_1, \cdots, s_{q_0} \tag{6.6}$$

of length $m + q_0 \le 2^{r+1}$ satisfies the conditions of the lemma for $r+1$.

Claim: We have

$$|J_\ell^*| = \max_{1 \le i \le m, 1 \le j \le q_0 - 1} (|J_i^*|, |\tilde{J}_j^*|). \tag{6.7}$$

Recall that $|J_\ell^*|$ was chosen to be maximal among $|J_1^*|, \cdots, |J_m^*|$. Hence, if the claim does not hold, then we can take j to be the least number such that $|\tilde{J}_j^*| > |J_\ell^*|$. Then by the induction hypothesis for (2),

$$\begin{aligned} |t_\ell - s_j| &\le |t_\ell - s_1| + |s_1 - s_j| \\ &\le Q(r+1)k|J_\ell^*| + 2Q(r)k \max_{1 \le i \le j-1} |\tilde{J}_i^*| \\ &\le (Q(r+1) + 2Q(r))k|J_\ell^*|. \end{aligned}$$

Now the collection $\{(I_i, J_i^*, J_i') : 1 \le i \le m\}$ intersects $(r+1)$ families $\mathcal{X}_1, \cdots, \mathcal{X}_{r+1}$ and $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$ belongs to one of these families, as $j \le q_0 - 1$. Hence there exists a triple (I_i, J_i^*, J_i') that belongs to the same family as $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$. Recall that the induction hypothesis for $t_1, \cdots, t_m$ gives us

$$|t_\ell - t_i| \le 2Q(r)k|J_\ell^*|.$$

Since $\beta > (Q(r+1) + 4Q(r))k$, we have

$$\begin{aligned} |t_i - s_j| &\le |t_i - t_\ell| + |t_\ell - s_j| \\ &\le (Q(r+1) + 4Q(r))k|J_\ell^*| \\ &< \beta|\tilde{J}_j^*|. \end{aligned}$$

Applying the β-regularity to the pair $(\tilde{I}_j, \tilde{J}_j^*)$, we conclude that

$$t_i \in \tilde{I}_j \cap J_i' \cap \mathsf{T}.$$

Since $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$ and (I_i, J_i^*, J_i') belong to the same family which is T-multiplicity free, they are equal to each other. This is a contradiction since $|\tilde{J}_j^*| > |J_\ell^*| \ge |J_i^*|$, proving the claim (6.7).

We next prove that (I_i, J_i^*, J_i') and $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$ are distinct for all $1 \le i \le m$ and $1 \le j \le q_0 - 1$. It suffices to check that J_i^* and $\tilde{J}_j^*$ are distinct. Note that we have

$$\max_{1\le i,j\le m} |t_i - t_j| < 2Q(r)k|J_\ell^*| \quad \text{and}$$
$$\max_{1\le i,j\le q_0} |s_i - s_j| < 2Q(r)k|J_\ell^*|$$

by the induction hypothesis together with claim (6.7). Now for $t_i \in J_i^*(1 \le i \le m)$ and $s_j \in \tilde{J}_j^*(1 \le j < q_0)$, we estimate:

$$\begin{aligned} |s_j - t_i| &\ge |s_1 - t_\ell| - |t_i - t_\ell| - |s_1 - s_j| \\ &> Q(r+1)|J_\ell^*| - 2Q(r)k|J_\ell^*| - 2Q(r)k|\tilde{J}_\ell^*| \\ &= (Q(r+1) - 4Q(r)k)|J_\ell^*| \\ &\ge |J_\ell^*|. \end{aligned} \tag{6.8}$$

This in particular means that $s_j \notin J_i^*$ and $t_i \notin \tilde{J}_j^*$. Hence $J_i^* \neq \tilde{J}_j^*$.

We now begin checking the conditions (1), (2) and (3).

(1): If $s_{q_0} \notin J'(\mathcal{X})$, there is nothing to check.

Now assume that $s_{q_0} \in \tilde{J}'_{q_0}$ for some $(\tilde{I}_{q_0}, \tilde{J}^*_{q_0}, \tilde{J}'_{q_0}) \in \mathcal{X}$. If $q_0 < m'$, then again there is nothing to prove, as the union

$$\{(I_i, J_i^*, J_i') : 1 \le i \le m\} \cup \{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le q_0\} \tag{6.9}$$

intersects a family other than $\mathcal{X}_1, \cdots, \mathcal{X}_{r+1}$. Hence we will assume $q_0 = m'$. By the induction hypothesis for r on the sequence $(s_1, \cdots, s_{m'})$, the family $\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le m'\}$ consists of pairwise distinct triples intersecting at least $(r+1)$ numbers of $\mathcal{X}_i$'s. Observe that in the estimate (6.8), there is no harm in allowing $j = q_0$ in addition to $j < q_0$. This shows that $\tilde{J}^*_{m'}$ is also distinct from all J_i^*'s. Hence the the triples in (6.9) are all distinct.

Now, unless the following inclusion

$$\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le m'\} \subset \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_{r+1}, \tag{6.10}$$

holds, we are done. Suppose that (6.10) holds. We will deduce a contradiction. Without loss of generality, we assume that

$$(I_\ell, J_\ell^*, J_\ell') \in \mathcal{X}_{r+1}.$$

We now claim that the following inclusion holds:

$$\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le m'\} \subset \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_r. \tag{6.11}$$

Note that this gives the desired contradiction, since $\{(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') : 1 \le j \le m'\}$ must intersect at least $(r+1)$ number of $\mathcal{X}_i$ by the induction hypothesis. In order to prove the inclusion (6.11), suppose on the contrary that $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j') \in \mathcal{X}_{r+1}$ for some $1 \le j \le m'$. Using $\beta > (Q(r+1) + 2Q(r))k$ and (6.7), we deduce

$$\begin{aligned} |t_\ell - s_j| &\le |t_\ell - s_1| + |s_1 - s_j| \\ &\le Q(r+1)k|J_\ell^*| + 2Q(r)k|J_\ell^*| \\ &< \beta|J_\ell^*| \end{aligned}$$

where we used the induction hypothesis for the sequence $(s_1, \cdots, s_{m'})$ in the second line, to estimate the term $|s_1 - s_j|$.

Next, applying the β-regularity to the pair (I_ℓ, J_ℓ^*), we conclude that $s_j \in I_\ell$. Since $s_j \in \tilde{J}_j'$, it follows that $I_\ell \cap \tilde{J}_j' \cap \mathsf{T} \neq \emptyset$. This contradicts the condition that $\mathcal{X}_{r+1}$ is of T-multiplicity free, as both $(\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')$ and $(I_\ell, J_\ell^*, J_\ell')$ belong to the same family $\mathcal{X}_{r+1}$. This completes the proof of (1).

(2): For $1 \leq i \leq m$ and $1 \leq j \leq q_0$, observe that

$$\begin{aligned} |t_i - s_j| &\leq |t_i - t_\ell| + |t_\ell - s_1| + |s_1 - s_j| \\ &\leq 2Q(r)k|J_\ell^*| + Q(r+1)k|J_\ell^*| + 2Q(r)k|J_\ell^*| \\ &< 2Q(r+1)k|J_\ell^*| \end{aligned}$$

as $Q(r+1) > 4Q(r)$. Hence we get the desired result by (6.7).

(3): We already have observed that the inequality $\beta > \varepsilon^{-1}Q(r+1)k$ implies that

$$(1-\varepsilon)^m |t_1| \leq |s_1| \leq (1+\varepsilon)^m |t_1|.$$

Combining this with the induction hypothesis, we deduce that

$$(1-\varepsilon)^{m+i-1}|t_1| \leq |s_i| \leq (1+\varepsilon)^{m+i-1}|t_1|$$

for all $1 \leq i \leq q_0$.

Finally, the last statement of the lemma is obtained from the case $r = n$, since there are only n-number of $\mathcal{X}_i$'s; hence the second possibility of (1) cannot arise for $r = n$. □

Proof of Proposition 6.3. We may assume that $\mathcal{X}_i$'s are all of degree 1, by replacing each $\mathcal{X}_i$'s with δ-number of families associated to it.

We set

$$\beta_0(n, k, 1) = (4k)^{n+1}\varepsilon^{-1}$$

where ε satisfies $\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2^n-1} \leq 2$. Note that $\beta_0(n, k, 1)$ is equal to the number given in (6.2). We may assume $x = 0$ without loss of generality. Let $\lambda > 0$. We need to find a point

$$t' \in \Big([-2k\lambda, -\lambda] \cup [\lambda, 2k\lambda]\Big) \cap \Big(\mathsf{T} - \bigcup_{i \in \Lambda} J'(\mathcal{X}_i)\Big). \tag{6.12}$$

Choose $s > 0$ such that

$$(1-\varepsilon)^{-(2^n-1)}\lambda \leq s \leq 2(1+\varepsilon)^{-(2^n-1)}\lambda. \tag{6.13}$$

Since T is globally k-thick, there exists

$$t \in ([-ks, -s] \cup [s, ks]) \cap \mathsf{T}.$$

If $t \notin \bigcup_{i=1}^n J'(\mathcal{X}_i)$, then by choosing $t' = t$, we are done. Now suppose $t \in \bigcup_{i=1}^n J'(\mathcal{X}_i)$. Since $0 \notin \bigcup_{i=1}^n I(\mathcal{X}_i)$, by applying Lemma 6.4 to $t \in \mathsf{T} \cap (\bigcup_{i=1}^n J'(\mathcal{X}_i))$, we obtain $t' \in \mathsf{T} - \bigcup_{i=1}^n J'(\mathcal{X}_i)$ such that

$$(1-\varepsilon)^{2^n-1}|t| \leq |t'| \leq (1+\varepsilon)^{2^n-1}|t|.$$

Note that

$$|t'| \leq (1+\varepsilon)^{2^n-1}|t| \leq (1+\varepsilon)^{2^n-1}ks \leq 2k\lambda.$$

Similarly, we have

$$|t'| \geq (1-\varepsilon)^{2^n-1}|t| \geq (1-\varepsilon)^{2^n-1}s \geq \lambda.$$

This completes the proof since t' satisfies (6.12).

7. Avoidance of the singular set

Let $\Gamma < G$ be a convex cocompact non-elementary subgroup and let

$$U = \{u_t\} < N$$

be a one-parameter subgroup. Let $\mathscr{S}(U)$, $\mathscr{G}(U)$, $X(H,U)$, and $\mathscr{H}^\star$ be as defined in section 5. In particular, $\mathscr{S}(U)$ is a countable union:

$$\mathscr{S}(U) = \bigcup_{H\in\mathscr{H}^\star} \Gamma\backslash\Gamma X(H,U).$$

The main goal of this section is to prove the avoidance Theorem 7.13 for any convex cocompact hyperbolic manifold with Fuchsian ends. For this, we extend the linearization method developed by Dani and Margulis [12] to our setting. Via a careful analysis of the graded self-intersections of the union $\bigcup_i \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\,\mathcal{M}$ for finitely many groups $H_i \in \mathscr{H}^\star$ and compact subsets $D_i \subset X(H_i,U)$, we construct families of triples of subsets of $\mathbb{R}$ satisfying the conditions of Proposition 6.3 relative to the global k-thick subset of the return time to $\mathrm{RF}\,\mathcal{M}$ under U given in Proposition 4.11.

Linearization. Let $H \in \mathscr{H}^\star$. Then H is reductive, algebraic, and is equal to $\mathrm{N}_G(H)$ by Proposition 5.3 and (3.1). There exists an $\mathbb{R}$-regular representation $\rho_H : G \to \mathrm{GL}(V_H)$ with a point $p_H \in V_H$, such that $H = \mathrm{Stab}_G(p_H)$ and the orbit $p_H G$ is Zariski closed [2, Theorem 3.5]. Since $\Gamma\backslash\Gamma H$ is closed, it follows that

$$p_H\Gamma$$

is a closed (and hence discrete) subset of V_H.

Let $\eta_H : G \to V_H$ denote the orbit map defined by

$$\eta_H(g) = p_H g \quad \text{for all } g \in G.$$

As H and U are algebraic subgroups, the set $X(H,U) = \{g \in G : gUg^{-1} \subset H\}$ is Zariski closed in G. Since $p_H G$ is Zariski closed in V_H, it follows that $A_H := p_H X(H,U)$ is Zariski closed in V_H and $X(H,U) = \eta_H^{-1}(A_H)$.

Following [16], for given $C > 0$ and $\alpha > 0$, a function $f : \mathbb{R} \to \mathbb{R}$ is called (C,α)-good if for any interval $I \subset \mathbb{R}$ and $\varepsilon > 0$, we have

$$\ell\{t \in I : |f(t)| \leq \varepsilon\} \leq C \cdot \left(\frac{\varepsilon}{\sup_{t\in I}|f(t)|}\right)^\alpha \cdot \ell(I)$$

where ℓ is a Lebesgue measure on $\mathbb{R}$.

Lemma 7.1. *For given $C > 1$ and $\alpha > 0$, consider continuous functions $p_1, p_2, \cdots, p_k : \mathbb{R} \to \mathbb{R}$ satisfying the (C, α)-good property. For $0 < \delta < 1$, set*

$$I = \{t \in \mathbb{R} : \max_i |p_i(t)| < 1\} \quad \text{and} \quad J(\delta) = \{t \in \mathbb{R} : \max_i |p_i(t)| < \delta\}.$$

For any $\beta > 1$, there exists $\delta = \delta(C, \alpha, \beta) > 0$ such that the pair $(I, J(\delta))$ is β-regular (see Def. 6.2).

Proof. We prove that the conclusion holds for $\delta := ((1+\beta)C)^{-1/\alpha}$. First, note that the function $q(t) := \max_i |p_i(t)|$ also has the (C, α)-good property. Let $J' = (a, b)$ be a component of $J(\delta)$, and I' be the component of I containing J'. Note that I' is an open interval and $(a, \infty) \cap I' = (a, c)$ for some $b \le c \le \infty$. We claim

$$J' + \beta|J'| \subset (a, \infty) \cap I' \subset I'. \tag{7.1}$$

We may assume that $c < \infty$; otherwise the inclusion is trivial. We claim that $q(c) = 1$. Since $\{t \in \mathbb{R} : q(t) < 1\}$ is open and c is the boundary point of I', we have $q(c) \ge 1$. If $q(c)$ were strictly bigger than 1, since $\{t \in \mathbb{R} : q(t) > 1\}$ is open, I' would be disjoint from an open interval around c, which is impossible. Hence $q(c) = 1$. Now that $\sup\{q(t) : t \in (a, \infty) \cap I'\} = q(c) = 1$, by applying the (C, α)-good property of q on the interval $(a, \infty) \cap I'$, we get

$$\begin{aligned} \ell(J') &\le \ell\{t \in (a, \infty) \cap I' : |q(t)| \le \delta\} \\ &\le C\delta^\alpha \cdot \ell((a, \infty) \cap I'). \end{aligned}$$

Now as $J' = (a, b)$ and $(a, \infty) \cap I'$ are nested intervals with one common endpoint, it follows from the equality $C\delta^\alpha = 1/(1+\beta)$ that

$$J' + \beta|J'| \subset (a, \infty) \cap I' \subset I',$$

proving (7.1). Similarly, applying the (C, α)-good property of q on $(-\infty, b) \cap I'$, we deduce that

$$J' - \beta|J'| \subset I'.$$

This proves that $(I, J(\delta))$ is β-regular. $\square$

Proposition 7.2. *Let V be a finite dimensional real vector space, $\theta \in \mathbb{R}[V]$ be a polynomial and $A = \{v \in V : \theta(v) = 0\}$. Then for any compact subset $D \subset A$ and any $\beta > 0$, there exists a compact neighborhood $D' \subset A$ of D which has a β-regular size with respect to D in the following sense: for any neighborhood Φ of D', there exists a neighborhood $\Psi \subset \Phi$ of D such that for any $q \in V - \Phi$ and for any one-parameter unipotent subgroup $\{u_t\} \subset \mathrm{GL}(V)$, the pair $(I(q), J(q))$ is β-regular where*

$$I(q) = \{t \in \mathbb{R} : qu_t \in \Phi\} \text{ and } J(q) = \{t \in \mathbb{R} : qu_t \in \Psi\}.$$

Furthermore, the degree of $(I(q), J(q))$ is at most $(\deg \theta + 2) \cdot \dim V$.

Proof. Choose a norm on V so that $\|\cdot\|^2$ is a polynomial function on V. Since D is compact, we can find $R > 0$ such that

$$D \subset \{v \in V : \|v\| < R\}.$$

Then we set

$$D' = \{v \in V : \theta(v) = 0,\ \|v\| < R/\sqrt{\delta}\},$$

where $0 < \delta < 1$ is to be specified later. Note that if Φ is a neighborhood of D', there exists $0 < \eta < 1$ such that

$$\{v \in V : \theta(v) < \eta,\ \|v\| < (R+\eta)/\sqrt{\delta}\} \subset \Phi.$$

We will take Ψ to be

$$\Psi = \{v \in V : \theta(v) < \eta\delta,\ \|v\| < (R+\eta)\}.$$

Set

$$\tilde{I}(q) = \{t \in \mathbb{R} : \theta(qu_t) < \eta,\ \|qu_t\| < (R+\eta)/\sqrt{\delta}\}.$$

Since $\tilde{I}(q) \subset I(q)$ for $0 < \delta < 1$, it suffices to find δ (and hence D' and Ψ) so that the pair $(\tilde{I}(q), J(q))$ is β-regular. If we set

$$\psi_1(t) := \frac{\theta(qu_t)}{\eta} \quad \text{and} \quad \psi_2(t) := \left(\frac{\|qu_t\|\sqrt{\delta}}{R+\eta}\right)^2,$$

then

$$\tilde{I}(q) = \{t \in \mathbb{R} : \max(\psi_1(t), \psi_2(t)) < 1\};$$
$$J(q) = \{t \in \mathbb{R} : \max(\psi_1(t), \psi_2(t)) < \delta\}.$$

As ψ_1 and ψ_2 are polynomials, they have the (C, α)-property for an appropriate choice of C and α. Therefore by applying Lemma 7.1, by choosing δ small enough, we can make the pair $(\tilde{I}(q), J(q))$ β-regular for any $\beta > 0$. Note that the degrees of ψ_1 and ψ_2 are bounded by $\deg\theta \cdot \dim V$ and $2\dim V$ respectively. Therefore $J(q)$ cannot have more than $(\deg\theta + 2)\cdot\dim V$ number of components. Hence the proof is complete. □

Collection $\mathcal{E}_U$. Recall the collection $\mathscr{H}^\star$ and the singular set:

$$\mathscr{S}(U) = \bigcup_{H \in \mathscr{H}^\star} \Gamma\backslash\Gamma X(H, U).$$

Definition 7.3. We define $\mathcal{E} = \mathcal{E}_U$ to be the collection of all compact subsets of $\mathscr{S}(U) \cap \mathrm{RF}\,\mathcal{M}$ which can be written as

$$E = \bigcup_{i \in \Lambda} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\,\mathcal{M} \tag{7.2}$$

where $\{H_i \in \mathscr{H}^\star : i \in \Lambda\}$ is a finite collection and $D_i \subset X(H_i, U)$ is a compact subset. In this expression, we always use the minimal index set Λ for E. When E is of the form (7.2), we will say that E is associated to the family $\{H_i : i \in \Lambda\}$.

Remark 7.4. We note that E can also be expressed as $\bigcup_{i\in\Lambda}\Gamma\backslash\Gamma H_i D_i\cap \mathrm{RF}\,\mathcal{M}$ where $H_i\in\mathscr{H}$ is a finite collection, and $D_i\subset X(H_i,U)$ is a compact subset which is left $\mathrm{C}(H_i)$-invariant.

Lemma 7.5. *In the expression* (7.2) *for $E\in\mathcal{E}$, the collection $\{H_i : i\in\Lambda\}$ is not redundant, in the sense that*

- *no $\gamma H_j\gamma^{-1}$ is equal to H_i for all triples $(i,j,\gamma)\in\Lambda\times\Lambda\times\Gamma$ except for the trivial cases of $i=j$ and $\gamma\in H_i$.*

Proof. Observe that if $\gamma H_j\gamma^{-1}=H_i$ for some $\gamma\in\Gamma$, then $\Gamma H_j D_j=\Gamma H_i\gamma D_j$, and hence by replacing D_i by $D_i\cup\gamma D_j\subset X(H_i,U)$, we may remove j from the index subset Λ. This contradicts the minimality of Λ. □

Observe that for any subgroups H_1,H_2 of G, and $g\in G$,

$$\begin{aligned}X(H_1\cap gH_2g^{-1},U)&=X(H_1,U)\cap X(gH_2g^{-1},U)\\&=X(H_1,U)\cap gX(H_2,U).\end{aligned}$$

Note that for $D_i\subset X(H_i,U)$, and $\gamma\in\Gamma$, the intersection $H_1D_1\cap\gamma H_2D_2$ only depends on the $(\Gamma\cap H_1,\Gamma\cap H_2)$-double coset of γ.

Proposition 7.6. *Let $H_1,H_2\in\mathscr{H}^\star$. Then for any compact subset $D_i\subset X(H_i,U)$ for $i=1,2$ and a compact subset $K\subset\Gamma\backslash G$, there exists a finite set $\Delta\subset(H_1\cap\Gamma)\backslash\Gamma/(H_2\cap\Gamma)$ such that*

$$\left\{K\cap\Gamma\backslash\Gamma(H_1D_1\cap\gamma H_2D_2)\right\}_{\gamma\in\Gamma}=\left\{K\cap\Gamma\backslash\Gamma(H_1D_1\cap\gamma H_2D_2)\right\}_{\gamma\in\Delta}$$

where the latter set consists of distinct elements.

Moreover for each $\gamma\in\Delta$, there exists a compact subset $C_0\subset H_1D_1\cap\gamma H_2D_2\subset X(H_1\cap\gamma H_2\gamma^{-1},U)$ such that

$$K\cap\Gamma\backslash\Gamma(H_1D_1\cap\gamma H_2D_2)=\Gamma\backslash\Gamma C_0.$$

Proof. For simplicity, write $\eta_{H_i}=\eta_i$ and $p_i=p_{H_i}$. Let $K_0\subset G$ be a compact set such that $K=\Gamma\backslash\Gamma K_0$. We fix $\gamma\in\Gamma$, and define for any $\gamma'\in\Gamma$,

$$K_{\gamma'}=\{g\in K_0:\gamma' g\in H_1D_1\cap\gamma H_1D_2\}.$$

We check that

$$K\cap\Gamma\backslash\Gamma(H_1D_1\cap\gamma H_2D_2)=\Gamma\backslash\Gamma\left(\cup_{\gamma'\in\Gamma}K_{\gamma'}\right).$$

If this set is non-empty, then $K_{\gamma'}\neq\emptyset$ for some $\gamma'\in\Gamma$ and

$$p_1\gamma' g\in p_1D_1,\quad p_2\gamma^{-1}\gamma' g\in p_2D_2$$

for some $g\in K_0$. In particular,

$$p_1\gamma'\in p_1DK_0^{-1},\quad p_2\gamma^{-1}\in p_2DK_0^{-1}\gamma'^{-1}. \tag{7.3}$$

As $p_1\Gamma$ is discrete, and $p_1D_1K_0^{-1}$ is compact, the first condition of (7.3) implies that there exists a finite set $\Delta_0\subset G$ such that $\gamma'\in(H_1\cap\Gamma)\Delta_0$. Writing $\gamma'=h\delta_0$ where $h\in H_1\cap\Gamma$, and $\delta_0\in\Delta_0$, the second condition of (7.3) implies

$$p_2\gamma^{-1}h\in p_2D_2K_0^{-1}\delta_0^{-1}.$$

As $p_2 D_2 K_0^{-1}\Delta_0^{-1}$ is compact and $p_2\Gamma$ is discrete, there exists a finite set $\Delta \subset G$ such that $\gamma^{-1}h \in (H_2\cap\Gamma)\Delta$. Hence, if $K\cap\Gamma\backslash\Gamma(H_1D_1\cap\gamma H_2D_2) \neq \emptyset$, then $\gamma \in (H_1\cap\Gamma)\Delta(H_2\cap\Gamma)$. This completes the proof of the first claim.

For the second claim, it suffices to set $C_0 := \bigcup_{\gamma'\in\Delta} K_{\gamma'}$. ☐

Proposition 7.7. *Let $H_1, H_2 \in \mathscr{H}^\star$ be such that $H_1 \cap H_2$ contains a unipotent element. Then there exists a unique smallest connected closed subgroup, say H_0, of $H_1 \cap H_2$ containing all unipotent elements of $H_1 \cap H_2$ such that $\Gamma\backslash\Gamma H_0$ is closed. Moreover, $H_0 \in \mathscr{H}$.*

Proof. The orbit $\Gamma\backslash\Gamma(H_1\cap H_2)$ is closed [41, Lemma 2.2]. Hence such H_0 exists. We need to show that $\Gamma\cap H_0$ is Zariski dense in H_0. Let L be the subgroup of H_0 generated by all unipotent elements in H_0. Note that L is a normal subgroup of H_0 and hence $(H_0\cap\Gamma)L$ is a subgroup of H_0. If F is the identity component of the closure of $(H_0\cap\Gamma)L$, then $\Gamma\backslash\Gamma F$ is closed. By the minimality assumption on H_0, we have $F = H_0$. Hence $\overline{(H_0\cap\Gamma)L} = H_0$; so $\overline{[e]L} = [e]H_0$. We can then apply [41, Corollary 2.12] and deduce the Zariski density of $H_0\cap\Gamma$ in H_0. ☐

Corollary 7.8. *Let $H_1, H_2 \in \mathscr{H}^\star$ and $\gamma\in\Gamma$ be satisfying that $X(H_1 \cap \gamma H_2\gamma^{-1}, U) \neq \emptyset$. Then there exists a subgroup $H \in \mathscr{H}^\star$ contained in $H_1 \cap \gamma H_2\gamma^{-1}$ such that for any compact subsets $D_i \subset X(H_i, U)$, $i = 1, 2$, there exists a compact subset $D_0 \subset X(H, U)$ such that*

$$K\cap\Gamma\backslash\Gamma(H_1D_1\cap\gamma H_2D_2) = K\cap\Gamma\backslash\Gamma HD_0.$$

Proof. Let $F\in\mathscr{H}$ be given by Proposition 7.7 for the subgroup $H_1 \cap \gamma H_2\gamma^{-1}$. Set $H := \mathrm{N}_G(F_{nc}) \in \mathscr{H}^\star$. Note that $X(H_1\cap\gamma H_2\gamma^{-1}, U) = X(H, U)$. Hence, by the second claim of Proposition 7.6, there exists a compact subset $D_0 \subset H_1D_1\cap\gamma H_2D_2$ such that

$$K\cap\Gamma\backslash\Gamma(H_1D_1\cap\gamma H_2D_2) = \Gamma\backslash\Gamma D_0. \tag{7.4}$$

We claim that

$$\Gamma\backslash\Gamma D_0 = K\cap\Gamma\backslash\Gamma HD_0.$$

The inclusion $\subset$ is clear. Let $g := hd \in HD_0$ with $h\in H$ and $d\in D_0$, and $[g]\in K$. Then by the condition on D_0, we have $g\in H_1D_1$ and $\gamma^{-1}g\in H_2D_2$. Therefore $g\in H_1D_1\cap\gamma H_2D_2$. By (7.4), this proves the inclusion $\supset$. ☐

Definition 7.9 (Self-intersection operator on $\mathcal{E}_U$)**.** We define an operator

$$\mathsf{s} : \mathcal{E}_U\cup\{\emptyset\} \to \mathcal{E}_U\cup\{\emptyset\}$$

as follows: we set $\mathsf{s}(\emptyset) = \emptyset$. For any

$$E = \bigcup_{i\in\Lambda}\Gamma\backslash\Gamma H_iD_i\cap\mathrm{RF}\,\mathcal{M}\in\mathcal{E}_U, \tag{7.5}$$

we define

$$\mathsf{s}(E) := \bigcup_{i,j\in\Lambda}\bigcup_{\gamma_{ij}\in\Gamma}\Gamma\backslash\Gamma(H_iD_i\cap\gamma_{ij}H_jD_j)\cap\mathrm{RF}\,\mathcal{M}$$

where $\gamma_{ij} \in \Gamma$ ranges over all elements of Γ satisfying

$$\dim(H_i \cap \gamma_{ij} H_j \gamma_{ij}^{-1})_{nc} < \min\{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}.$$

By Proposition 7.6 and Corollary 7.8, we have:

Corollary 7.10. (1) *For $E \in \mathcal{E}_U$, we have $\mathsf{s}(E) \in \mathcal{E}_U$.*
(2) *For $E_1, E_2 \in \mathcal{E}_U$, we have $E_1 \cap E_2 \in \mathcal{E}_U$.*

Hence for $E \in \mathcal{E}_U$ as in (7.5), $\mathsf{s}(E)$ is of the form

$$\mathsf{s}(E) = \bigcup_{i \in \Lambda'} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\,\mathcal{M}$$

where Λ' is a (minimal) finite index set, $H_i \in \mathscr{H}$ with $X(H_i, U) \neq \emptyset$ and

$$\max\{\dim(H_i)_{nc} : i \in \Lambda'\} < \max\{\dim(H_i)_{nc} : i \in \Lambda\}.$$

Hence, s maps $\mathcal{E}_U$ to $\mathcal{E}_U \cup \{\emptyset\}$ and for any $E \in \mathcal{E}_U$,

$$\mathsf{s}^{\dim G}(E) = \emptyset.$$

Definition 7.11. For a compact subset $K \subset \Gamma\backslash G$ and $E \in \mathcal{E}_U$, we say that K does not have any self-intersection point of E, or simply say that K is *E-self intersection-free*, if

$$K \cap \mathsf{s}(E) = \emptyset.$$

Proposition 7.12. *Let $E = \bigcup_{i \in \Lambda} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\,\mathcal{M} \in \mathcal{E}$ where $D_i \subset X(H_i, U)$ is a compact subset and Λ is a finite subset. Let $K \subset \mathrm{RF}\,\mathcal{M}$ be a compact subset which is E-self intersection-free. Then there exists a collection of open neighborhoods Ω_i of D_i, $i \in \Lambda$, such that for $\mathcal{O} := \bigcup_{i \in \Lambda} \Gamma\backslash\Gamma H_i \Omega_i$, the compact subset K is $\mathcal{O}$-self intersection free, in the sense that, if $\dim H_i = \dim H_j$ and*

$$K \cap \Gamma\backslash\Gamma(H_i \Omega_i \cap \gamma H_j \Omega_j) \neq \emptyset$$

for some $(i, j, \gamma) \in \Lambda \times \Lambda \times \Gamma$, then $i = j$ and $\gamma \in H_i \cap \Gamma$.

Proof. For each $k \in \mathbb{N}$ and $i \in \Lambda$, let $\Omega_i(k)$ be the $1/k$-neighborhood of the compact subset D_i. Since Λ is finite, if the proposition does not hold, by passing to a subsequence, there exist $i, j \in \Lambda$ with $\dim H_i = \dim H_j$ and a sequence $\gamma_k \in \Gamma$ such that

$$K \cap \Gamma\backslash\Gamma(H_i \Omega_i(k) \cap \gamma_k H_j \Omega_j(k)) \neq \emptyset$$

and

$$(i, j, \gamma_k) \notin \{(i, i, \gamma) : i \in \Lambda, \gamma \in H_i \cap \Gamma\}. \tag{7.6}$$

Hence there exist $g_k = h_k w_k \in H_i \Omega_i(k)$ and $g_k' = h_k' w_k' \in H_j \Omega_j(k)$ such that $g_k = \gamma_k g_k'$ where $[g_k] \in K$. Now as $k \to \infty$, we have $w_k \to w \in D_i$ and $w_k' \to w' \in D_j$. There exists $\delta_k \in \Gamma$ such that $\delta_k g_k \in \tilde{K}$ where $\tilde{K}$ is a compact subset of G such that $K = \Gamma\backslash\Gamma\tilde{K}$, so the sequence $\delta_k g_k$ converges to g_0 as $k \to \infty$. Since ΓH_i and ΓH_j are closed, we have $\delta_k h_k \to \delta_0 h_i$ and

$\delta_k\gamma_k h'_k \to \delta'_0 h_j$ where $\delta_0, \delta'_0 \in \Gamma$, $h_i \in H_i$ and $h_j \in H_j$. As $\Gamma[H_i]$ and $\Gamma[H_j]$ are discrete in the spaces G/H_i and G/H_j respectively, we have

$$\delta_0^{-1}\delta_k \in H_i \quad \text{and} \quad (\delta'_0)^{-1}\delta_k\gamma_k \in H_j \tag{7.7}$$

for all sufficiently large k. Therefore $g_0 = \delta_0 h_i w = \delta'_0 h_j w' \in \delta_0(H_i D_i \cap \delta_0^{-1}\delta'_0 H_j D_j)$ and $[g_0] \in K$. Hence

$$K \cap \Gamma\backslash\Gamma(H_i D_i \cap \delta_0^{-1}\delta'_0 H_j D_j) \neq \emptyset.$$

Set $\delta := \delta_0^{-1}\delta'_0 \in \Gamma$.

Since $K \cap \mathsf{s}(E) = \emptyset$, this implies that $\mathrm{RF}\,\mathcal{M} \cap \Gamma\backslash\Gamma(H_i D_i \cap \delta H_j D_j) \not\subset \mathsf{s}(E)$. By the definition of $\mathsf{s}(E)$,

$$\dim(H_i \cap \delta H_j \delta^{-1})_{nc} = \min\{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}.$$

Since $H_i = \mathrm{N}_G(H_i) = \mathrm{N}_G((H_i)_{nc})$, and similarly for H_j, we have $H_i \cap \delta H_j \delta^{-1}$ is either H_i or $\delta H_j \delta^{-1}$. Since $\dim H_i = \dim H_j$, $\delta H_j \delta^{-1} = H_i$ or $H_i = \delta H_j \delta^{-1}$.

By Lemma 7.5, this implies that $i = j$ and $\delta \in \mathrm{N}_G(H_i) \cap \Gamma$. It follows from (7.7) that

$$\gamma_k \in \mathrm{N}_G(H_i) \cap \Gamma = H_i \cap \Gamma$$

for all large k. This is a contradiction to (7.6), completing the proof. □

In the rest of this section, we assume that $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ is a convex cocompact hyperbolic manifold with Fuchsian ends, and let k be as given by Proposition 4.11.

Theorem 7.13 (Avoidance theorem I)**.** *Let $U = \{u_t\} < N$ be a one-parameter subgroup. For any $E \in \mathcal{E}_U$, there exists $E' \in \mathcal{E}_U$ such that the following holds: If $F \subset \mathrm{RF}\,\mathcal{M}$ is a compact set disjoint from E', then there exists a neighborhood $\mathcal{O}^\diamond$ of E such that for all $x \in F$, the following set*

$$\{t \in \mathbb{R} : xu_t \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}^\diamond\}$$

is $2k$-thick. Moreover, if E is associated to $\{H_i : i \in \Lambda\}$, then E' is also associated to the same family $\{H_i : i \in \Lambda\}$ in the sense of Definition 7.3.

Proof. ♠1. **The constant** β_0**:** We write $\mathscr{H}^\star = \{H_i\}$. For simplicity, set $V_i = V_{H_i}$ and $p_i = p_{H_i}$. Let θ_i be the defining polynomial of the algebraic variety A_{H_i}.

Set

$$m := \dim(G)^2; \text{ and}$$
$$\delta := \max_{H_i \in \mathscr{H}^\star} (\deg \theta_i + 2) \dim V_i.$$

Note that if H_i is conjugate to H_j, then θ_i and θ_j have same degree and $\dim V_i = \dim V_j$. Since there are only finitely many conjugacy classes in $\mathscr{H}^\star$ by Proposition 5.3, the constant δ is finite. Now let

$$\beta_0 := \beta_0(m\delta, k, 1) = (4k)^{m\delta+1}\varepsilon^{-1}$$

be given as in Proposition 6.3 where $\varepsilon = \varepsilon_{m\delta}$ satisfies $\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{2^{m\delta}-1} \leq 2$.

♠2. **Definition of E_n and E'_n:** We write

$$E = \bigcup_{i\in\Lambda_0} \Gamma\backslash\Gamma H_i D_i \cap \operatorname{RF}\mathcal{M}$$

for some finite minimal set Λ_0. Set

$$\ell := \max_{i\in\Lambda_0} \dim(H_i)_{nc}.$$

We define $E_n, E'_n \in \mathcal{E}_U$ for all $1 \le n \le \ell$ inductively as follows: set

$$E_\ell := E \quad \text{and} \quad \Lambda_\ell := \Lambda_0.$$

For each $i \in \Lambda_\ell$, let D'_i be a compact subset of $X(H_i, U)$ containing D_i such that $p_i D'_i$ has a β_0-regular size with respect to $p_i D_i$ as in Proposition 7.2. Set

$$E'_\ell := \bigcup_{i\in\Lambda_\ell} \Gamma\backslash\Gamma H_i D'_i \cap \operatorname{RF}\mathcal{M}.$$

Suppose that $E_{n+1}, E'_{n+1} \in \mathcal{E}_U$ are given for $n \ge 1$. Then, define

$$E_n := E \cap \mathsf{s}(E'_{n+1}).$$

Then by Corollary 7.10, E_n belongs to $\mathcal{E}_U$ and hence can be written as

$$E_n = \bigcup_{i\in\Lambda_n} \Gamma\backslash\Gamma H_i D_i \cap \operatorname{RF}\mathcal{M}$$

where D_i is a compact subset of $X(H_i, U)$, so that Λ_n is a minimal index set. For each $i \in \Lambda_n$, let D'_i be a compact subset of $X(H_i, U)$ containing D_i such that $p_i D'_i$ has a β_0-regular size with respect to $p_i D_i$ as in Proposition 7.2. Set

$$E'_n := \bigcup_{i\in\Lambda_n} \Gamma\backslash\Gamma H_i D'_i \cap \operatorname{RF}\mathcal{M}.$$

Hence we get a sequence of compact (possibly empty) subsets of E:

$$E_1, E_2, \cdots, E_{\ell-1}, E_\ell = E,$$

and a sequence of compact sets

$$E'_1, E'_2, \cdots, E'_{\ell-1}, E'_\ell = E'.$$

Note that $\mathsf{s}(E_1) = \mathsf{s}(E'_1) = \emptyset$ by the dimension reason.[9]

♠3. **Outline of the plan:** Let $F \subset \operatorname{RF}\mathcal{M}$ be a compact set disjoint from E'. For $x \in F$, we set

$$\mathsf{T}(x) := \{t \in \mathbb{R} : xu_t \in \operatorname{RF}\mathcal{M}\}$$

which is a globally k-thick set by Proposition 4.11. We will construct

- a neighborhood $\mathcal{O}'$ of E' disjoint from F, and
- a neighborhood $\mathcal{O}^\diamond$ of E

[9] In fact $E_{\ell-i} = \emptyset$ for all $i \ge d-1$, but we won't use this information

such that for any $x \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}'$, we have

$$\{t \in \mathbb{R} : xu_t \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}^{\diamond}\} \supset \mathsf{T}(x) - J'(\mathcal{X})$$

where $\mathcal{X} = \mathcal{X}(x)$ is the union of at most m-number of β_0-regular families $\mathcal{X}_i$ of triples $(I(q), J^*(q), J'(q))$ of subsets of $\mathbb{R}$ with degree δ and of $\mathsf{T}(x)$-multiplicity free. Once we do that, the theorem is a consequence of Proposition 6.3. Construction of such $\mathcal{O}'$ and $\mathcal{O}^{\diamond}$ requires an inductive process on E_n's.

♠4. **Inductive construction of K_n, $\mathcal{O}'_{n+1}$, $\mathcal{O}_{n+1}$, and $\mathcal{O}^{\star}_{n+1}$:** Let

$$K_0 := \mathrm{RF}\,\mathcal{M}.$$

For each $i \in \Lambda_1$, there exists a neighborhood Ω'_i of D'_i such that for

$$\mathcal{O}'_1 := \bigcup_{i \in \Lambda_1} \Gamma\backslash\Gamma H_i \Omega'_i,$$

the compact subset K_0 is $\mathcal{O}'_1$-self intersection free by Lemma 7.12, since $\mathsf{s}(E'_1) = \emptyset$. By Proposition 7.2, there exists a neighborhood Ω_i of D_i such that the pair $(I(q), J(q))$ is β_0-regular for all $q \in V_i - p_i\Omega'_i$ where

$$I(q) = \{t \in \mathbb{R} : qu_t \in p_i\Omega'_i\} \quad \text{and} \quad J(q) = \{t \in \mathbb{R} : qu_t \in p_i\Omega_i\}. \tag{7.8}$$

Set

$$\mathcal{O}_1 := \bigcup_{i \in \Lambda_1} \Gamma\backslash\Gamma H_i \Omega_i.$$

Since $E_1 = \bigcup_{i \in \Lambda_1} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\,\mathcal{M}$, $\mathcal{O}_1$ is a neighborhood of $E_1 = \mathsf{s}(E'_2) \cap E$. Now the compact subset $\mathsf{s}(E'_2) - \mathcal{O}_1$ is contained in $\mathsf{s}(E'_2) - E$, which is relatively open in $\mathsf{s}(E'_2)$. Therefore we can take a neighborhood $\mathcal{O}^{\star}_1$ of $\mathsf{s}(E'_2) - \mathcal{O}_1$ so that

$$\overline{\mathcal{O}^{\star}_1} \cap E = \emptyset.$$

We will now define the following quadruple $K_n, \mathcal{O}'_{n+1}, \mathcal{O}_{n+1}$ and $\mathcal{O}^{\star}_{n+1}$ for each $1 \le n \le \ell - 1$ inductively:

- a compact subset $K_n = K_{n-1} - (\mathcal{O}_n \cup \mathcal{O}^{\star}_n) \subset \mathrm{RF}\,\mathcal{M}$,
- a neighborhood $\mathcal{O}'_{n+1}$ of E'_{n+1},
- a neighborhood $\mathcal{O}_{n+1}$ of E_{n+1} and
- a neighborhood $\mathcal{O}^{\star}_{n+1}$ of $\mathsf{s}(E'_{n+2}) - \mathcal{O}_{n+1}$ such that

$$\overline{\mathcal{O}^{\star}_{n+1}} \cap E = \emptyset.$$

Assume that the sets K_{n-1}, $\mathcal{O}'_n$, $\mathcal{O}_n$ and $\mathcal{O}^{\star}_n$ are defined. We define

$$K_n := K_{n-1} - (\mathcal{O}_n \cup \mathcal{O}^{\star}_n) = \mathrm{RF}\,\mathcal{M} - \bigcup_{i=1}^{n} (\mathcal{O}_i \cup \mathcal{O}^{\star}_i).$$

For each $i \in \Lambda_{n+1}$, let Ω'_i be a neighborhood of D'_i in G such that for $\mathcal{O}'_{n+1} := \bigcup_{i \in \Lambda_{n+1}} \Gamma\backslash\Gamma H_i \Omega'_i$, K_n is $\mathcal{O}'_{n+1}$-self intersection free. Since $\mathcal{O}_n \cup \mathcal{O}^{\star}_n$ is a neighborhood of $\mathsf{s}(E'_{n+1})$, which is the set of all self-intersection points of E'_{n+1}, such collection of Ω'_i, $i \in \Lambda_{n+1}$ exists by Lemma 7.12.

Since $F \subset \mathrm{RF}\,\mathcal{M}$ is compact and disjoint from E', we can also assume $\Gamma\backslash\Gamma H_i\Omega_i'$ is disjoint from F, by shrinking Ω_i' if necessary. More precisely, writing $F = \Gamma\backslash\Gamma\tilde{F}$ for some compact subset $\tilde{F} \subset G$, this can be achieved by choosing a neighborhood Ω_i' of D_i' so that $p_i\Omega_i'$ is disjoint from $p_i\Gamma\tilde{F}$; and this is possible since $p_i\Gamma\tilde{F}$ is a closed set disjoint from a compact subset p_iD_i'. After choosing Ω_i' for each $i \in \Lambda_{n+1}$, define the following neighborhood of E_{n+1}':

$$\mathcal{O}_{n+1}' := \bigcup_{i\in\Lambda_{n+1}} \Gamma\backslash\Gamma H_i\Omega_i'.$$

We will next define $\mathcal{O}_{n+1}$. By Lemma 7.2, there exists a neighborhood Ω_i of D_i such that the pair $(I(q), J(q))$ is β_0-regular for all $q \in V_i - p_i\Omega_i'$ where

$$I(q) = \{t \in \mathbb{R} : qu_t \in p_i\Omega_i'\} \text{ and } J(q) = \{t \in \mathbb{R} : qu_t \in p_i\Omega_i\}.$$

We then define the following neighborhood of $E_{n+1} = \mathsf{s}(E_{n+2}') \cap E$:

$$\mathcal{O}_{n+1} := \bigcup_{i\in\Lambda_{n+1}} \Gamma\backslash\Gamma H_i\Omega_i.$$

Since the compact subset $\mathsf{s}(E_{n+2}') - \mathcal{O}_{n+1}$ is contained in the set $\mathsf{s}(E_{n+2}') - E$, which is relatively open inside $\mathsf{s}(E_{n+2}')$, we can take a neighborhood $\mathcal{O}_{n+1}^\star$ of $\mathsf{s}(E_{n+2}') - \mathcal{O}_{n+1}$ so that

$$\overline{\mathcal{O}_{n+1}^\star} \cap E = \emptyset.$$

This finishes the inductive construction.

♠5. **Definition of $\mathcal{O}'$ and $\mathcal{O}^\diamond$:** We define:

$$\mathcal{O}' := \bigcup_{n=1}^{\ell} \mathcal{O}_n', \quad \mathcal{O} := \bigcup_{n=1}^{\ell} \mathcal{O}_n \quad \mathcal{O}^\star := \bigcup_{n=1}^{\ell} \overline{\mathcal{O}_n^\star}.$$

Note that $\mathcal{O}'$ and $\mathcal{O}$ are neighborhoods of E' and E respectively. Since $E \cap \mathcal{O}^\star = \emptyset$, the following defines a neighborhood of E:

$$\mathcal{O}^\diamond := \mathcal{O} - \mathcal{O}^\star. \tag{7.9}$$

♠6. **Construction of β_0-regular families of $\mathsf{T}(x)$-multiplicity free:** Fix $x \in F \subset \mathrm{RF}\,\mathcal{M} - \mathcal{O}'$. Choose a representative $g \in G$ of x. We write each Λ_n as the disjoint union

$$\Lambda_n = \bigcup_{j\in\theta_n} \Lambda_{n,j}$$

where $\Lambda_{n,j} = \{i \in \Lambda_n : \dim H_i = j\}$ and $\theta_n = \{j : \Lambda_{n,j} \neq \emptyset\}$. Note that $\#\theta_n < \dim G$.

Fix $1 \le n \le \ell$, $j \in \theta_n$ and $i \in \Lambda_{n,j}$. For each $q \in p_i\Gamma g$, we define the following subsets of $\mathbb{R}$:

- $I(q) := \{t : qu_t \in p_i\Omega_i'\}$ and
- $J(q) := \{t : qu_t \in p_i\Omega_i\}$.

In general, $I(q)$'s have high multiplicity among q's in $\bigcup_{i\in\Lambda_{n,j}} p_i\Gamma g$, but the following subset $I'(q)$'s will be multiplicity-free, and this is is why we defined K_{n-1} as carefully as above:

- $I'(q) := \{t : \text{for some } a \geq 0,\ [t, t+a] \subset I(q) \text{ and } xu_{t+a} \in K_{n-1}\}$;
- $J^*(q) := I'(q) \cap J(q)$;
- $J'(q) := \{t \in J(q) : xu_t \in K_{n-1}\}$.

Observe that $I'(q)$ and $J^*(q)$ are unions of finitely many intervals, $J'(q) \subset \mathsf{T}(x)$ and that

$$J'(q) \subset J^*(q) \subset I'(q).$$

Now, for each $1 \leq n \leq \ell$ and $j \in \theta_n$, define the family

$$\mathcal{X}_{n,j} = \big\{(I(q), J^*(q), J'(q))) : q \in \bigcup_{i\in\Lambda_{n,j}} p_i\Gamma g\big\}. \tag{7.10}$$

We claim that each $\mathcal{X}_{n,j}$ is a β_0-regular family with degree at most δ and $\mathsf{T}(x)$-multiplicity free.

Note for each $q \in p_i\Gamma g$, the number of connected components of $J^*(q)$ is less than or equal to that of $J(q)$. Now that $J^*(q) \subset J(q)$ and all the pairs $(I(q), J(q))$ are β_0-regular pairs of degree at most δ, it follows that $\mathcal{X}_{n,j}$'s are β_0-regular families with degree at most δ.

We now claim that $\mathcal{X}_{n,j}$ has $\mathsf{T}(x)$-multiplicity free, that is, for any distinct indices $q_1, q_2 \in \bigcup_{i\in\Lambda_{n,j}} p_i\Gamma g$ of $\mathcal{X}_{n,j}$,

$$I(q_1) \cap J'(q_2) = \emptyset.$$

We first show that

$$I'(q_1) \cap I'(q_2) = \emptyset.$$

Suppose not. Then there exists $t \in I'(q_1) \cap I'(q_2)$ for some $q_1 = p_i\gamma_1 g$ and $q_2 = p_k\gamma_2 g$, where $i, k \in \Lambda_{n,j}$. Then for some $a \geq 0$, we have $[t, t+a] \subset I(q_1) \cap I(q_2)$ and $xu_{t+a} \in K_{n-1}$. In particular,

$$xu_{t+a} \in \Gamma\backslash\Gamma(\gamma_1^{-1} H_i\Omega_i' \cap \gamma_2^{-1} H_k\Omega_k') \cap K_{n-1}.$$

Since K_{n-1} is $\mathcal{O}_n'$-self intersection free, and $\dim H_i = \dim H_k = j$, we deduce from Proposition 7.12 that this may happen only when $i = k$, and $\gamma_1\gamma_2^{-1} \in H_i \cap \Gamma$. Hence we have

$$q_1 = q_2.$$

This shows that $I'(q)$'s are pairwise disjoint. Now suppose that there exists an element $t \in I(q_1) \cap J'(q_2)$. Then by the disjointness of $I'(q_1)$ and $I'(q_2)$, it follows that

$$t \in (I(q_1) - I'(q_1)) \cap J'(q_2).$$

By the definition of $I'(q_1)$, we have $xu_t \notin K_{n-1}$. This contradicts the assumption that $t \in J'(q_2)$.

♠7. **Completing the proof:** Let $\mathcal{X} := \bigcup_{1\leq i\leq \ell, j\in\theta_n} \mathcal{X}_{n,j}$. In view of Proposition 6.3, it remains to check that the condition $t \in \mathsf{T}(x) - J'(\mathcal{X})$ implies that $xu_t \notin \mathcal{O}^\diamond$ where $\mathcal{O}^\diamond$ is given in (7.9).

Suppose that there exists $t \in \mathsf{T}(x) - J'(\mathcal{X})$ such that $xu_t \in \mathcal{O}^\diamond$. Write the neighborhood $\mathcal{O}^\diamond$ as the disjoint union

$$\mathcal{O}^\diamond = \bigcup_{n=1}^{\ell} (\mathcal{O}_n - (\cup_{i\le n-1}\mathcal{O}_i \cup \mathcal{O}^\star)).$$

Let $n \le \ell$ be such that

$$xu_t \in \mathcal{O}_n - (\bigcup_{i=1}^{n-1} \mathcal{O}_i \cup \mathcal{O}^\star).$$

Since $t \in \mathsf{T}(x) - J'(\mathcal{X})$, we have $xu_t \in \mathrm{RF}\,\mathcal{M} - K_{n-1}$. Since $K_{n-1} = \mathrm{RF}\,\mathcal{M} - \bigcup_{i=1}^{n-1} (\mathcal{O}_i \cup \mathcal{O}_i^\star)$,

$$xu_t \in \bigcup_{i=1}^{n-1} \mathcal{O}_i \cup \mathcal{O}_i^\star.$$

This is a contradiction, since $\bigcup_{i=1}^{\ell} \mathcal{O}_i^\star \subset \mathcal{O}^\star$. □

As $\mathscr{H}^\star$ is countable and $X(H_i, U)$ is σ-compact, the intersection $\mathscr{S}(U) \cap \mathrm{RF}\,\mathcal{M}$ can be exhausted by the union of the increasing sequence of $E_j \in \mathcal{E}_U$'s. Therefore, we deduce:

Corollary 7.14. *There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}_U$ with $\mathscr{S}(U) \cap \mathrm{RF}\,\mathcal{M} = \bigcup_{j=1}^{\infty} E_j$ which satisfies the following: Let $x_i \in \mathrm{RF}\,\mathcal{M}$ be a sequence converging to $x \in \mathscr{G}(U) \cap \mathrm{RF}\,\mathcal{M}$. Then for each $j \in \mathbb{N}$, there exist a neighborhood $\mathcal{O}_j$ of E_j and $i_j \ge 1$ such that*

$$\{t \in \mathbb{R} : x_i u_t \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}_j\}$$

is $2k$-thick for all $i \ge i_j$.

Proof. For each $j \ge 1$, we may assume $E_{j+1} \supset E_j'$ where E_j' is given by Theorem 7.13. For each $j \ge 1$, there exists $i_j \in \mathbb{N}$ such that $x_i \notin E_{j+1}$ for all $i \ge i_j$. Applying Proposition 7.13 to a compact subset $F = \{x_i : i \ge i_j\}$ of $\mathrm{RF}\,\mathcal{M}$, we obtain a neighborhood $\mathcal{O}_j$ of E_j such that

$$\{t \in \mathbb{R} : x_i u_t \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}_j\}$$

is $2k$-thick for all $i \ge i_j$. □

Indeed we will apply Corollary 7.14 for the sequence $\{x_i\}$ contained in a closed orbit $x_0 L$ of a proper connected closed subgroup $L < G$, which can be proved in the same way:

Theorem 7.15 (Avoidance Theorem II)**.** *Consider a closed orbit $x_0 L$ for some $x_0 \in \mathrm{RF}\,\mathcal{M}$ and $L \in \mathcal{Q}_U$. There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}_U$ with $\mathscr{S}(U, x_0 L) \cap \mathrm{RF}\,\mathcal{M} = \bigcup_{j=1}^{\infty} E_j$, which satisfies the following: if $x_i \to x$ in $\mathrm{RF}\,\mathcal{M} \cap x_0 L$ with $x \in \mathscr{G}(U, x_0 L)$,* *then for each $j \in \mathbb{N}$, there exist $i_j \geq 1$ and an open neighborhood $\mathcal{O}_j \subset x_0 L$ of E_j such that*

$$\{t \in \mathbb{R} : x_i u_t \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}_j\}$$

is a $2k$-thick set for all $i \geq i_j$.

8. Limits of RF $\mathcal{M}$-points in F^* and generic points

Let $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Recall that $\Lambda \subset \mathbb{S}^{d-1}$ denotes the limit set of Γ. In this section, we collect some geometric lemmas which are needed in modifying a sequence limiting on an RF $\mathcal{M}$ point (resp. limiting on a point in RF $\mathcal{M} \cap \mathscr{G}(U)$) to a sequence of RF $\mathcal{M}$-points (resp. whose limit still remains inside $\mathscr{G}(U)$). Recall from Defintion 4.1 that $\Omega = \mathbb{S}^{d-1} - \Lambda$.

Lemma 8.1. *Let $C_n \to C$ be a sequence of convergent circles in $\mathbb{S}^{d-1}$. If $C \not\subset \overline{B}$ for any component B of Ω, then*

$$\#\limsup_{n\to\infty} C_n \cap \Lambda \geq 2.$$

Proof. Without loss of generality, we may assume that $\infty \notin \Lambda$ and hence consider Λ as a subset of the Euclidean space $\mathbb{R}^{d-1}$. Note that there is one component, say, B_1 of Ω which contains ∞ and all other components of Ω are contained in the complement of B_1, which is a (bounded) round ball in $\mathbb{R}^{d-1}$. It follows that there are only finitely many components of Ω whose diameters are bounded from below by a fixed positive number; this follows from the fact that ΓB is closed for each component B of Ω, and that there are only finitely many Γ-orbits of components of Ω.

Let $\delta = 0.5\,\mathrm{diam}(C)$ so that we may assume $\mathrm{diam}(C_n) > \delta$ for all sufficiently large $n \gg 1$. It suffices to show that there exists $\varepsilon_0 > 0$ such that $C_n \cap \Lambda$ contains ξ_n, ξ_n' with $d(\xi_n, \xi_n') \geq \varepsilon_0$ for all sufficiently large n. Suppose not. Then for any $\varepsilon > 0$, there exists an interval $I_n \subset C_n$ such that $\mathrm{diam}(I_n) \leq \varepsilon$ and $C_n - I_n \subset \Omega$ for some infinite sequence of n's. Since $C_n - I_n$ is connected, there exists a component B_n of Ω such that $C_n \subset \mathcal{N}_\varepsilon(B_n)$, where $\mathcal{N}_\varepsilon(B_n)$ denotes the ε-neighborhood of B_n. In particular, we have $\mathrm{diam}(B_n) + \varepsilon > \delta$. Taking ε smaller than 0.5δ, this means that $\mathrm{diam}(B_n) > 0.5\delta$. On the other hand, there are only finitely many components of Ω whose diameters are greater than 0.5δ, say $B_1, \cdots, B_\ell$. Let $\varepsilon_0 > 0$ be such that $\mathcal{N}_{\varepsilon_0}(B_1), \cdots, \mathcal{N}_{\varepsilon_0}(B_\ell)$ are all disjoint. Then by passing to a subsequence, there exists B_i such that $C_n \subset \mathcal{N}_\varepsilon(B_i)$ for all small $0 < \varepsilon < \varepsilon_0$ and all $n \geq 1$; hence $C \subset \mathcal{N}_\varepsilon(B_i)$. Since this holds for all sufficiently small $\varepsilon > 0$, we get that $C \subset \overline{B_i}$, yielding a contradiction to the hypothesis on C. ☐

In the next two lemmas, we set $U^- = U$ and $U^+ = U^t$.

Lemma 8.2. *Let $U < N$ be a connected closed subgroup. Let $[g]L$ be a closed orbit for some $L \in \mathcal{L}_U$ and $[g] \in \mathrm{RF}\,\mathcal{M}$. Let S_0 and S^* denote the* *boundaries of $\pi(gH(U))$ and $\pi(gL)$ respectively. If S is a sphere such that $S_0 \subset S \subsetneq S^*$ and ΓS is closed, then $[g] \in \mathscr{S}(U^{\pm}, [g]L)$.*

Proof. Write $L = H(\widehat{U})C \in \mathcal{L}_U$. Since $S_0 \subset S \subsetneq S^*$, there exists a connected proper subgroup $\tilde{U}$ of $\widehat{U}$, containing U such that S is the boundary of $\pi(gH(\tilde{U}))$. Since ΓS is closed, $[g]H'(\tilde{U})$ is closed by Proposition 3.9. Now the claim follows from Proposition 4.9 and the definition of $\mathscr{S}(U^{\pm}, [g]L)$. $\square$

Recall from Section 3 that $\mathcal{C}^k$ denotes the space of all oriented k-spheres in $\mathbb{S}^{d-1}$.

Lemma 8.3. *Let $U < N$ be a connected closed subgroup with dimension $m \geq 1$, and let $U_{\pm}^{(1)}, \cdots, U_{\pm}^{(m)}$ be one-parameter subgroups generating $U^{\pm}$. Consider a closed orbit yL where $L \in \mathcal{L}_U$ and*

$$y \in F^*_{H(U)} \cap \mathrm{RF}\,\mathcal{M} \cap \bigcap_{i=1}^{m} \mathscr{G}(U_{\pm}^{(i)}, yL).$$

If $x_n \to y$ in yL, then, by passing to a subsequence, there exists a sequence $h_n \to h$ in $H(U)$ so that

$$x_n h_n \in \mathrm{RF}\,\mathcal{M} \cap yL \quad \text{ and } yh \in \mathrm{RF}\,\mathcal{M} \cap \bigcap_{i=1}^{m} \mathscr{G}(U_{\pm}^{(i)}, yL).$$

Proof. Let $g_0 \in G$ be such that $y = [g_0]$ and S^* denote the boundary of $\pi(g_0 L)$. Let $\mathcal{Q}$ be the collection of all spheres $S \subsetneq S^*$ such that $S \cap \Lambda \neq \emptyset$ and ΓS is closed in $\mathcal{C}^{\dim S}$. By Corollary 5.8 and Remark 5.9, $\mathcal{Q}$ is countable. Choose a sequence $g_n \to g_0$ in G as $n \to \infty$, so that $x_n = [g_n]$. Let S_n and S_0 denote the boundaries of $\pi(g_n H(U))$ and $\pi(g_0 H(U))$ respectively so that $S_n \to S_0$ in $\mathcal{C}^m$ as $n \to \infty$.

We will choose a circle $C_0 \subset S_0$ and a sequence of circles $C_n \subset S_n$ so that $C_n \to C_0$ and $\limsup(C_n \cap \Lambda)$ contains two distinct points outside of $\cup_{S \in \mathcal{Q}} S$. If $m = 1$, we set $C_0 = S_0$. When $m \geq 2$, we choose a circle $C_0 \subset S_0$ as follows. Note that S_0 is not contained in any sphere in $\mathcal{Q}$ by the assumption on y and Lemma 8.2. Hence for any $S \in \mathcal{Q}$, $S_0 \cap S$ is a proper sub-sphere of S_0. Since $y \in F^*_{H(U)}$, for any component B_i of Ω, $S_0 \not\subset \overline{B_i}$ and hence $S_0 \cap \partial B_i$ is a proper sub-sphere of S_0. Choose a circle $C_0 \subset S_0$ such that $\{g_0^+, g_0^-\} \subset C_0 \cap \Lambda$, $C_0 \not\subset S$ for any $S \in \mathcal{Q}$, and $C_0 \not\subset \partial B_i \cap S_0$ for all i. This is possible, since $\mathcal{Q}$ is countable. Since $S_n \to S_0$, we can find a sequence of circles $C_n \subset S_n$ such that $C_n \to C_0$. We claim that $\limsup_n (C_n \cap \Lambda)$ is uncountable. Since $\# C_0 \cap \Lambda \geq 2$ and $C_0 \not\subset \partial B_i$, $C_0 \not\subset \overline{B_i}$ for all i. Therefore, by Lemma 8.1, for any infinite subsequence C_{n_k} of C_n,

$$\# \limsup_k (C_{n_k} \cap \Lambda) \geq 2.$$

By passing to a subsequence, we can find two distinct points $\xi_n, \xi'_n \in C_n \cap \Lambda$ which converge to two distinct points ξ, ξ' of $C_0 \cap \Lambda$ respectively as $n \to \infty$. Choose a sequence $p_n \to p \in G$ such that $p_n^+ = \xi_n$, $p_n^- = \xi'_n$, $p^+ = \xi$ and $p^- =$

ξ'. Let $\langle u_t\rangle < N$ be a one-parameter subgroup such that $p_n u_t^- = C_n - \{\xi_n\}$. By Proposition 4.11, $T_n = \{t : [p_n]u_t \in \mathrm{RF}\,\mathcal{M}\}$ is a global k-thick subset, and hence $\mathcal{T} := \limsup_n T_n$ is a global k-thick subset contained in the set $\{t : [p]u_t \in \mathrm{RF}\,\mathcal{M}\}$. Then $C_n \cap \Lambda$ converges, in the Hausdorff topology, to a compact subset $L \subset C_0 \cap \Lambda$ homeomorphic to the one-point compactification of $\mathcal{T}$. Therefore L is uncountable, so is $\limsup_n (C_n \cap \Lambda)$, proving the claim.

Let $\Psi := \cup_{S\in\mathcal{Q}} C_0 \cap S$, i.e., the union of all possible intersection points of C_0 and spheres in $\mathcal{Q}$. Since $C_0 \not\subset S$ for any $S \in \mathcal{Q}$, $\#C_0 \cap S \le 2$. Hence Ψ is countable, and hence $\limsup_n (C_n \cap \Lambda) - \Psi$ is uncountable. Note that this works for any infinite subsequence of C_n's. Therefore we can choose sequences $\xi_n^-, \xi_n^+ \in C_n \cap \Lambda$ converging to distinct points ξ^-, ξ^+ of $(C_0 \cap \Lambda) - \Psi$ respectively, by passing to a subsequence. As $\xi^-, \xi^+ \in C_0$ and $C_0 \subset S_0$, there exists a frame $g_0 h = (v_0, \cdots, v_{d-1}) \in g_0 H(U)$ whose first vector v_0 is tangent to the geodesic $[\xi^-, \xi^+]$. Setting $g := g_0 h$, we claim that

$$[g] \in \bigcap_i \mathscr{G}(U_\pm^{(i)}, yL).$$

Suppose that $[g] \in \mathscr{S}(U_\pm^{(i)}, yL)$ for some i. We will assume $[g] \in \mathscr{S}(U_-^{(i)}, yL)$, as the case when $[g] \in \mathscr{S}(U_+^{(i)}, yL)$ can be dealt similarly, by changing the role of g^- and g^+ below. For simplicity, set $U^{(i)} := U_-^{(i)}$. Now by Proposition 5.13, there exist $L_0 \in \mathcal{L}_{U^{(i)}}$ and $\alpha \in N \cap L$ such that $(L_0)_{nc} \lneq L_{nc}$ and $[g]\alpha L_0$ is closed. Let S denote the boundary of $\pi(g\alpha L_0)$. Since $\alpha \in N \cap L$, we have $(g\alpha)^+ = g^+ = \xi^+ \in S \cap \Lambda \cap C_0$. Since $S \subsetneq S^*$, $S \cap \Lambda \neq \emptyset$ and ΓS is closed, we have $S \in \mathcal{Q}$. It follows that $\xi^+ \in \Psi$, contradicting the choice of ξ^+. This proves the claim.

Now choose a vector $v_0^{(n)}$ which is tangent to the geodesic $[\xi_n^-, \xi_n^+]$. We then extend $v_0^{(n)}$ to a frame $g_n h_n \in g_n H(U)$ so that $g_n h_n$ converges to $g = g_0 h$ as $n \to \infty$. Since $\{\xi_n^\pm\} \subset \Lambda$, we have $[g_n h_n] \in \mathrm{RF}\,\mathcal{M}$. This completes the proof. □

We will need the following lemma later.

Lemma 8.4. *Let $k \ge 1$. Let χ be a k-horosphere in $\mathbb{H}^{k+1}$ resting at $p \in \partial\mathbb{H}^{k+1}$, and $\mathcal{P}$ be the geodesic k-plane in $\mathbb{H}^{k+1}$. Let $\xi \in \partial\mathcal{P}$, δ be a geodesic joining ξ and p, and $q = \delta \cap \chi$. There exists $R_0 > 1$ such that for any $R > R_0$, if $d(\chi, \mathcal{P}) < R - 1$, then $d(q, \mathcal{P}) < R$.*

Proof. For $k = 1$, this is shown in [27, Lemma 4.2]. Now let $k \ge 2$. Consider a geodesic plane $\mathbb{H}^2 \subset \mathbb{H}^{k+1}$ which passes through q and orthogonal to $\mathcal{P}$. Then $\chi \cap \mathbb{H}^2$ and $\mathcal{P} \cap \mathbb{H}^2$ are a horocycle and a geodesic in $\mathbb{H}^2$ respectively. As $d_{\mathbb{H}^{k+1}}(\chi, \mathcal{P}) = d_{\mathbb{H}^2}(\chi \cap \mathbb{H}^2, \mathcal{P} \cap \mathbb{H}^2)$ and $d_{\mathbb{H}^{k+1}}(q, \mathcal{P}) = d_{\mathbb{H}^2}(q, \mathcal{P} \cap \mathbb{H}^2)$, the conclusion follows from the case $k = 1$. □

Recall the definition of $\check{H} = H(U_{d-2})$ from Section 4.

Lemma 8.5. *Let $U < \check{H} \cap N$ be a non-trivial connected closed subgroup. If the boundary of $\pi(gH(U))$ is contained in ∂B for some component B of Ω, then $[g] \in \mathrm{BF}\,\mathcal{M} \cdot \mathrm{C}(H(U))$.*

Proof. As U is equal to $mU_k m^{-1}$ for some $m \in \check{H} \cap M$ and $1 \le k \le d-2$, the general case is easily reduced to the case when $U = U_k$. Since $g = (v_0, \cdots, v_d)$ has its first $(k+1)$-vectors tangent to the geodesic $(k+1)$-plane $\pi(gH(U_k))$ and $\partial(\pi(gH(U_k)) \subset \partial B$, we can use an element $c \in \mathrm{C}(H(U_k)) = \mathrm{SO}(d-k-2)$ to modify the next $(d-k-2)$-vectors so that gc has its first $(d-1)$-vectors tangent to $\mathrm{hull}(\partial B)$. Then $[gc] \in \mathrm{BF}\,\mathcal{M}$, proving the claim. ∎

Lemma 8.6. *Let $U < \check{H} \cap N$ be a non-trivial connected closed subgroup. If $x_n \in \mathrm{RF}\,\mathcal{M} \cdot U$ is a sequence converging to some $x \in \mathrm{RF}\,\mathcal{M}$, then passing to a subsequence, there exists $u_n \in U$ such that $x_n u_n \in \mathrm{RF}\,\mathcal{M}$ and at least one of the following holds:*

(1) *$u_n \to e$ and hence $x_n u_n \to x$, or*
(2) *$x = zc$ for some $z \in \mathrm{BF}\,\mathcal{M}$ with $c \in \mathrm{C}(H(U))$, and $x_n u_n$ accumulates on $z\check{H}c$.*

Proof. If x_n belongs to $\mathrm{RF}\,\mathcal{M}$ for infinitely many n, we simply take $u_n = e$. So assume that $x_n \notin \mathrm{RF}\,\mathcal{M}$ for all n. Choose a sequence $g_n \to g_0$ in G so that $x_n = [g_n]$ and $x = [g_0]$. As $x \in \mathrm{RF}\,\mathcal{M}$, we have $\{g_0(0), g_0(\infty)\} \subset \Lambda$. As $x_n \in \mathrm{RF}_+\mathcal{M} - \mathrm{RF}\,\mathcal{M}$, we have $g_n(\infty) \in \Lambda$ and $g_n(0) \in \Omega$. For each n, choose an element $u_n \in U$ so that $0 < \alpha_n := \|u_n\| \le \infty$ is the minimum of $\|u\|$ for all $u \in U$ satisfying $g_n u(0) \in \Lambda$. Set

$$\alpha := \limsup_n \alpha_n.$$

If $\alpha = 0$, then we are in case (1). Hence we will assume $0 < \alpha \le \infty$. Let C_n denote the boundary of $\pi(g_n H(U))$ and C_0 the boundary of $\pi(g_0 H(U))$. Then $C_n \to C_0$ in $\mathcal{C}^{\dim U}$. Recall that $B_U(r)$ denotes the ball of radius r centered at 0 inside U. Set

$$\mathcal{B}_n := g_n B_U(\alpha_n)(0) \quad \text{and} \quad \mathcal{B}_0 := g_0 B_U(\alpha)(0).$$

Then $\mathcal{B}_n \subset C_n \cap \Omega$, and $\partial \mathcal{B}_n \cap \Lambda \neq \emptyset$ by the choice of u_n. By passing to a subsequence, we have $\alpha_n \to \alpha$ and $\mathcal{B}_n \to \mathcal{B}_0$ as $n \to \infty$ and hence the diameter of $\mathcal{B}_n$ in $\mathbb{S}^{d-1}$ is bounded below by some positive number. Hence, passing to a subsequence, we may assume that $\mathcal{B}_n$ are all contained in the same component, say B of Ω. Consequently, $\mathcal{B}_0 \subset \overline{B}$.

We claim that $\#\overline{\mathcal{B}_0} \cap \partial B \ge 2$. First note that $g_0(0) \in \Lambda$. If $\alpha = \infty$, then $g_n u_n(0) \to g_0(\infty) \in \Lambda \cap \overline{\mathcal{B}_0}$. If $\alpha < \infty$, then u_n converges to some $u \in U$, passing to a subsequence, and $u \neq e$, as $\alpha > 0$. Now, $g_n u_n(0) \to g_0 u(0) \in \Lambda \cap \overline{\mathcal{B}_0}$. Since $\Lambda \cap \overline{B} \subset \partial B$, this proves the claim.

Therefore $\mathcal{B}_0$ is contained in ∂B, and hence so is C_0. By Lemma 8.5, this implies that $x = zc$ for some $z \in \mathrm{BF}\,\mathcal{M}$ and $c \in \mathrm{C}(H(U))$. We proceed to show that $x_n u_n$ accumulates on $z\check{H}c$. Since $c \in \mathrm{C}(H(U))$, we may assume $c = e$ by replacing x with xc^{-1}, and x_n with $x_n c^{-1}$.

We claim that the distance between $\pi(g_n u_n)$ and the plane $\pi(g_0\check{H})$ tends to 0 as $n \to \infty$. Since $x\check{H} = [g_0]\check{H}$ is compact, $g_n u_n \in g_n\check{H}$ and $\pi(g_n\check{H})$ is a geodesic plane nearly parallel to $\pi(g_0\check{H})$ for all large n, this claim implies that $[g_n]u_n$ accumulates on $z\check{H}$, completing the proof.

Now, to prove the claim, let $D_n := C_n \cap \partial B$, and $\mathcal{P}_n := \mathrm{hull}(D_n)$. Let $k = \dim U$. Since C_n is a k-sphere meeting the $(d-2)$-sphere $\partial B \subset \mathbb{S}^{d-1}$, and $C_n \not\subset \partial B$, it follows that D_n is a $(k-1)$-sphere. We set $\mathcal{H}_n := \mathrm{hull}(C_n)$, $\mathcal{H}_0 := \mathrm{hull}(C_0)$ and $\mathcal{H} := \mathrm{hull}(\partial B) = \pi(g_0\check{H})$. Then $\mathcal{H}_n \cap \mathcal{H} = \mathcal{P}_n$. Let $\varepsilon > 0$ be arbitrary, and $\mathcal{N}_\varepsilon(\mathcal{H})$ denote the ε-neighborhood of $\mathcal{H}$ in $\mathbb{H}^d$. Letting $d_{\mathcal{H}_n}(\cdot,\cdot)$ denote the hyperbolic distance in $\mathcal{H}_n$, we may write

$$\mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n = \{p \in \mathcal{H}_n : d_{\mathcal{H}_n}(p, \mathcal{P}_n) < R_n\}$$

for some $R_n > 0$. This is because $\mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n$ is convex and invariant under family of isometries, whose axes of translation and rotation are contained in $\mathcal{P}_n$. As $C_n \to C_0 \subset \partial B$ as $n \to \infty$, it follows that $R_n \to \infty$ as $n \to \infty$. Let $\chi_n := \pi(g_n U)$, and $\chi_0 := \pi(g_0 U)$, which are k-horospheres contained in $\mathcal{H}_n$ and $\mathcal{H}_0$ respectively.

We next show that there is a uniform upper bound for $d_{\mathcal{H}_n}(\mathcal{P}_n, \chi_n)$, $n \in \mathbb{N}$. To see this, we only need to consider those $\mathcal{P}_n$'s which are disjoint from χ_n, as $d_{\mathcal{H}_n}(\mathcal{P}_n, \chi_n) = 0$ otherwise. Since $\chi_n \to \chi_0$ and $C_n \to C_0$ as $n \to \infty$, it suffices to check that the diameters of D_n with respect to the spherical metric on $\mathbb{S}^{d-1}$ have a uniform positive lower bound. Let us write $C_n - D_n = E_n \cup E_n'$, where E_n is a connected component of $C_n - D_n$ meeting B, and E_n' is the other component. Since $C_n \to C_0$ as $n \to \infty$, a uniform lower bound for both $\mathrm{diam}(E_n)$ and $\mathrm{diam}(E_n')$ will give a uniform upper bound for $\mathrm{diam}(D_n)$. Since $\mathcal{B}_n \subset E_n$, $\mathrm{diam}(E_n) > \mathrm{diam}(\mathcal{B}_0)/2$ for all sufficiently large n. On the other hand, note that $\chi_n \subset \mathcal{H}_n$ is a horosphere resting at a point in E_n'. Since χ_n converges to χ, the condition that $\mathcal{P}_n \cap \chi_n = \emptyset$ implies that $\mathrm{diam}(E_n')$ is also bounded below by some positive constant. Since $R_n \to \infty$, we conclude that $d_{\mathcal{H}_n}(\mathcal{P}_n, \chi_n) < R_n - 1$ for all sufficiently large n. Applying Lemma 8.4 to $\mathbb{H}^{k+1} = \mathcal{H}_n$, $\chi = \chi_n$, $\mathcal{P} = \mathcal{P}_n$, $\xi = g_n^+$ and $q = \pi(g_n u_n)$, we have

$$d_{\mathcal{H}_n}(\pi(g_n u_n), \mathcal{P}_n) < R_n$$

and hence $\pi(g_n u_n) \in \mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n$, for all sufficiently large n. As $\varepsilon > 0$ was arbitrary, this proves that $\pi(g_n u_n)$ goes arbitrarily close to $\pi(g_0\check{H})$ as $n \to \infty$. This finishes the proof. □

Lemma 8.7. *Let $U < N$ be a non-trivial connected closed subgroup. If $x_n \to x$ in $F^* \cap \mathrm{RF}_+\mathcal{M}$, and $x \in F^* \cap \mathrm{RF}\,\mathcal{M}$, then there exists $u_n \to e$ in U such that $x_n u_n \in \mathrm{RF}\,\mathcal{M}$; in particular, $x_n u_n \to x$ in $F^* \cap \mathrm{RF}\,\mathcal{M}$.*

Proof. The general case easily reduces to the case when $U < \check{H} \cap N$. Then the claim follows from Lemma 8.6 and Lemma 4.5. □

Obtaining limits in F^*. For $\varepsilon > 0$, we set

$$\operatorname{core}_\varepsilon(\mathcal{M}) := \left\{x \in \Gamma\backslash G : \pi(x) \in \operatorname{core}\mathcal{M} \text{ and } d(\pi(x), \partial \operatorname{core}\mathcal{M}) \geq \varepsilon\right\}. \tag{8.1}$$

We note that $\operatorname{core}_\varepsilon(\mathcal{M})$ is a compact of F^* for all sufficiently large $\varepsilon > 0$. In the rest of the section, let $U < N$ denote a non-trivial connected closed subgroup.

Lemma 8.8. *Let $x \in \operatorname{RF}\mathcal{M}$, and $V = \{v_t : t \in \mathbb{R}\} < N$ be a one-parameter subgroup. If $\pi(xV) \not\subset \partial \operatorname{core}\mathcal{M}$, and $xv_{t_i} \in \operatorname{RF}\mathcal{M}$ for some sequence $t_i \to +\infty$, then there exists a sequence $s_i \to +\infty$ such that xv_{s_i} converges to a point in F^*.*

Proof. It suffices to show that there exists a sequence $s_i \to +\infty$ such that $xv_{s_i} \in \operatorname{core}_{\eta/3}\mathcal{M}$ where η is as given in (4.11). Let $x = [g]$, and set $o = (1, 0, \cdots, 0) \in \mathbb{H}^d = \mathbb{R}^+ \times \mathbb{R}^{d-1}$. We may assume $g = (e_0, \cdots, e_{d-1})_o \in \operatorname{F}\mathbb{H}^d$ where e_i are standard basis vectors in $\operatorname{T}_o\mathbb{H}^d \simeq \mathbb{R}^d$. Note that for $V^+ = \{v_t : t > 0\}$, gV^+ is a translation of the frame g along a horizontal ray emanating from o along the V^+-direction. By the definition of η, the $\eta/3$-neighborhoods of $\operatorname{hull} B_i$'s are mutually disjoint. For each i, set $s_i := t_i$ if $xv_{t_i} \in \operatorname{core}_{\eta/3}\mathcal{M}$. Otherwise, there exists a unique j such that $d(\pi(gv_{t_i}), \operatorname{hull} B_j) < \eta/3$. If $\pi(gV_{[t_i,\infty)})$ were contained in the $\eta/3$-neighborhood of $\operatorname{hull} B_j$, then the unique geodesic 2-plane which contains $\pi(gV_{[t_i,\infty)})$ must lie in $\partial \operatorname{hull} B_j$, and hence $\pi(xV) \subset \partial \operatorname{core}\mathcal{M}$; this contradicts the hypothesis. Therefore there exists $t_i < s_i < \infty$ such that $d(\pi(gv_{s_i}), \operatorname{hull} B_j) = \eta/3$. The sequence s_i satisfies the claim. □

Lemma 8.9. *Let $x_nL_nv_n$ be a sequence of closed orbits with $x_n \in \operatorname{RF}_+\mathcal{M}$, $L_n \in \mathcal{L}_U$ and $v_n \in (L_n \cap N)^\perp$. Suppose that either*

(1) $x_n \in F^*$ for all n; or
(2) $x_nL_nv_n \cap \operatorname{RF}_+\mathcal{M} \cap F^* \neq \emptyset$ for all n.

Then

$$F^* \cap \limsup_n (x_nL_nv_n \cap \operatorname{RF}_+\mathcal{M}) \neq \emptyset.$$

Proof. We claim that if $x_n \in F^*$, then $x_nL_nv_n \cap \operatorname{RF}_+\mathcal{M} \cap F^* \neq \emptyset$, that is, the hypothesis (1) implies (2). Suppose not. Then, since $A \subset L_n$, $(x_nAv_nA \cap \operatorname{RF}_+\mathcal{M}) \subset \operatorname{RF}_+\mathcal{M} - F^*$. Since the set $\operatorname{RF}_+\mathcal{M} - F^*$ is a closed A-invariant set and $e \in \overline{Av_nA}$, we would have $x_n \in \operatorname{RF}_+\mathcal{M} - F^*$, yielding a contradiction. It follows from the claim that there exists $z_n \in x_nL_n \cap \operatorname{RF}_+\mathcal{M}$ such that $\pi(z_nv_nU) \not\subset \partial \operatorname{core}\mathcal{M}$ for all n. In particular, there exists $u_n \in U$ such that $z_nv_nu_n \in \operatorname{core}_{\eta/3}(\mathcal{M})$. Since $\operatorname{core}_{\eta/3}(\mathcal{M})$ is a compact subset of F^*, $z_nv_nu_n = z_nu_nv_n$ converges to a point in F^*, finishing the proof. □

Lemma 8.10. *Let x_0L be a closed orbit with $x_0 \in \operatorname{RF}\mathcal{M}$ and $L \in \mathcal{L}_U$. Suppose that E is a closed U-invariant subset containing $x_0Lv_n \cap \operatorname{RF}_+\mathcal{M}$ for some sequence $v_n \to \infty$ in $(L \cap N)^\perp$. If $x_0 \in F^*$ or $x_0Lv_n \cap \operatorname{RF}_+\mathcal{M} \cap F^* \neq \emptyset$*

for all n, then there exist $y \in \mathrm{RF}\,\mathcal{M} \cap F^*$ and a one parameter subgroup $V \subset (L \cap N)^\perp$ such that

$$E \supset y(L \cap N)V.$$

Proof. Note that

$$(x_0 L v_n \cap \mathrm{RF}_+ \mathcal{M})(v_n^{-1} A v_n) \subset E.$$

By Lemma 8.9, there exists

$$y \in F^* \cap \limsup_{n\to\infty} (x_0 L v_n \cap \mathrm{RF}_+ \mathcal{M}).$$

Since $y \in F^* \cap \mathrm{RF}_+ \mathcal{M} \subset \mathrm{RF}\,\mathcal{M} \cdot U$, we may assume $y \in F^* \cap \mathrm{RF}\,\mathcal{M}$ by modifying y using an element of U. Note that $\liminf_{n\to\infty} (x_0 L v_n \cap \mathrm{RF}_+ \mathcal{M}) \supset y(L \cap N)$, passing to a subsequence. Since $\limsup_{n\to\infty}(v_n^{-1} A v_n)$ contains a one-parameter subgroup $V \subset (L \cap N)^\perp$ by Lemma 3.4, we obtain that $y(L \cap N)V \subset E$. □

Lemma 8.11. *If $yLv_0 \cap \mathrm{RF}\,\mathcal{M} \cap F^* \neq \emptyset$ for some $v_0 \in N$ and $L \in \mathcal{L}_U$, then $yLv \cap F^* \cap \mathrm{RF}\,\mathcal{M} \neq \emptyset$ for all $v \in Av_0A$.*

Proof. Let $y_0 := y\ell v_0 \in yLv_0 \cap F^* \cap \mathrm{RF}\,\mathcal{M}$, and $v = av_0b \in Av_0A$. Then $(y\ell a^{-1})v = y\ell v_0 b \in F^* \cap \mathrm{RF}\,\mathcal{M}$ as $F^* \cap \mathrm{RF}\,\mathcal{M}$ is A-invariant. Since $y\ell a^{-1} v \in yLv$, the claim is proved. □

Lemma 8.12. *Let $x_0 L$ be a closed orbit with $x_0 \in \mathrm{RF}\,\mathcal{M}$ and $L \in \mathcal{L}_U$. Suppose that E is a closed AU-invariant subset containing $x_0 Lv \cap \mathrm{RF}_+ \mathcal{M}$ for some non-trivial element $v \in (L \cap N)^\perp$. If $x_0 \in F^*$ or $x_0 Lv \cap \mathrm{RF}\,\mathcal{M} \cap F^* \neq \emptyset$, then there exist $y \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and a one parameter subgroup $V \subset (L \cap N)^\perp$ such that*

$$E \supset y(L \cap N)VA.$$

Proof. Since X is A-invariant, we get

$$(x_0 L \cap \mathrm{RF}_+ \mathcal{M})AvA \subset E.$$

Choose a sequence $v_n := a_n v a_n^{-1} \in AvA$ tending to ∞. Note that either $x_0 \in F^*$ or for all n, $x_0 L v_n \cap \mathrm{RF}\,\mathcal{M} \cap F^* \neq \emptyset$ by Lemma 8.11. Therefore the claim follows from Lemma 8.10. □

9. Limits of unipotent blowups

Let $\mathcal{M}$ be a convex cocompact hyperbolic manifold with Fuchsian ends and fix $k > 1$ as given by Proposition 4.11. In the whole section, we fix a non-trivial connected subgroup $U < N$. For a given sequence $g_i \to e$, and a sequence of k-thick subsets T_i of a one-parameter subgroup $U_0 < U$, we study the following set

$$\limsup \mathsf{T}_i g_i U$$

under certain conditions on the sequence g_i. The basic tool used here is the so-called *quasi-regular map* associated to the sequence g_i introduced in the work of Margulis-Tomanov [24] to study the object $\limsup U_0 g_i U$ in the finite volume case. For our application, we need a somewhat more precise information on the shape of the set $\limsup U_0 g_i U$ as well as $\limsup \mathsf{T}_i g_i U$ than discussed in [24].

Let $U^\perp$ denote the orthogonal complement of U in $N \simeq \mathbb{R}^{d-1}$ as defined in section 3. Recall from (3.2) that

$$\mathrm{N}(U) = AN\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$$

where $\mathrm{C}_1(U) = \mathrm{C}(H(U))$ and $\mathrm{C}_2(U) = H(U) \cap M \cap \mathrm{C}(U^\perp)$. Since $\mathrm{N}(U)$ is the identity component of $\mathrm{N}_G(U)$, for a sequence $g_i \to e$, the condition $g_i \in \mathrm{N}_G(U)$ means $g_i \in \mathrm{N}(U)$ for all sufficiently large $i \gg 1$. Note that the product $AU^\perp\,\mathrm{C}_2(U)$ is a connected subgroup of G, since $\mathrm{C}_2(U)$ commutes with $U^\perp$, and A normalizes $U^\perp\,\mathrm{C}_2(U)$.

Lemma 9.1. *For a given sequence $g_i \to e$ in $G - \mathrm{N}(U)$, there exists a one-parameter subgroup $U_0 < U$ such that the following holds; for any given sequence of k-thick subsets $\mathsf{T}_i \subset U_0$, there exist sequences $t_i \in \mathsf{T}_i$, and $u_i \in U$ such that as $i \to \infty$,*

$$u_i g_i u_{t_i} \to \alpha$$

for some non-trivial element $\alpha \in AU^\perp\,\mathrm{C}_2(U) - \mathrm{C}_2(U)$. Moreover, α can be made arbitrarily close to e.

Proof. Set $L := AU^\perp M N^+$. Note that

$$\mathrm{N}(U) \cap L = AU^\perp\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$$

and that the product map from $U \times L$ to G is a diffeomorphism onto a Zariski open neighborhood of e in G.

Following [24], we will construct a quasi-regular map

$$\psi : U \to \mathrm{N}(U) \cap L$$

associated to the sequence g_i. Except for a Zariski closed subset of U, the product $g_i u$ can be written as an element of UL in a unique way. We denote by $\psi_i(u) \in L$ its L-component so that

$$g_i u \in U\psi_i(u).$$

By Chevalley's theorem, there exists an $\mathbb{R}$-regular representation $G \to \mathrm{GL}(W)$ with a distinguished point $p \in W$ such that $U = \mathrm{Stab}_G(p)$. Then pG is locally closed, and

$$\mathrm{N}_G(U) = \{g \in G : pgu = pg \text{ for all } u \in U\}.$$

For each i, the map $\tilde\phi_i : U \to W$ defined by

$$\tilde\phi_i(u) = pg_i u$$

is a polynomial map in $U = \mathbb{R}^m$ of degree uniformly bounded, and $\tilde\phi_i(e)$ converges to p as $i \to \infty$. As $g_i \notin \mathrm{N}_G(U)$, $\tilde\phi_i$ is non-constant. Denote by

$B(p,r)$ the ball of radius r centered at p, fixing a norm $\|\cdot\|$ on W. Since pG is open in its closure, we can find $\lambda_0 > 0$ such that

$$B(p,\lambda_0) \cap \overline{pG} \subset pG. \tag{9.1}$$

Without loss of generality, we may assume that $\lambda_0 = 2$ by renormalizing the norm. Now define

$$\lambda_i := \sup\{\lambda \geq 0 : \tilde{\phi}_i(B_U(\lambda)) \subset B(p,2)\}.$$

Note that $\lambda_i < \infty$ as ϕ_i is nonconstant, and $\lambda_i \to \infty$ as $i \to \infty$, as $g_i \to e$. We define $\phi_i : U \to W$ by

$$\phi_i(u) := \tilde{\phi}_i(\lambda_i u).$$

This forms an equi-continuous family of polynomials on U. Therefore, after passing to a subsequence, ϕ_i converges to a non-constant polynomial ϕ uniformly on every compact subset of U. Moreover $\sup\{\|\phi(u) - p\| : u \in B_U(1)\} = 1$, $\phi(B_U(1)) \subset pL$, and $\phi(0) = p$. Now the following map ψ defines a non-constant rational map defined on a Zariski open dense neighborhood of $\mathcal{U}$ of e in U:

$$\psi := \rho_L^{-1} \circ \phi$$

where ρ_L is the restriction to L of the orbit map $g \mapsto p.g$. We have $\psi(e) = e$ and

$$\psi(u) = \lim_i \psi_i(\lambda_i u)$$

where the convergence is uniform on compact subsets of $\mathcal{U}$ and

$$\psi(u) \in L \cap \mathrm{N}(U) = AU^\perp\, \mathrm{C}_1(U)\, \mathrm{C}_2(U).$$

Since ψ is non-constant, there exists a one-parameter subgroup $U_0 < U$ such that $\psi|_{U_0}$ is non-constant. Now let T_i be a sequence of k-thick sets in $U_0 \simeq \mathbb{R}$. Then T_i/λ_i is also a k-thick set, and so is

$$\mathsf{T}_\infty := \limsup_{i\to\infty} (\mathsf{T}_i/\lambda_i) \subset U_0.$$

Finally, for all $t \in \mathsf{T}_\infty$, there exists a sequence $t_i \in \mathsf{T}_i$ such that $t_i/\lambda_i \to t$ as $i \to \infty$ (by passing to a subsequence). Since $\psi_i \circ \lambda_i \to \psi$ uniformly on compact subsets,

$$\psi(t) = \lim_{i\to\infty} (\psi_i \circ \lambda_i)\,(t_i/\lambda_i) = \lim_{i\to\infty} \psi_i(t_i).$$

By the definition of ψ_i, this means that there exists $u_i \in U$ such that

$$\psi(t) = \lim_{i\to\infty} u_i g_i u_{t_i}.$$

Since $\psi|_{U_0}$ is a non-constant continuous map, and an uncountable set T_∞ accumulates on 0, the image $\psi(\mathsf{T}_\infty)$ contains a non-trivial element α of $AU^\perp\, \mathrm{C}_1(U)\, \mathrm{C}_2(U)$ which can be taken arbitrarily close to e.

We now claim that if α is sufficiently close to e, then it belongs to $AU^\perp\, \mathrm{C}_2(U)$. Consider $H'(U) := H(U)\, \mathrm{C}_1(U)$, and let $\mathfrak{h}$ denote its Lie algebra. Now for all i large enough, using the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ in (3.5), we can write $g_i = c_i d_i r_i$ where $c_i \in \mathrm{C}_1(U)$, $d_i \in H(U)$ and $r_i \in \exp \mathfrak{h}^\perp$, Since c_i commutes with U, we can write

$$u_i g_i u_{t_i} = (u_i u_{t_i}) c_i (u_{t_i}^{-1} d_i u_{t_i})(u_{t_i}^{-1} r_i u_{t_i}).$$

On the other hand, we have

$$\lim_i p u_i g_i u_{t_i} = \lim_i p c_i (u_{t_i}^{-1} d_i u_{t_i})(u_{t_i}^{-1} r_i u_{t_i}) = p\alpha.$$

Since $c_i \to e$, $u_{t_i} d_i u_{t_i}^{-1} \in H(U)$, and $u_{t_i} r_i u_{t_i}^{-1} \in \exp \mathfrak{h}^\perp$, it follows that both sequences $u_{t_i} d_i u_{t_i}^{-1}$ and $u_{t_i} r_i u_{t_i}^{-1}$ must converge, say to $h \in H(U)$ and to $q \in \exp \mathfrak{h}^\perp$, respectively. Hence $\alpha = hq$ by replacing h by uh for some $u \in U$. On the other hand, we can write $\alpha = avc_1c_2 \in AU^\perp \,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$. So $hq = avc_1c_2$. Note that $c := c_1c_2 \in \mathrm{C}(H(U))H(U) = H'(U)$. We get

$$(a^{-1}hc^{-1})(cqc^{-1}) = v. \tag{9.2}$$

Now, when α is sufficiently close to e, all elements appearing in (9.2) are also close to e. Recall that the map $H'(U) \times \mathfrak{h}^\perp \to G$ given by $(h', X) \to h' \exp X$ is a local diffeomorphism onto a neighborhood of e. Since $(a^{-1}hc^{-1}) \in H'(U)$, and $cqc^{-1}, v \in \exp \mathfrak{h}^\perp$, we have $a^{-1}hc^{-1} = e$ and $cqc^{-1} = v$ for α sufficiently small. In particular,

$$a^{-1}hc_2^{-1} = c_1^{-1} \in H(U) \cap \mathrm{C}(H(U)) = \{e\}.$$

Hence $c_1 = e$. It follows that $\alpha \in AU^\perp \,\mathrm{C}_2(U)$, as desired.

We further claim that we can choose α outside of $\mathrm{C}_2(U)$. As $\mathrm{C}_2(U)$ is a compact subgroup, we can choose a $\mathrm{C}_2(U)$-invariant Euclidean norm $\|\cdot\|$ on W. If $\alpha = \psi(t) \in \mathrm{C}_2(U)$ for some $t \in \mathsf{T}_\infty \subset U_0$, then t is one of finitely many solutions of the polynomial equation $\|\phi(t)\|^2 = \|p\|^2$. Therefore, except for finitely many $t \in \mathsf{T}_\infty$, $\alpha = \psi(t) \in AU^\perp \,\mathrm{C}_2(U) - \mathrm{C}_2(U)$. This finishes the proof. □

The following lemma is similar to Lemma 9.1, but here we consider the case when U is the whole horospherical subgroup N. In this restrictive case, the limiting element can be taken inside A.

Lemma 9.2. *Let $\mathsf{T}_i \subset N$ be a sequence of k-thick subsets in the sense that for any one-parameter subgroup $U_0 < N$, $\mathsf{T}_i \cap U_0$ is a k-thick subset of $U_0 \simeq \mathbb{R}$. For any sequence $g_i \to e$ in $G - \mathrm{N}_G(N)$, there exist $t_i \to \infty$ in T_i and $u_i \in N$ such that*

$$u_i g_i u_{t_i} \to a$$

for some non-trivial element $a \in A$. Moreover, a can be chosen to be arbitrarily close to e.

Proof. We first consider the case when g_i belongs to the opposite horospherical subgroup N^+. We will use the notations u^+ and u^- defined in Section 3. Write $g_i = \exp u^+(w_i)$ for some $w_i \in \mathbb{R}^{d-1}$. For $x \in \mathbb{R}^{d-1}$, set

$u_x := \exp u^-(x) \in N$. Let $\varepsilon > 0$ be arbitrary. Since T_i is a k-thick subset of N, there exists $\alpha_i \in \mathbb{R}$ such that $\alpha_i w_i \in \mathsf{T}_i$ and

$$\varepsilon < \frac{|\alpha_i|\|w_i\|^2}{2} < k\varepsilon.$$

Setting $u_{x_i} := u_{\alpha_i w_i} \in \mathsf{T}_i$ and $y_i := -\alpha_i w_i \left(1 + \frac{\alpha_i \|w_i\|^2}{2}\right)^{-1}$, we compute:

$$u_{y_i} g_i u_{x_i} = \begin{pmatrix} \left(1 + \frac{\alpha_i\|w_i\|^2}{2}\right)^{-2} & 0 & 0 \\ \left(1 + \frac{\alpha_i\|w_i\|^2}{2}\right)^{-1} w_i & \mathrm{I}_{d-1} & 0 \\ -\frac{\|w_i\|^2}{2} & -\left(1 + \frac{\alpha_i\|w_i\|^2}{2}\right) w_i^t & \left(1 + \frac{\alpha_i\|w_i\|^2}{2}\right)^2 \end{pmatrix}.$$

The condition for the size of α_i guarantees that, by passing to a subsequence, the sequence $u_{x_i} g_i u_{y_i}$ converges to an element

$$\mathrm{diag}(\alpha, \mathrm{I}_{d-1}, \alpha^{-1}) \in A, \quad \text{for } \alpha \in [(1-\varepsilon)^{-2}, (1-k\varepsilon)^{-2}] \cup [(1+k\varepsilon)^{-2}, (1+\varepsilon)^{-2}]$$

as $i \to \infty$. This proves the claim when $g_i \in N^+$.

Since the product map $A \times M \times N^+ \times N \to G$ is a diffeomorphism onto a Zariski-open neighborhood of e in G, we can write $g_i = a_i m_i u_i^+ u_i^-$ for some $a_i \in A$, $m_i \in M$, $u_i^+ \in N^+$ and $u_i^- \in N$ all of which converge to e as $i \to \infty$. By the previous case, we can find $u_{t_i} \in \mathsf{T}_i$ and $u_i \in N$ such that $u_i u_i^+ u_{t_i}$ converges to a non-trivial element $a \in A$. Let $\tilde{u}_i := (a_i m_i) u_i (a_i m_i)^{-1} \in N$. Then $\tilde{u}_i g_i u_{t_i} = a_i m_i u_i u_i^+ u_i^- u_{t_i} = a_i m_i (u_i u_i^+ u_{t_i}) u_i^- \to a$ as $i \to \infty$, proving the claim. □

Lemma 9.3. *Let L be any connected reductive subgroup of G normalized by A. Let U_0 be a one-parameter subgroup of $L \cap N$. Let $\mathsf{T}_i \subset U_0$ be a sequence of k-thick subsets. For a given sequence $r_i \to e$ in $\exp(\mathfrak{l}^\perp) - \mathrm{N}(U_0)$, there exists a sequence $t_i \in \mathsf{T}_i$ such that as $i \to \infty$,*

$$u_{t_i}^{-1} r_i u_{t_i} \to v$$

for some non-trivial element $v \in (L \cap N)^\perp$, and v can be chosen arbitrarily close to e. Moreover, for all n large enough, we can make v so that

$$n \leq \|v\| \leq 2k^2 n.$$

Proof. Without loss of generality, by Proposition 3.7, we may assume that $L_{nc} = H(U)$ for $U = U_k = \mathbb{R}^k$ some $k \geq 1$ and $U_0 := \mathbb{R}e_1$. We write $r_i = \exp(q_i)$ where $q_i \to 0$ in $\mathfrak{l}^\perp$. Using the notations introduced in section 3 and setting $\mathfrak{u}^\perp = \mathrm{Lie}(U^\perp) = \mathbb{R}^{d-1-k}$, we can write

$$q_i = u^-(x_i) + u^+(y_i) + m(C_i)$$

where $x_i \in \mathfrak{u}^\perp$, $y_i \in (\mathfrak{u}^\perp)^t$, and $C_i = \begin{pmatrix} 0_k & B_i \\ -B_i^t & A_i \end{pmatrix}$ is a skew symmetric matrix, all of which converge to 0 as $i \to \infty$. We consider $U_0 = \mathbb{R}e_1$ as $\{u_s = se_1 \in \mathbb{R}^{d-1}\}$ and define the map $\psi_i : \mathbb{R} \to \mathfrak{l}^\perp$ by

$$\psi_i(s) = u_s^{-1} q_i u_s \quad \text{for all } s \in \mathbb{R};$$

this is well-defined since $\mathfrak{l}^\perp$ is $\mathrm{Ad}(L)$-invariant. Then a direct computation shows

$$\psi_i(s) = u^-(x_i + sB_i^t e_1 + s^2 y_i/2) + u^+(y_i) + m(\tilde{C}_i) \tag{9.3}$$

where $\tilde{C}_i$ is a skew-symmetric matrix of the form

$$\tilde{C}_i = \begin{pmatrix} 0_k & B_i + se_1 y_i^t \\ -B_i^t - sy_i e_1^t & A_i \end{pmatrix}.$$

Since $r_i \notin \mathrm{N}(U_0)$, it follows that either $y_i \neq 0$ or $y_i = 0$ and $B_i^t e_1 \neq 0$. Hence ψ_i is a non-constant polynomial of degree at most 2, and $\psi_i(0) \to 0$. Let $\lambda_i \in \mathbb{R}$ be defined by

$$\lambda_i = \sup\{\lambda > 0 : |\psi_i[-\lambda, \lambda]| \leq 1\}.$$

Then $0 < \lambda_i < \infty$ and $\lambda_i \to \infty$. Now the rescaled polynomials $\phi_i = \psi_i \circ \lambda_i : \mathbb{R} \to \mathfrak{l}^\perp$ form an equicontinuous family of polynomials of degree at most 2 and $\lim_{i\to\infty} \phi_i(0) = 0$. Therefore ϕ_i converges to a polynomial $\phi : \mathbb{R} \to \mathfrak{l}^\perp$ uniformly on compact subsets. Since $\phi(0) = 0$ and $\sup\{|\phi(\lambda)| : \lambda \in [-1, 1]\} = 1$, ϕ is a non-constant polynomial. From (9.3), it can be easily seen that $\mathrm{Im}(\phi)$ is contained $\mathrm{Lie}(N) \cap \mathfrak{l}^\perp$, by considering the two cases of $y_i \neq 0$, and $y_i = 0$ and $B_i^t e_1 \neq 0$ separately. For a given sequence T_i of k-thick subsets of U_0, set

$$\mathsf{T}_\infty := \limsup_{i\to\infty} (\mathsf{T}_i/\lambda_i),$$

which is also a k-thick subset of U_0.

Let $s \in \mathsf{T}_\infty$. By passing to a subsequence, there exists $t_i \in \mathsf{T}_i$ such that $t_i/\lambda_i \to s$ as $i \to \infty$. As $\phi_i \to \phi$ uniformly on compact subsets, it follows that

$$\phi(s) = \lim_{i\to\infty} \psi_i(\lambda_i \cdot t_i/\lambda_i) = \lim_{i\to\infty} u_{t_i}^{-1} q_i u_{t_i}.$$

Since T_∞ accumulates on 0, so does $\phi(\mathsf{T}_\infty)$. Taking the exponential map to each side of the above, the first part of the lemma follows.

The second part of the lemma holds by applying Lemma 9.4 below for the non-constant polynomial $p(s) = \|\phi(s)\|^2$ of degree at most 4. □

Lemma 9.4. *If $p \in \mathbb{R}[s]$ is a polynomial of degree $\delta \geq 1$ and $\mathsf{T} \subset \mathbb{R}$ is a k-thick subset, then $p(\mathsf{T})$ is $2k^\delta$-thick at ∞.*

Proof. Let C be the coefficient of s^δ term of the polynomial p. Then there exists $s_0 > 1$ such that $\frac{1}{\sqrt{2}} \leq \frac{|p(s)|}{|Cs^\delta|} \leq \sqrt{2}$ for all $|s| > s_0$. Let $r > \frac{|C|s_0^\delta}{\sqrt{2}}$. Since T is k-thick, there exists $t \in \mathsf{T}$ such that $(\sqrt{2}r/|C|)^{1/\delta} < |t| < k(\sqrt{2}r/|C|)^{1/\delta}$. We compute that $r \leq |p(t)| \leq 2k^\delta r$, proving the claim. □

10. Translates of relative U-minimal sets

Assume that $\mathcal{M}$ is a convex cocompact hyperbolic manifold with Fuchsian ends and fix $k > 1$ as given by Proposition 4.11. In this section, we fix a non-trivial connected closed subgroup $U < N$. Unless mentioned otherwise, we let R be a compact A-invariant subset of $\mathrm{RF}\,\mathcal{M}$ such that for every $x \in R$, and for any one-parameter subgroup $U_0 = \{u_t\}$ of U, the following set

$$\{t \in \mathbb{R} : xu_t \in R\}$$

is k-thick. In practice, R will be either $\mathrm{RF}\,\mathcal{M}$ or a compact subset of the form $\mathrm{RF}\,\mathcal{M} \cap F^*_{H(U)} \cap X$ for a closed $H(U)$-invariant subset X.

The main aim of this section is to prove Propositions 10.6 and 10.9 using the results of section 9. The results in this section are needed in the step of finding a closed orbit in a given $H(U)$-orbit closure of an $\mathrm{RF}\,\mathcal{M}$-point.

Definition 10.1.
- A U-invariant closed subset $Y \subset \Gamma\backslash G$ is U-minimal if yU is dense in Y for any $y \in Y$.
- A U-invariant closed subset $Y \subset \Gamma\backslash G$ is U-minimal with respect to R if $Y \cap R \neq \emptyset$ and for any $y \in Y \cap R$, yU is dense in Y.

A U-minimal subset may not exist, but a U-minimal subset with respect to a compact subset R always exists by Zorn's lemma. In this section, we study how to find an additional invariance of Y beyond U under certain conditions.

Lemma 10.2. *Let $Y \subset \Gamma\backslash G$ be a U-minimal subset with respect to R. For any $y \in Y \cap R$, there exists a sequence $u_n \to \infty$ in U such that $yu_n \to y$.*

Proof. The set $Z := \{z \in Y : yu_n \to z \text{ for some } u_n \to \infty \text{ in } U\}$ is U-invariant and closed. By the assumption on R, there exists $u_n \to \infty$ in U such that $yu_n \in Y \cap R$. Since $Y \cap R$ is compact, yu_n converges to some $z \in Y \cap R$, by passing to a subsequence. Hence Z intersects $Y \cap R$ non-trivially. Therefore $Z = Y$, by the U-minimality of Y with respect to R. □

A subset S of a topological space is said to be *locally closed* if S is open in its closure $\overline{S}$.

Lemma 10.3. *Let Y be a U-minimal subset of $\Gamma\backslash G$ with respect to R, and S be a closed subgroup of $\mathrm{N}(U)$ containing U. For any $y_0 \in Y \cap R$, the orbit $y_0 S$ is not locally closed.*

Proof. Suppose that $y_0 S$ is locally closed for some $y_0 \in Y \cap R$. Since Y is U-minimal with respect to R, there exists $u_n \to \infty$ in U such that $y_0 u_n \to y_0$ by Lemma 10.2. We may assume that $y_0 = [e]$ without loss of generality. Since $y_0 S$ is locally closed, $y_0 S$ is homeomorphic to $(S \cap \Gamma)\backslash S$ (cf. [49, Theorem 2.1.14]). Therefore there exists $\delta_n \in S \cap \Gamma$ such that $\delta_n u_n \to e$ as $n \to \infty$. Since $\mathrm{N}(U) = AN\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, writing $\delta_n = a_n r_n$ for $a_n \in A$ and $r_n \in N\,\mathrm{C}_1(U)\,\mathrm{C}_2(U)$, it follows that $a_n \to e$. On the other hand, note that a_n is non-trivial as Γ does not contain any elliptic or parabolic element. This is a contradiction, as there exists a positive lower bound for the translation lengths of elements of Γ, which is given by the minimal length of a closed geodesic in M. □

In the rest of this section, we use the following notation:

$$H = H(U),\ H' = H'(U),\ \text{and } F^* = F^*_{H(U)}.$$

Lemma 10.4. *For every U-minimal subset $Y \subset \Gamma\backslash G$ with respect to $\operatorname{RF}\mathcal{M}$ such that $Y \cap F^* \cap \operatorname{RF}\mathcal{M} \neq \emptyset$, and for any $y_0 \in Y \cap F^* \cap \operatorname{RF}\mathcal{M}$, there exists a sequence $g_n \to e$ in $G - \operatorname{N}(U)$ such that $y_0 g_n \in Y \cap \operatorname{RF}\mathcal{M}$ for all n.*

Proof. Let $y_0 \in Y \cap F^* \cap \operatorname{RF}\mathcal{M}$. As $Y = \overline{y_0 U}$, $Y \subset \operatorname{RF}_+\mathcal{M}$. Using Lemma 4.5 and the fact that F^* is open, we get that there exists an open neighborhood $\mathcal{O}$ of e such that

$$y_0\mathcal{O} \cap Y \subset Y \cap F^* \subset Y \cap \operatorname{RF}\mathcal{M} \cdot U. \tag{10.1}$$

Without loss of generality, we may assume that the map $g \mapsto y_0 g \in \Gamma\backslash G$ is injective on $\mathcal{O}$, by shrinking $\mathcal{O}$ if necessary. We claim that there exists $g_n \to e$ in $G - \operatorname{N}(U)$ such that $y_0 g_n \in Y \cap F^*$. Suppose not. Then there exists a neighborhood $\mathcal{O}' \subset \mathcal{O}$ of e such that

$$y_0\mathcal{O}' \cap Y \subset y_0 \operatorname{N}(U). \tag{10.2}$$

Set

$$S := \{g \in \operatorname{N}(U) : Yg = Y\}$$

which is a closed subgroup of $\operatorname{N}(U)$ containing U. We will show that $y_0 S$ is locally closed; this contradicts Lemma 10.3. We first claim that

$$y_0\mathcal{O}' \cap Y \subset y_0 S. \tag{10.3}$$

If $g \in \mathcal{O}'$ such that $y_0 g \in Y$, then $g \in \operatorname{N}(U)$. Therefore $\overline{y_0 g U} = \overline{y_0 U} g = Yg \subset Y$. Moreover, $Yg \cap \operatorname{RF}\mathcal{M} \neq \emptyset$ by (10.1). Hence $Yg = Y$, proving that $g \in S$. Now, (10.3) implies that $y_0 S$ is open in Y. On the other hand, since $U \subset S$, we get $Y = \overline{y_0 S}$. Therefore, $y_0 S$ is locally closed.

Hence we have $g_n \to e$ in $G - \operatorname{N}(U)$ such that $y_0 g_n \in Y \cap F^*$. Since $y_0 g_n \in F^* \cap \operatorname{RF}_+\mathcal{M}$ converges to $y_0 \in F^* \cap \operatorname{RF}\mathcal{M}$, by Lemma 8.7, there exists a sequence $u_n \to e$ in U such that $y_0 g_n u_n \in \operatorname{RF}\mathcal{M}$. Therefore, by replacing g_n with $g_n u_n$, this finishes the proof. ☐

Lemma 10.5. *Let Y be a U-minimal subset with respect to R, and let W be a connected closed subgroup of $\operatorname{N}(U)$. Suppose that there exists a sequence $\alpha_i \to e$ in W such that $Y\alpha_i \subset Y$. Then there exists a one-parameter subsemigroup $S < W$ such that $YS \subset Y$.*

Moreover if W_0 is a compact Lie subgroup of W and $\alpha_i \in W - W_0$ for all i, then S can be taken so that $S \not\subset W_0$.

Proof. The set $S_0 = \{g \in W : Yg \subset Y\}$ is a closed subsemigroup of W. Write $\alpha_i = \exp \xi_i$ for some $\xi_i \in \operatorname{Lie}(W)$. Then the sequence $v_i := \|\xi_i\|^{-1}\xi_i$ of unit vectors has a limit, say, v. It suffices to note that $S := \{\exp(tv) : t \geq 0\}$ is contained in the closure of the subsemigroup generated by α_i's. Now suppose that $\alpha_i \in W - W_0$. Set $M_0 := \{g \in W_0 : Yg = Y\}$. This is a closed Lie subgroup of W_0. Write $\operatorname{Lie} W = \mathfrak{m}_0 \oplus \mathfrak{m}_0^{\perp}$ where $\mathfrak{m}_0 = \operatorname{Lie} M_0$. By modifying α_i by elements of M_0, we may assume $\alpha_i = \exp \xi_i$ for $\xi_i \to 0$

in $\mathfrak{m}_0^\perp$. Letting $v \in \mathfrak{m}_0^\perp$ be a limit of $\xi_i/\|\xi_i\|$, it remains to check $v \notin W_0$. Suppose not. Since W_0 is compact, we have $\{\exp tv : t \geq 0\} = \exp \mathbb{R}v$. Hence for all $t \geq 0$, $Y \exp tv \subset Y$ as well as $Y \exp(-tv) \subset Y$. Therefore $Y \exp tv = Y$. Hence $\exp v \in M_0$. This is a contradiction, since $v \in \mathfrak{m}_0^\perp$. ☐

Proposition 10.6 (Translate of Y inside of Y)**.** *Let Y be a U-minimal set of $\Gamma\backslash G$ with respect to* $\mathrm{RF}\,\mathcal{M}$ *such that* $Y \cap F^* \cap \mathrm{RF}\,\mathcal{M} \neq \emptyset$.

Then there exists an unbounded one-parameter subsemigroup S inside the subgroup $AU^\perp \mathrm{C}_2(U)$ such that

$$YS \subset Y.$$

Proof. Choose $y_0 \in Y \cap \mathrm{RF}\,\mathcal{M} \cap F^*$. By Lemma 10.4, there exists $g_i \to e$ in $G - \mathrm{N}(U)$ such that $y_0 g_i \in Y \cap \mathrm{RF}\,\mathcal{M}$. Let $U_0 = \{u_t\}$ be a one-parameter subgroup of U as given by Lemma 9.1, with respect to the sequence g_i.

Let

$$\mathsf{T}_i := \{u_t \in U_0 : y_0 g_i u_t \in Y \cap \mathrm{RF}\,\mathcal{M}\}$$

which is a k-thick subset of U_0. By Lemma 9.1, there exist sequences $u_{t_i} \to \infty$ in T_i, and $u_i \in U$ such that

$$u_i g_i u_{t_i} \to \alpha$$

for some element $\alpha \in AU^\perp \mathrm{C}_2(U) - \mathrm{C}_2(U)$. Note that $y_0 g_i u_{t_i} \in Y \cap \mathrm{RF}\,\mathcal{M}$ converges to some $y_1 \in Y \cap \mathrm{RF}\,\mathcal{M}$ by passing to a subsequence. Hence as $i \to \infty$,

$$y_0 u_i^{-1} = y_0 g_i u_{t_i} (u_i g_i u_{t_i})^{-1} \to y_1 \alpha^{-1}.$$

So $y_1\alpha^{-1} \in Y$, and hence $Y\alpha^{-1} \subset Y$, since $y_1 \in Y \cap \mathrm{RF}\,\mathcal{M}$. Since α can be made arbitrarily close to e in Lemma 9.1, the claim follows from Lemma 10.5. ☐

Proposition 10.7 (Translate of Y inside of X)**.** *Let X be a closed H'-invariant set such that $X \cap R \neq \emptyset$. Let $Y \subset X$ be a U-minimal subset with respect to R, and assume that there exists $y \in Y \cap R$ and a sequence $g_n \to e$ in $G - H'$ such that $yg_n \in X$ for all n. Then there exists some non-trivial $v \in U^\perp$ such that*

$$Yv \subset X.$$

Proof. Let $\mathfrak{h}$ denote the Lie algebra of H'. We may write $g_n = r_n h_n$ where $h_n \in H'$ and $r_n \in \exp \mathfrak{h}^\perp$. By replacing g_n with $g_n h_h^{-1}$, we may assume $g_n = r_n$. If $r_n \in U^\perp$ for some n, then the claim follows since $y_0 r_n \in X$ and hence $Yr_n \subset X$. Hence we assume that $r_n \notin U^\perp$ for all n. We have from (3.5)

$$\mathfrak{h}^\perp \cap \mathrm{Lie}(\mathrm{N}(U)) = \mathrm{Lie}\, U^\perp.$$

Hence $r_n \notin \mathrm{N}(U)$ for all n. Therefore there exists a one-parameter subgroup $U_0 = \{u_t\} < U$ such that $r_n \notin \mathrm{N}(U_0)$. Let

$$\mathsf{T} = \{t \in \mathbb{R} : yu_t \in R\}.$$

Since $y \in R$, it follows that T is a k-thick subset of $\mathbb{R}$ by the assumption on R. Hence, by Lemma 9.3, there exists $t_n \in \mathsf{T}$ such that $u_{t_n}^{-1} r_n u_{t_n} \to v$ for some non-trivial $v \in U^\perp$. Observe

$$(yu_{t_n})(u_{t_n}^{-1} r_n u_{t_n}) = yr_n u_{t_n} \in X.$$

Passing to a subsequence, $yu_{t_n} \to y_0$ for some $y_0 \in Y \cap R$, and hence $y_0 v \in X$. It follows $Yv \subset X$. $\square$

For a one-parameter subgroup $V = \{v_t : t \in \mathbb{R}\}$ and a subset $I \subset \mathbb{R}$, the notation V_I means the subset $\{v_t : t \in I\}$.

Lemma 10.8. *Let X be a closed AU-invariant set of $\Gamma\backslash G$, and V be a one-parameter subgroup of $U^\perp$. Assume that $R := X \cap \operatorname{RF}\mathcal{M} \cap F^*$ is non-empty and compact. If $x_0 V_I \subset X$ for some $x_0 \in R$ and a closed interval I containing 0, then X contains a V-orbit of a point in R.*

Proof. Choose a sequence $a_n \in A$ such that $\liminf_{n\to\infty} a_n V_I a_n^{-1}$ contains a subsemigroup V^+ of V as $n \to \infty$. Then

$$(x_0 a_n^{-1})(a_n V_I a_n^{-1}) = x_0 V_I a_n^{-1} \subset X.$$

By passing to a subsequence, we have $x_0 a_n^{-1}$ converges to some $x_1 \in \operatorname{RF}\mathcal{M}$; so $x_1 V^+ \subset X$. Since R is compact, so is $\overline{x_0 A} \cap F^*$, which implies that $x_1 \in \overline{x_0 A} \cap F^*$. Since x_1 belongs to the open set F^*, it follows $x_1 v_s \in F^*$ for all sufficiently small $s \in \mathbb{R}$. By Lemma 4.5, this implies that $x_1 v_s U \cap \operatorname{RF}\mathcal{M} \neq \emptyset$ for some $s > 0$ with $v_s \in V^+$. Note that

$$(x_1 v_s U)(v_s^{-1} V^+) = x_1 U V^+ \subset X.$$

Choose $x_2 \in x_1 v_s U \cap \operatorname{RF}\mathcal{M} \subset X \cap \operatorname{RF}\mathcal{M} \cap F^*$. Then $x_2(v_s^{-1}V^+) \subset X$. Similarly as before, let $a_n \in A$ be a sequence such that $\liminf_{n\to\infty} a_n(v_s^{-1}V^+)a_n^{-1} = V$ and such that $x_2 a_n^{-1}$ converges to some $x_3 \in R$. From

$$(x_2 a_n^{-1})(a_n v_s^{-1} V^+ a_n^{-1}) = x_2 v_s^{-1} V^+ a_n^{-1} \subset X,$$

we conclude that $x_3 V \subset X$. This finishes the proof. $\square$

Proposition 10.9. *Let X be a closed H'-invariant set. Assume that $R := X \cap F^* \cap \operatorname{RF}\mathcal{M}$ is a non-empty compact set, and let $Y \subset X$ be a U-minimal subset with respect to R. Suppose that there exists $y \in Y \cap R$ such that $X - yH'$ is not closed. Then there exist an element $z \in R$ and a non-trivial connected closed subgroup $V < U^\perp$ such that*

$$zUV \subset X.$$

Proof. Since $X - yH'$ is not closed, there exists a sequence $g_n \to e$ in $G - H'$ such that $yg_n \in X$ for all $n \geq 1$. By Lemma 10.8, it suffices to find $x_0 \in R$ and a one-parameter subgroup $V < U^\perp$ such that $x_0 V_I \subset X$ for some interval $I < \mathbb{R}$ containing 0. It follows from Propositions 10.6 and 10.7 that

$$Yv_0 \subset X \quad \text{ and } \quad YS \subset Y$$

where $v_0 \in U^\perp - \{e\}$ and S is an unbounded one-parameter subsemigroup of $AU^\perp \operatorname{C}_2(U)$. By Lemma 3.3, S is either of the form

(1) $S = \{\exp(t\xi_V)\exp(t\xi_C) : t \geq 0\}$, or
(2) $S = \{\big(v\exp(t\xi_A)v^{-1}\big)\exp(t\xi_C) : t \geq 0\}$

for some $\xi_A \in \mathrm{Lie}(A)-\{0\}, \xi_C \in \mathrm{Lie}(\mathrm{C}_2(U)), \xi_V \in \mathrm{Lie}(V)-\{0\}$, and $v \in U^\perp$.
Case (1)**:** Since X is $H'(U)$-invariant, we may assume $YS \subset X$ with $\xi_C = 0$; so the claim follows.
Case (2)**:** Set

$$Y_0 := Y\,\mathrm{C}_2(U).$$

It is easy to check that Y is a $U\,\mathrm{C}_2(U)$-minimal subset of X with respect to R. First suppose that $v = e$. Let $A^+ := \{\exp(t\xi_A) : t \geq 0\}$. Since $YS \subset Y$ and $\xi_C \in \mathrm{Lie}(\mathrm{C}_2(U))$, it follows that $Y_0A^+ \subset Y_0$. Choose $y \in Y \cap R$, and let $a_n \to \infty$ be a sequence in A^+. Since R is compact and A-invariant, ya_n converges to some $z_0 \in R$ by passing to a subsequence. Since $Y_0A^+ \subset Y_0$, we have $z_0 \in Y_0 \cap R$. Since $\liminf a_{-n}A^+ = A$, we get $z_0A \subset Y_0$. Since $z_0AU\,\mathrm{C}_2(U) = z_0U\,\mathrm{C}_2(U)A$, and Y_0 is $U\,\mathrm{C}_2(U)$-minimal with respect to R, we obtain $Y_0A \subset Y_0$. Since v_0 commutes with $\mathrm{C}_2(U)$, we also get $Y_0v_0 \subset X$. Therefore $Y_0Av_0 \subset Y_0v_0 \subset X$. By the A-invariance of X, it follows $Y_0(Av_0A) \subset X$. Since Av_0A contains some V^+, the claim follows.

Next suppose $v \neq e$. Since $\mathrm{C}_2(U)$ commutes with v, it follows that

$$Y_0vA^+v^{-1} \subset Y_0.$$

Since X is A-invariant, we get

$$Y_0(vA^+v^{-1})A \subset Y_0A \subset X.$$

Set $V := \exp\mathbb{R}(\log v)$. Since $vA^+v^{-1}A$ contains V_I for some interval I containing 0 for any subsemigroup A^+ of A. we get $Y_0V_I \subset X$, finishing the proof. ☐

11. Closures of orbits inside ∂F and non-homogeneity

Let $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ be a convex cocompact hyperbolic manifold with non-empty Fuchsian ends. Let U be a connected closed subgroup of $\check{H} \cap N$ and set $H := H(U)$ as before. Then

$$\partial F = \mathrm{BF}\,\mathcal{M}\cdot\check{V}^+\cdot H'(U) \quad\text{and}\quad \partial F \cap \mathrm{RF}\,\mathcal{M} = \mathrm{BF}\,\mathcal{M}\cdot\mathrm{C}(H(U)).$$

In this section, we classify closures of $xH(U)$ and xAU for $x \in \partial F - \mathrm{RF}\,\mathcal{M}$ (Theorem 11.5); they are never homogeneous.

Theorem 11.1. *If $x = zc \in \mathrm{BF}\,\mathcal{M}\cdot\mathrm{C}(H(U))$ with $z \in \mathrm{BF}\,\mathcal{M}$ and $c \in \mathrm{C}(H(U))$. Then*

(1) *$\overline{xU} = xL$ for some $L \in \mathcal{Q}_U$ contained in $c^{-1}\check{H}c$;*
(2) *$\overline{xH(U)} = xL$ for some $L \in \mathcal{L}_U$ contained in $c^{-1}\check{H}c$, and for any $y \in \mathscr{G}(U, xL)$, $\overline{yU} = xL$;*
(3) *$\overline{xAU} = \overline{xH(U)}$.*

Proof. Since x is contained in the compact homogeneous space $xc^{-1}\check{H}c$, the claims (1) and (2) are special cases of Ratner's theorem [37], which were also proved by Shah independently [44]. So we only need to discuss the proof of (3). We show that $\overline{xAU} = xL$ where L is given by (2). If $U = L \cap N$, then the claim follows from Theorem 13.1. Suppose that U is a proper subgroup of $L \cap N$. Since $\overline{xAU}(K \cap H(U)) = \overline{xH(U)} = xL$ and $\mathscr{S}(U, xL) \cdot (K \cap H(U))$ is a proper subset of xL (cf. Lemma 5.15), there exists $y \in \overline{xAU} \cap \mathscr{G}(U, xL)$. Hence (3) follows from (2). $\square$

Lemma 11.2. *Let $V^+ \subset N$ be a one-parameter subsemigroup which is not contained in $\check{H}$. Then $V^+H(U)$ is a closed subset of G.*

Proof. Since the product map $A \times N \to AN$ is a diffeomorphism and AN is closed, the product subset AW is closed in G for any closed subset W of N. Hence AUV^+ is a closed subset of AN. We use Iwasawa decompositions $H(U) = UA(K \cap H(U))$, and the fact that $AV^+ = V^+A$ in order to write $V^+H(U) = AUV^+(K \cap H(U))$. Hence the conclusion follows from compactness of $K \cap H(U)$. $\square$

Lemma 11.3. *Let $V^+ \subset N$ be as in Lemma 11.2. If $g_i \in \check{H}$ is a sequence such that $g_i v_i h_i$ converges for some $v_i \in V^+$ and $h_i \in H(U)$ as $i \to \infty$, then, after passing to a subsequence, there exists $p_i \in AU$ such that $g_i p_i$ converges to an element of $\check{H}$ as $i \to \infty$.*

Proof. We write $g_i = \tilde{k}_i \tilde{a}_i \tilde{n}_i \in (K \cap \check{H})A(N \cap \check{H})$ and $h_i = u_i a_i k_i \in UA(K \cap H(U))$. Since $K \cap \check{H}$ and $K \cap H(U)$ are compact, we may assume without loss of generality that $\tilde{k}_i = k_i = e$ for all i. Observe that

$$\begin{aligned} g_i v_i h_i &= \tilde{a}_i \tilde{n}_i v_i u_i a_i \\ &= \tilde{a}_i a_i (a_i^{-1} \tilde{n}_i u_i a_i)(a_i^{-1} v_i a_i) \end{aligned}$$

where $\tilde{a}_i a_i \in A$, $a_i^{-1} \tilde{n}_i u_i a_i \in N \cap \check{H}$, and $a_i^{-1} v_i a_i \in V^+$. Since $g_i v_i h_i$ converges as $i \to \infty$ and the product map $A \times (N \cap \check{H}) \times V^+ \to G$ is an injective proper map, it follows that all three sequences $\tilde{a}_i a_i$, $a_i^{-1} \tilde{n}_i u_i a_i$ and $a_i^{-1} v_i a_i$ are convergent as $i \to \infty$. Noting that

$$g_i u_i a_i = \tilde{a}_i \tilde{n}_i u_i a_i = \tilde{a}_i a_i (a_i^{-1} \tilde{n}_i u_i a_i),$$

it remains to set $p_i := u_i a_i \in AU$ to finish the proof. $\square$

For $z \in \operatorname{BF} \mathcal{M}$, $\pi(z\check{H}\check{V}^+\check{H}) = \pi(z\check{H}\check{V}^+)$ is the closure of a Fuchsian end, of the form $S_0 \times [0, \infty)$ where $S_0 = \pi(z\check{H})$.

Lemma 11.4. *Let $z \in \operatorname{BF} \mathcal{M}$. Let zL be a closed orbit contained in $\operatorname{BF} \mathcal{M}$ for some $L \in \mathcal{L}_U$ contained in $\check{H}$, and $V^+ \subset N$ be a one-parameter subsemigroup such that $\check{H}V^+ = \check{H}\check{V}^+$. Then both $zLV^+H(U)$ and zLV^+ are closed.*

Proof. Without loss of generality, we assume $z = [e]$. Let B denote the component of Ω such that $\operatorname{hull}(\partial B) = \pi(\check{H})$ for the projection map π :

$G \to \mathbb{H}^d$. Since $\check{H}V^+ = \check{H}\check{V}^+$, we have $\pi(\check{H}V^+\check{H}) = \operatorname{hull}\overline{B}$. Note that if $\gamma(\operatorname{hull}(B)) \cap \operatorname{hull}(B) \neq \emptyset$ for $\gamma \in \Gamma$, then $\gamma \in \check{H} \cap \Gamma = \operatorname{Stab}_\Gamma(B)$.

Suppose that $\gamma_i \ell_i v_i h_i$ converges to some element $g \in G$ where $\gamma_i \in \Gamma$, $\ell_i \in L$, $v_i \in V^+$ and $h_i \in H(U)$. Since $\pi(\gamma_i \ell_i v_i h_i) \in \Gamma \operatorname{hull}\overline{B}$, and $\Gamma \operatorname{hull}\overline{B}$ is a closed subset of $\mathbb{H}^d$, we have $\pi(g) \in \Gamma \operatorname{hull}\overline{B}$. Without loss of generality, we may assume $\pi(g) \in \operatorname{hull}\overline{B}$ by replacing γ_i by $\gamma\gamma_i$ for some $\gamma \in \Gamma$ if necessary.

We claim that by passing to a subsequence,

$$\gamma_i \in \check{H} \cap \Gamma.$$

Let $\mathcal{O}$ be a neighborhood of $\pi(g)$ such that

$$\mathcal{O} \cap \Gamma \operatorname{hull}\overline{B} \subset \operatorname{hull}\overline{B};$$

such $\mathcal{O}$ exists, since $d(\operatorname{hull}(\gamma B), \operatorname{hull}(B)) \geq \eta$ for all $\gamma \in \Gamma - (\check{H} \cap \Gamma)$ where $\eta > 0$ is given in (4.11). By passing to a subsequence, we may assume that $\pi(\gamma_i \ell_i v_i h_i) \in \mathcal{O}$. Since $\pi(\ell_i v_i h_i) \in \operatorname{hull}\overline{B}$ for all i, it follows that $\pi(\gamma_i \ell_i v_i h_i) \in \operatorname{hull}\overline{B}$ for all n. Therefore $\gamma_i \in \check{H} \cap \Gamma$. Applying Lemma 11.3 to the sequence $(\gamma_i \ell_i) v_i h_i \to g$, there exists $p_i \in AU$ such that $\gamma_i \ell_i p_i \to h$ in $\check{H}$ as $i \to \infty$. Since ΓL is closed, we have $h \in \Gamma L$.

Since $p_i^{-1} v_i h_i \in AUV^+H(U) = V^+H(U)$ and

$$\lim_{i \to \infty} p_i^{-1} v_i h_i = h^{-1} g, \tag{11.1}$$

we have $h^{-1}g \in V^+H(U)$ by Lemma 11.2. Therefore, $g = h(h^{-1}g) \in \Gamma L V^+ H(U)$. This proves that $\Gamma L V^+ H(U)$ is closed. Note that in the above argument, if $h_i = e$ for all i, then $h^{-1} g = \lim p_i^{-1} v_i \in AUV^+$. Hence $g = h(h^{-1}g) \in \Gamma LAUV^+ = \Gamma L V^+$. This proves that $\Gamma L V^+$ is closed. □

Note that $x \in \mathrm{RF}_+\mathcal{M} - \mathrm{RF}\,\mathcal{M} \cdot H(U)$ if and only if $x \in (\mathrm{RF}_+\mathcal{M} \cap \partial F_{H(U)}) - \mathrm{BF}\,\mathcal{M} \cdot \mathrm{C}(H(U))$.

Theorem 11.5. *Let $x \in \mathrm{RF}_+\mathcal{M} - \mathrm{RF}\,\mathcal{M} \cdot H(U)$. Then there exist a compact orbit $zL \subset \mathrm{BF}\,\mathcal{M}$ with $L \in \mathcal{L}_U$, an element $c \in \mathrm{C}(H(U))$ and a one-parameter subsemigroup $V^+ \subset N$ with $\check{H}V^+ = \check{H}\check{V}^+$ such that*

(1) $\overline{xH(U)} = zLV^+H(U)c$;
(2) $\overline{xAU} = zLV^+c$.

Moreover the closure of the geodesic plane $\pi(xH(U))$ is diffeomorphic to a properly immersed submanifold $S \times [0, \infty)$ where $S = \pi(zL)$ is a compact geodesic plane inside $\mathrm{BF}\,\mathcal{M}$.

Proof. We write $x = z_0 v c$ for some non-trivial $v \in \check{V}^+$, $z_0 \in \mathrm{BF}\,\mathcal{M}$ and $c \in \mathrm{C}(H(U))$. Without loss of generality, we may assume $c = e$. By Theorem 11.1, $\overline{z_0 U} = z_0 v_0^{-1} L v_0$ where $L \in \mathcal{L}_U$ is contained in $\check{H}$ and $v_0 \in \check{H} \cap N$. Hence $\overline{xH(U)}$ contains $zL(v_0 v)H(U)$ for $z := z_0 v_0^{-1} \in \mathrm{BF}\,\mathcal{M}$. Set $V^+ := \{\exp t(\log(v v_0)) : t \geq 0\}$.

Note that V^+ is contained in $A(v_0 v)A \cup \{e\}$, and hence

$$zL \cup zLv_0 v H(U) = zLV^+H(U)$$

and $\check{H}V^+ = \check{H}\check{V}^+$ since $v \neq e$.

Since $\overline{xH(U)} \subset zL \cup zL(v_0 v)H(U)$, and zL lies in the closure of $zL(v_0v)H(U)$, the claim (1) follows since $zLV^+H(U)$ is closed by Lemma 11.4. For the claim (2), note that $\overline{xAU} \supset \overline{z_0 U} vA = zLV^+$. By Lemma 11.4, zLV^+ is AU-invariant and closed. Since $x \in zLV^+$, we conclude $\overline{xAU} = zLV^+$.

To see the last claim, observe that $\pi(zLV^+H(U)) = \pi(zLV^+AU) = \pi(zLV^+)$ since $V^+AU = AUV^+$, and $AU < L$. Since $\check{H}V^+ = \check{H}\check{V}^+$, and $\pi(zL)$ is a compact geodesic plane (without boundary) in $\pi(z\check{H})$, we get $\pi(z\check{H}V^+) \simeq \pi(z\check{H}) \times [0, \infty)$ and $\pi(zLV^+) \simeq \pi(zL) \times [0, \infty)$. □

Remark 11.6. An immediate consequence of Theorem 11.5 is that if $P \subset \mathcal{M}$ is a geodesic plane such that $P \cap \operatorname{core} \mathcal{M} = \emptyset$ but $\overline{P} \cap \operatorname{core} \mathcal{M} \neq \emptyset$, then P is not properly immersed in $\mathcal{M}$ and $\overline{P}$ is a properly immersed submanifold with non-empty boundary.

12. Density of almost all U-orbits

Let $\Gamma < G = \mathrm{SO}^\circ(d, 1)$ be a Zariski dense convex cocompact subgroup. The action of N on $\mathrm{RF}_+ \mathcal{M}$ is minimal, and hence any N-orbit is dense in $\mathrm{RF}_+ \mathcal{M}$ [48]. Given a non-trivial connected closed subgroup U of N, there exists a dense U-orbit in $\mathrm{RF}_+ \mathcal{M}$ [29]. In this section, we deduce from [30] and [29] that almost every U-orbit is dense in $\mathrm{RF}_+ \mathcal{M}$ with respect to the Burger-Roblin measure in the case of a convex cocompact hyperbolic manifold with Fuchsian ends (Corollary 12.4).

The critical exponent $\delta = \delta_\Gamma$ of Γ is defined to be the infimum $s \geq 0$ such that the Poincare series $\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma(o))}$ converges for any $o \in \mathbb{H}^d$. It is known that δ is equal to the Hausdorff dimension of the limit set Λ and $\delta = d - 1$ if and only if Γ is a lattice in G [45].

Denote by m^{BR} the N-invariant Burger-Roblin measure supported on $\mathrm{RF}_+ \mathcal{M}$; it is characterized as a unique locally finite Borel measure supported on $\mathrm{RF}_+ \mathcal{M}$ (up to a scaling) by ([6], [40], [48]). We won't give an explicit formula of this measure as we will only use the fact that its support is equal to $\mathrm{RF}_+ \mathcal{M}$, together with the following theorem: recall that a locally finite U-invariant measure μ is ergodic if every U-invariant measurable subset has either zero measure or zero co-measure, and is conservative if for any measurable subset S with positive measure, $\int_U 1_S(xu)du = \infty$ for μ-almost all x, where du denotes the Haar measure on U.

Theorem 12.1 ([30], [29])**.** *Let $U < N$ be a connected closed subgroup, and let Γ be a convex cocompact Zariski dense subgroup of G. Then m^{BR} is U-ergodic and conservative if $\delta >$ co-$\dim_N(U)$.*

Lemma 12.2. *Suppose that $\Gamma_1 < \Gamma_2$ are convex cocompact subgroups of G with $[\Gamma_1 : \Gamma_2] = \infty$. Then $\delta_{\Gamma_1} < \delta_{\Gamma_2}$.*

Proof. Note that a convex cocompact subgroup is of divergent type ([45], [40]). Hence the claim follows from [8, Proposition 9] if we check that $\Lambda_{\Gamma_1} \neq \Lambda_{\Gamma_2}$.

If $\Lambda := \Lambda_{\Gamma_1} = \Lambda_{\Gamma_2}$, then their convex hulls are the same, and hence the convex core of the manifold $\Gamma_i\backslash\mathbb{H}^d$ is equal to $\Gamma_i\backslash\text{hull}(\Lambda)$, which is compact. Since we have a covering map $\Gamma_1\backslash\text{hull}(\Lambda) \to \Gamma_2\backslash\text{hull}(\Lambda)$, it follows that $[\Gamma_1 : \Gamma_2] < \infty$. □

Lemma 12.3. *If $\Gamma\backslash\mathbb{H}^d$ is a convex cocompact hyperbolic manifold with Fuchsian ends, then $\delta > d-2$.*

Proof. If Γ is a lattice, then $\Lambda = \mathbb{S}^{d-1}$ and $\delta = d-1$. If $\Gamma\backslash\mathbb{H}^d$ is a convex cocompact hyperbolic manifold with non-empty Fuchsian ends, then Γ contains a cocompact lattice Γ_0 in a conjugate of $\text{SO}(d-1,1)$ whose limit set is equal to ∂B_i for some i. Now $[\Gamma : \Gamma_0] = \infty$; otherwise, $\Lambda = \partial B_i$. Hence $\delta > \delta_{\Gamma_0} = d-2$ by Lemma 12.2. □

Corollary 12.4. *Let $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Let $U < N$ be any non-trivial connected closed subgroup. Then for m^{BR}-almost every $x \in \text{RF}_+\mathcal{M}$,*

$$\overline{xU} = \text{RF}_+\mathcal{M}.$$

Proof. Without loss of generality, we may assume that $U = \{u_t\}$ is a one-parameter subgroup. By Lemma 12.3 and Theorem 12.1, m^{BR} is U-ergodic and conservative. Since $\text{RF}_+\mathcal{M}$ is second countable and the U-action on it is continuous, the claim follows. □

Since $F^*_{H(U)} \cap \text{RF}_+\mathcal{M}$ is a non-empty open subset, it follows that almost all U-orbits in $F^*_{H(U)} \cap \text{RF}_+\mathcal{M}$ are dense in $\text{RF}_+\mathcal{M}$.

13. Horospherical action in the presence of a compact factor

Let $\mathcal{M} = \Gamma\backslash\mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends and fix a non-trivial connected closed subgroup U of N. Consider a closed orbit xL for $x \in \text{RF}\,\mathcal{M}$ where $L \in \mathcal{Q}_U$ and $U = L\cap N$. The subgroup U is a horospherical subgroup of L, which is known to act minimally on $xL\cap\text{RF}_+\mathcal{M}$ provided $L = L_{nc}$. In this section, we extend the U-minimality on xL in the case when L has a compact factor.

Theorem 13.1. *Let $X := xL$ be a closed orbit where $x \in \text{RF}_+\mathcal{M}$, and $L \in \mathcal{Q}_U$. Let $U := L \cap N$. Then the following holds:*

1. *$X \cap \text{RF}_+\mathcal{M}$ is U-minimal.*
2. *X is L_{nc}-minimal.*
3. *If $L \in \mathcal{L}_U$ and $x \in \text{RF}\,\mathcal{M}$, then $X\cap\text{RF}\,\mathcal{M}$ contains a dense A-orbit.*
4. *For any non-trivial connected closed subgroup $U_0 < U$, for m^{BR}_X-almost all $x \in X$,*

$$\overline{xU_0} = X \cap \text{RF}_+\mathcal{M}.$$

The subgroup $L \in \mathcal{Q}_U$ is of the form $v^{-1}H(U)Cv$ where $H(U)C \in \mathcal{L}_U$ and $v \in N$. A general case can be easily reduced to the case where $L \in \mathcal{L}_U$.

In the following, we assume $L = H(U)C \in \mathcal{L}_U$. As before, we set

$$H = H(U),\ H' = H'(U), \text{ and } F^* = F^*_{H(U)}$$

and let $\pi_1 : H' \to H$ and $\pi_2 : H' \to \mathrm{C}(H)$ be the canonical projections. In order to define $\mathsf{m}^{\mathrm{BR}}_X$, choose $g \in G$ so that $[g] = x$. If we identify $H \simeq \mathrm{SO}^\circ(k,1)$, then by Proposition 4.9, $S := \pi_1(g^{-1}\Gamma g \cap HC)\backslash\mathbb{H}^k$ is a convex cocompact hyperbolic manifold with Fuchsian ends. Now $\pi_1(g^{-1}\Gamma g \cap HC)\backslash H$ is the frame bundle of S, on which there exists the Burger-Roblin measure as discussed in section 12. In the above statement, the notation $\mathsf{m}^{\mathrm{BR}}_X$ means the C-invariant lift of this measure to $X = xHC$.

We first prove the following, which is a more concrete version of Proposition 10.6 in the case at hand:

Proposition 13.2. *Let X be as in Theorem 13.1. Any U-minimal set Y of X with respect to $\mathrm{RF}\,\mathcal{M}$ such that $Y \cap F^* \cap \mathrm{RF}\,\mathcal{M} \neq \emptyset$ is A-invariant.*

Proof. Let Y be a U-minimal set of X with respect to $\mathrm{RF}\,\mathcal{M}$. Let $y_0 \in Y \cap F^* \cap \mathrm{RF}\,\mathcal{M}$. By Lemma 10.4, there exists a sequence $g_i \to e$ in $HC - \mathrm{N}(U)$ such that $y_0 g_i \in Y \cap \mathrm{RF}\,\mathcal{M}$ for all $i \geq 1$.

Since U is a horospherical subgroup of H and C commutes with H, we can apply Lemma 9.2 to the sequence g_i^{-1} and the sequence of k-thick sets $\mathsf{T}_i := \{u \in U : y_0 g_i u \in Y \cap \mathrm{RF}\,\mathcal{M}\}$ of U. This gives us sequences $u_{t_i} \to \infty$ in T_i and $u_i \in U$ such that as $i \to \infty$,

$$u_{t_i}^{-1} g_i u_i \to a$$

for some non-trivial element $a \in A$. Since $y_0 u_{t_i}$ converges to some $y_1 \in Y \cap \mathrm{RF}\,\mathcal{M}$ by passing to a subsequence, we have

$$y_1 a = \lim (y_0 u_{t_i})(u_{t_i}^{-1} g_i u_i) \in Y.$$

Since $\overline{y_1 U} = Y$, we get $Ya \subset Y$. Since a can be made arbitrarily close to e by Lemma 9.2, there exists a subsemigroup A_+ of A such that $YA_+ \subset Y$ by Lemma 10.5. Moreover, for any $a \in A_+$, $Ya \cap \mathrm{RF}\,\mathcal{M} \neq \emptyset$ as $\mathrm{RF}\,\mathcal{M}$ is A-invariant. Therefore, $Ya = Y$. It follows that $Ya^{-1} = Y$ as well. Hence Y is A-invariant. $\square$

We now present:

Proof of Theorem 13.1. First suppose that $xL \cap F^* \neq \emptyset$. We may then assume $x \in F^* \cap \mathrm{RF}\,\mathcal{M}$. Let Y be a U-minimal set of X with respect to $\mathrm{RF}\,\mathcal{M}$. If Y were contained in ∂F, then $Y \subset \partial F \cap \mathrm{RF}\,\mathcal{M}$. Since $\mathrm{Stab}_L(x)$ is Zariski dense in L by the definition of $\mathcal{L}_U$, it follows from [5, Lemma 4.13] that $X \cap \mathrm{RF}_+\,\mathcal{M}$ is AU-minimal. Therefore we have

$$\overline{YA} = X \cap \mathrm{RF}_+\,\mathcal{M}$$

and hence X has to be contained in the closed A-invariant subset $\partial F \cap \mathrm{RF}\,\mathcal{M}$ as well, yielding a contradiction. Therefore,

$$Y \cap F^* \cap \mathrm{RF}\,\mathcal{M} \neq \emptyset.$$

Hence, by Proposition 13.2, Y is A-invariant. Therefore the claim (1) follows from the AU-minimality of $X \cap \mathrm{RF}_+ \mathcal{M}$ if $x \in F^*$. Now suppose $xL \subset \partial F$. In this case, it suffices to consider the case when U is a proper subgroup of N; otherwise $L = G$ and has no compact factor. Hence we may assume without loss of generality that $U \subset \check{H} \cap N$. As xL is closed, Theorem 11.5 implies that $xL \subset \mathrm{BF}\, \mathcal{M} \cdot \mathrm{C}(H(U))$. Hence by modifying x by an element of $\mathrm{C}(H(U))$, we may assume that X is contained in a compact homogeneous space of $\check{H} = \mathrm{SO}^\circ(d-1, 1)$, which is the frame bundle of a convex cocompact hyperbolic manifold with empty Fuchsian ends. Therefore the claim (1) follows from the previous case of $x \in F^*$, since $F^* = \mathrm{RF}\, \mathcal{M}$ in the finite volume case.

Claim (2) follows from (1) since $\mathrm{RF}_+ \mathcal{M} \cdot H$ is closed, and $X \subset \mathrm{RF}_+ \mathcal{M} \cdot H$.

For the claim (3), it suffices to show that the A action on $X \cap \mathrm{RF}\, \mathcal{M}$ is topologically transitive (cf. [7]). Let x, $y \in X \cap \mathrm{RF}\, \mathcal{M}$ be arbitrary, and $\mathcal{O}$, $\mathcal{O}'$ be open neighborhoods of e in H. The set $UU^t A(M \cap H)$ is a Zariski open neighborhood of e in H where U^t is the expanding horospherical subgroup of H for the action of A. Choose an open neighborhood Q_0 of e in U, and an open neighborhood P_0 of e in $U^t A(M \cap H)$ such that $Q_0 P_0 \subset \mathcal{O}$.

We claim that $xQ_0 A \cap y\mathcal{O}' \neq \emptyset$, which implies $x\mathcal{O}A \cap y\mathcal{O}' \neq \emptyset$. Suppose that this is not true. Then

$$xQ_0 A \subset \Gamma \backslash G - y\mathcal{O}'$$

where the latter is a closed set. Now, choose a sequence $a_n \in A$ such that $a_n Q_0 a_n^{-1} \to U$ as $n \to \infty$, and observe

$$xa_n^{-1}(a_n Q_0 a_n^{-1}) = xQ_0 a_n^{-1} \subset \Gamma \backslash G - y\mathcal{O}'.$$

Passing to a subsequence, $xa_n^{-1} \to x_0$ for some $x_0 \in \mathrm{RF}\, \mathcal{M}$, and we obtain that $x_0 U$ is contained in the closed subset $\Gamma \backslash G - y\mathcal{O}'$. This contradicts the U-minimality of $X \cap \mathrm{RF}_+ \mathcal{M}$, which is claim (1). This proves (3).

For the claim (4), note that by Corollary 12.4, almost all U_0-orbits in $\pi_1(g^{-1}\Gamma g \cap HC) \backslash H$ are dense in the corresponding $\mathrm{RF}_+ \mathcal{M}$-set. It follows that for almost all x, the closure $\overline{xU_0}$ contains a U-orbit of X. Hence (4) follows from the claim (1).

14. Orbit closure theorems: beginning of the induction

In the rest of the paper, let $\mathcal{M} = \Gamma \backslash \mathbb{H}^d$ be a convex cocompact hyperbolic d-manifold with Fuchsian ends, and $G = \mathrm{SO}^\circ(d, 1)$. Let $U < N$ be a non-trivial connected proper closed subgroup, and $H(U)$ be its associated simple Lie subgroup of G.

Let $\mathcal{L}_U$ and $\mathcal{Q}_U$ be as defined in (5.8) and (5.9). The remainder of the paper is devoted to the proof of the next theorem from which Theorem 1.2 follows:

Theorem 14.1. (1) *For any* $x \in \mathrm{RF}\, \mathcal{M}$,

$$\overline{xH(U)} = xL \cap F_{H(U)}$$

where xL is a closed orbit of some $L \in \mathcal{L}_U$.

(2) *Let $x_0\widehat{L}$ be a closed orbit for some $\widehat{L} \in \mathcal{L}_U$ and $x_0 \in \mathrm{RF}\,\mathcal{M}$.*

(a) *For any $x \in x_0\widehat{L} \cap \mathrm{RF}_+\mathcal{M}$,*

$$\overline{xU} = xL \cap \mathrm{RF}_+\mathcal{M}$$

where xL is a closed orbit of some $L \in \mathcal{Q}_U$.

(b) *For any $x \in x_0\widehat{L} \cap \mathrm{RF}\,\mathcal{M}$,*

$$\overline{xAU} = xL \cap \mathrm{RF}_+\mathcal{M}$$

where xL is a closed orbit of some $L \in \mathcal{L}_U$.

(3) *Let $x_0\widehat{L}$ be a closed orbit for some $\widehat{L} \in \mathcal{L}_U$ and $x_0 \in \mathrm{RF}\,\mathcal{M}$. Let $x_iL_i \subset x_0\widehat{L}$ be a sequence of closed orbits intersecting* RF $\mathcal{M}$ *where $x_i \in \mathrm{RF}_+\mathcal{M}$, $L_i \in \mathcal{Q}_U$. Assume that no infinite subsequence of x_iL_i is contained in a subset of the form y_0L_0D where y_0L_0 is a closed orbit of $L_0 \in \mathcal{L}_U$ with $\dim L_0 < \dim \hat{L}$ and $D \subset \mathrm{N}(U)$ is a compact subset. Then*

$$\lim_{i\to\infty} (x_iL_i \cap \mathrm{RF}_+\mathcal{M}) = x_0\widehat{L} \cap \mathrm{RF}_+\mathcal{M}.$$

We will prove (1), (2), and (3) of Theorem 14.1 by induction on the co-dimension of U in N and the co-dimension of U in $\widehat{L} \cap N$, respectively.

For simplicity, let us say $(1)_m$ holds, if (1) is true for all U satisfying co-$\dim_N(U) \leq m$. We will say $(2)_m$ (resp. $(2.a)_m$, $(2.b)_m$) holds, if (2) (resp. (a) of (2), (b) of (2)) is true for all U and $\widehat{L}$ satisfying co-$\dim_{\widehat{L}\cap N}(U) \leq m$ and similarly for $(3)_m$.

Base case of $m = 0$. Note that the bases cases $(1)_0$, and $(3)_0$ are trivial, and that $(2)_0$ follows from Theorem 13.1.

We will deduce $(1)_{m+1}$ from $(2)_m$ and $(3)_m$ in section 16, and $(2)_{m+1}$ from $(1)_{m+1}$, $(2)_m$, and $(3)_m$ in section 17, and finally deduce $(3)_{m+1}$ from $(1)_{m+1}$, $(2)_{m+1}$ and $(3)_m$ in section 18.

Remark 14.2. When co-$\dim_{\widehat{L}\cap N}(U) \geq 1$ and $\widehat{L} \in \mathcal{L}_U$, we may assume without loss of generality that

$$U \subset \widehat{L} \cap N \cap \check{H}$$

by replacing U and $\widehat{L}$ by their conjugates using an element $m \in M$.

Remark 14.3. In the case when $x \in \partial F_{H(U)}$, Theorem 14.1 (1) and (2) follow from Theorem 11.1, and if $x_0 \in \partial F_{H(U)}$, (3) follows from the work of Mozes-Shah [29]. So the main new cases of Theorem 14.1 are when $x, x_0 \in F^*_{H(U)}$.

We will use following observation:

Singular U-orbits under the induction hypothesis. Recall the notation $\mathscr{S}(U, x\widehat{L})$ and $\mathscr{G}(U, x\widehat{L})$ from (5.7).

Lemma 14.4. *Suppose that $(2.a)_m$ is true and that for $x \in \mathrm{RF}\,\mathcal{M}$, xU is contained in a closed orbit $x\widehat{L}$ for some $\widehat{L} \in \mathcal{L}_U$.*

(1) *If* $\text{co-dim}_{\widehat{L}\cap N}(U) \le m+1$, *then for any* $x_0 \in \mathscr{S}(U, x\widehat{L}) \cap \mathrm{RF}_+\,\mathcal{M}$,
$$\overline{x_0U} = x_0L \cap \mathrm{RF}_+\,\mathcal{M}$$
where x_0L is a closed orbit of some subgroup $L < \widehat{L}$ contained in $\mathcal{Q}_U$, satisfying $\dim L_{nc} < \dim \widehat{L}_{nc}$.

(2) *If* $\text{co-dim}_{\widehat{L}\cap N}(U) \le m$, *then for any* $x_0 \in \mathscr{G}(U, x\widehat{L})$,
$$\overline{x_0U} = x_0\widehat{L} \cap \mathrm{RF}_+\,\mathcal{M}.$$

Proof. Suppose that $\text{co-dim}_{\widehat{L}\cap N}(U) \le m+1$ and that $x_0 \in \mathscr{S}(U, x\widehat{L}) \cap \mathrm{RF}_+\,\mathcal{M}$. By Proposition 5.13, we get
$$\overline{x_0U} \subset x_0Q$$
for some closed orbit x_0Q where $Q \in \mathcal{Q}_U$ satisfies $\dim Q_{nc} < \dim \widehat{L}_{nc}$.

Now $Q = vL_0v^{-1}$ for some $L_0 \in \mathcal{L}_U$ and $v \in U^{\perp}$. We have $x_0Uv = x_0vU \subset x_0vL_0$. Since $\text{co-dim}_{N\cap L_0}(U) = \text{co-dim}_{N\cap Q}(U) \le m$, by applying $(2)_m$, we get
$$\overline{x_0vU} = x_0vL \cap \mathrm{RF}_+\,\mathcal{M}$$
for some closed orbit x_0vL where $L \in \mathcal{Q}_U$ is contained in L_0. Therefore
$$\overline{x_0U} = x_0vLv^{-1} \cap \mathrm{RF}_+\,\mathcal{M}.$$
As $vLv^{-1} \in \mathcal{Q}_U$ and $\dim L_{nc} \le \dim Q_{nc} < \dim \widehat{L}_{nc}$, the claim (1) is proved.

To prove (2), note that by $(2.a)_m$, we get $\overline{x_0U} = x_0L \cap \mathrm{RF}_+\,\mathcal{M}$ for some closed orbit x_0L with $L \in \mathcal{Q}_U$ such that $L \subset \widehat{L}$. Since $x_0 \in \mathscr{G}(U, x\widehat{L})$, we have $\dim L_{nc} = \dim \widehat{L}_{nc}$.

Since $L \subset \widehat{L}$, $L \cap N$ is a horospherical subgroup of $\widehat{L}$. By Theorem 13.1, $L \cap N$ acts minimally on $x\widehat{L}$, and hence $L = \widehat{L}$. □

15. Generic points, uniform recurrence and additional invariance

The primary goal of this section is to prove Propositions 15.1 and 15.2 in obtaining additional invariances using a sequence converging to a generic point of an intermediate closed orbit; the main ingredient is Theorem 7.15 (Avoidance theorem II). The results in this section are main tools in the enlargement steps of the proof of Theorem 14.1.

In this section, we let $U < N$ be a non-trivial connected closed subgroup. We suppose that

- $(2)_m$ and $(3)_m$ are true;
- $x\widehat{L}$ is a closed orbit for some $x \in \mathrm{RF}\,\mathcal{M}$, and $\widehat{L} \in \mathcal{L}_U$;
- $\text{co-dim}_{\widehat{L}\cap N}(U) \le m+1$.

We let $\{U^{(i)}\}$ be a finite collection of one-parameter subgroups generating U. In the next two propositions, we let X be a closed U-invariant subset of $x_0\widehat{L}$ such that

$$X \supset xL \cap \mathrm{RF}_+ \mathcal{M}$$

for some closed orbit xL where $L \in \mathcal{Q}_U$ is a proper subgroup of $\widehat{L}$ and

$$x \in \bigcap_i \mathscr{G}(U^{(i)}, xL) \cap \mathrm{RF}\,\mathcal{M}.$$

Proposition 15.1 (Additional invariance I)**.** *Suppose that there exists a sequence $x_i \to x$ in X where $x_i = x\ell_i r_i$ with $x\ell_i \in xL \cap \mathrm{RF}\,\mathcal{M}$ and $r_i \in \exp \mathfrak{l}^\perp - \mathrm{N}(U)$.*

Then there exists a sequence $v_n \to \infty$ in $(L \cap N)^\perp$ such that

$$xLv_n \cap \mathrm{RF}_+ \mathcal{M} \subset X.$$

Proof. Since $r_i \notin \mathrm{N}(U)$, we can fix a one-parameter subgroup $U_0 = \{u_t : t \in \mathbb{R}\}$ in the family $\{U^{(i)}\}$ such that $r_i \notin \mathrm{N}(U_0)$ by passing to a subsequence.

Let E_j, $j \in \mathbb{N}$, be a sequence of compact subsets in $\mathscr{S}(U_0, xL) \cap \mathrm{RF}\,\mathcal{M}$ given by Theorem 7.15. Set $z_i := x\ell_i \in xL \cap \mathrm{RF}\,\mathcal{M}$. Fix $j \in \mathbb{N}$ and $n \gg 1$. Since $z_i \to x$ and $x \in \mathscr{G}(U_0, xL)$, there exist $i_j \geq 1$ and an open neighborhood $\mathcal{O}_j$ of E_j such that for each $i \geq i_j$, the set

$$\mathsf{T}_i = \{t \in \mathbb{R} : z_i u_t \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}_j\},$$

is $2k$-thick by loc. cit. We apply Lemma 9.3 to the sequence T_i. We can find a sequence $t_i = t_i(n) \in \mathsf{T}_i$, $i \geq i_j$ and elements $y_j = y_j(n), v_j = v_j(n)$ satisfying that as $i \to \infty$,

- $z_i u_{t_i} \to y_j \in (\mathrm{RF}\,\mathcal{M} \cap xL) - \mathcal{O}_j$;
- $u_{t_i}^{-1} r_i u_{t_i} \to v_j \in (L \cap N)^\perp$ with $n \leq \|v_j\| \leq (2k^2)n$.

So as $i \to \infty$,

$$x_i u_{t_i} = z_i r_i u_{t_i} \to y_j v_j \quad \text{in } X.$$

Note that since L is a proper subgroup of $\widehat{L}$, we have $\text{co-dim}_{L\cap N}(U) \leq m$ by Lemma 5.11.

If y_j belongs to $\mathscr{G}(U, xL)$, then $\overline{y_j U} v_j = xL \cap \mathrm{RF}_+ \mathcal{M}$ by Lemma 14.4(2), and hence

$$X \supset \overline{y_j v_j U} = \overline{y_j U} v_j = xLv_j \cap \mathrm{RF}_+ \mathcal{M}.$$

Hence the claim follows if $y_j(n) \in \mathscr{G}(U, xL)$ for an infinite subsequence of n's.

Now we may suppose that for all $n \geq 1$ and $j \geq 1$, $y_j(n) \in \mathscr{S}(U, xL) \cap \mathrm{RF}_+ \mathcal{M}$, after passing to a subsequence. Fix n, and set $y_j = y_j(n)$ and $v_j = v_j(n)$. Then, since $\dim_{L\cap N} U \leq m$, by $(2)_m$, we have

$$\overline{y_j U} = y_j L_j \cap \mathrm{RF}_+ \mathcal{M} \tag{15.1}$$

for some closed $y_j L_j$ where $L_j \in \mathcal{Q}_U$ is contained in $\widehat{L}$ and $\dim(L_j)_{nc} < \dim \widehat{L}_{nc}$. Write $L_j = w_j^{-1} L_j' w_j$ for $L_j' \in \mathcal{L}_U$ and $w_j \in U^\perp$. We claim that the sequence $y_j L_j = y_j w_j^{-1} L_j' w_j$ satisfies the hypothesis of $(3)_m$. It follows from the condition $y_j \in (\mathrm{RF}\,\mathcal{M} \cap xL) - \mathcal{O}_j$ for all j that no infinite subsequence of $y_j L_j$ is contained in a subset of the form $y_0 L_0 D \subset \mathscr{S}(U, xL)$ where $y_0 L_0$ is closed, $L_0 \subset \mathcal{Q}_U$ and $D \subset \mathrm{N}(U)$ is a compact subset. Hence, by $(3)_m$, we have

$$\limsup_j y_j L_j \cap \mathrm{RF}_+ \mathcal{M} = xL \cap \mathrm{RF}_+ \mathcal{M}.$$

Therefore for each fixed $n \gg 1$ and $y_j = y_j(n)$,

$$\limsup_j \overline{y_j U} = xL \cap \mathrm{RF}_+ \mathcal{M}.$$

By passing to a subsequence, there exists $u_j \in U$ such that $y_j u_j$ converges to x. As $n \le \|v_j(n)\| \le (2k^2)n$, the sequence $v_j(n)$ converges to some $v_n \in (L \cap N)^\perp$ as $j \to \infty$, after passing to a subsequence. Therefore

$$\limsup_j \overline{y_j(n) v_j(n) U} = \limsup_j \overline{y_j(n) U} v_j(n) \supset \overline{xU} v_n = xLv_n \cap \mathrm{RF}_+ \mathcal{M}$$

where the last equality follows from Lemma 14.4(2), since $\text{co-dim}_{L\cap N}(U) \le m$. □

Note that in the above proposition, $y_i = x\ell_i r_i$ is not necessarily in $\mathrm{RF}\,\mathcal{M}$, and hence we cannot apply the avoidance theorem 7.15 to the sequence y_i directly. We instead applied it to the sequence $x\ell_i$.

In the proposition below, we will consider a sequence $x_i \to y$ inside $\mathrm{RF}\,\mathcal{M}$, and apply Theorem 7.15 to the sequence x_i.

Proposition 15.2 (Additional invariance II). *Suppose that there exists a sequence $x_i \in X \cap \mathrm{RF}\,\mathcal{M} - xL \cdot \mathrm{N}(U)$, converging to x as $i \to \infty$. Then there exists a sequence $v_j \to \infty$ in $(N \cap L)^\perp$ such that*

$$xLv_j \cap \mathrm{RF}_+ \mathcal{M} \subset X \quad \textit{and} \quad xLv_j \cap \mathrm{RF}\,\mathcal{M} \neq \emptyset.$$

The same works for a sequence $x_i \in \mathrm{RF}\,\mathcal{M} - xL\cdot\mathrm{N}(U)$ such that $\limsup x_i U \subset X$.

Proof. Let $x_i \in \mathrm{RF}\,\mathcal{M} - xL \cdot \mathrm{N}(U)$ be a sequence converging to x such that $\limsup x_i U \subset X$. Write $x_i = xg_i$ for $g_i \to e$ in $\widehat{L}$. Since L is reductive, we can write $g_i = \ell_i r_i$ where $\ell_i \to e$ in L and $r_i \to e$ in $\exp \mathfrak{l}^\perp$ as $i \to \infty$. By the assumption on x_i, there exists a one-parameter subgroup $U_0 = \{u_t : t \in \mathbb{R}\}$ among $U^{(i)}$ such that $r_i \notin \mathrm{N}(U_0)$ by passing to a subsequence.

For $R > 0$, we set $B(R) := \{v \in (L \cap N)^\perp \cap \widehat{L} : \|v\| \le R\}$. Fix j and $n \in \mathbb{N}$. Let $E_j, \mathcal{O}_j$ be given by Theorem 7.15 for xL with respect to U_0. Then E_j is of the form

$$E_j = \bigcup_{i \in \Lambda_j} \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\,\mathcal{M}$$

where $H_i \in \mathscr{H}^\star$ satisfies $\dim(H_i)_{nc} < \dim L_{nc}$ and D_i is a compact subset of $X(H_i, U_0) \cap L$. As $B(2k^2 n) \subset \mathrm{C}(U_0)$, we have $D_j^* := D_j B(2k^2 n)$ is a compact subset of $X(H_i, U_0)$. Hence the following set

$$\tilde{E}_j := \bigcup_{i\in\Lambda_j} \Gamma\backslash\Gamma H_i D_i^* \cap \mathrm{RF}\,\mathcal{M}$$

belongs to $\mathcal{E}_{U_0}$ and is associated to the family $\{H_i : i \in \Lambda_j\}$, as defined in (7.3).

Let $\tilde{E}'_j \in \mathcal{E}_{U_0}$ be a compact subset given by Theorem 7.13, which is also associated to the same family $\{H_i : i \in \Lambda_j\}$. Note that for any $z \in \tilde{E}'_j$, the closure $\overline{zU_0}$ is contained in $\Gamma\backslash\Gamma H_i D_i^*$ for some $i \in \Lambda_j$. In particular, $\tilde{E}'_j$ is a compact subset disjoint from $\mathscr{G}(U_0, xL)$. Since $x_i \to x$ and $x \in \mathscr{G}(U_0, xL)$, there exists $i_j \geq 1$ such that $x_i \notin \tilde{E}'_j$ for all $i \geq i_j$. By Theorem 7.13, there exists a neighborhood $\tilde{\mathcal{O}}_j$ of $\tilde{E}_j$ such that for each $i \geq i_j$, the set

$$\mathsf{T}_i = \{t \in \mathbb{R} : x_i u_t \in \mathrm{RF}\,\mathcal{M} - \tilde{\mathcal{O}}_j\}$$

is $2k$-thick. Applying Lemma 9.3 to T_i, and $r_i \to e$, we can find $t_i = t_i(n) \in \mathsf{T}_i$ such that $u_{t_i}^{-1} r_i u_{t_i} \to v_j$ for some $v_j = v_j(n) \in (L \cap N)^\perp$, with $n \leq \|v_j\| \leq 2k^2 \cdot n$. Passing to a subsequence, $x_i u_{t_i}$ converges to some $\tilde{x}_j(n) \in \mathrm{RF}\,\mathcal{M} - \tilde{\mathcal{O}}_j$ as $i \to \infty$. Set

$$z_i := x\ell_i, \text{ and } \mathcal{O}_j := \tilde{\mathcal{O}}_j B(2k^2 n) \cap xL.$$

Since $x_i u_{t_i} = z_i u_{t_i}(u_{t_i}^{-1} r_i u_{t_i})$, we have

$$z_i u_{t_i} \to y_j \in (\mathrm{RF}_+\,\mathcal{M} \cap xL) - \mathcal{O}_j$$

where $y_j = y_j(n) := \tilde{x}_j(n) v_j^{-1}$.

We check that $E_j \subset \mathcal{O}_j$ as $B(2k^2 n)B(2k^2 n)$ contains e. It follows that $y_j \notin E_j$. Since $\tilde{x}_j(n) \in \overline{y_j U} v_j \subset X$, we have $\overline{y_j U} v_j \cap \mathrm{RF}\,\mathcal{M} \neq \emptyset$. Given these, we can now repeat verbatim the proof of Proposition 15.1 to complete the proof. □

Theorem 2.2 in the introduction can be proved similarly to the proof of Proposition 15.1.

Proof of Theorem 2.2 Let E_j, $j \in \mathbb{N}$, be a sequence of compact subsets of $\mathscr{S}(U_0) \cap \mathrm{RF}\,\mathcal{M}$ given by Thoerem 7.15. Fix $j \in \mathbb{N}$. Then there exist $i_j \geq 1$ and a neighborhood $\mathcal{O}_j$ of E_j such that

$$\{t \in \mathbb{R} : x_i u_t \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}_j\}$$

is $2k$-thick for all $i \geq i_j$. Hence we can find a sequence $t_i \in [-2kT_i, -T_i] \cup [T_i, 2kT_i]$ such that $x_i u_{t_i} \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}_j$ for all $i \geq i_j$. Hence, by passing to a subsequence, $x_i u_{t_i}$ converges to some $y_j \in \mathrm{RF}\,\mathcal{M} - \mathcal{O}_j$ as $i \to \infty$. If $y_j \in \mathscr{G}(U)$ for some j, then $(2)_m$ and Lemma 14.4(2) imply that $\overline{y_j U} = \mathrm{RF}_+\,\mathcal{M}$, proving the claim.

Now, we assume that $y_j \in \mathscr{S}(U, x\widehat{L})$ for all j. Then by $(2)_m$ and Lemma 14.4(1), we have

$$\overline{y_j U} = y_j L_j \cap \mathrm{RF}_+\,\mathcal{M}$$

for some closed y_jL_j where $L_j \in \mathcal{Q}_U$ is a proper subgroup of G. Similarly to the proof of Theorem 15.1, we can show that the sequence y_jL_j satisfies the hypothesis of $(3)_m$. Hence, by applying $(3)_m$ to the sequence y_jL_j, we get

$$\limsup \, (y_jL_j \cap \mathrm{RF}_+ \mathcal{M}) = \mathrm{RF}_+ \mathcal{M}.$$

Therefore $\limsup \; y_jU = \limsup \; \overline{y_jU} = \mathrm{RF}_+ \mathcal{M}$. This, together with Theorem 13.1(4), finishes the proof.

16. $H(U)$-orbit closures: proof of $(1)_{m+1}$

We fix a non-trivial connected proper subgroup $U < N$. Without loss of generality, we may assume

$$U < N \cap \check{H}$$

using a conjugation by an element of M. We set

$$H = H(U), \;\; H' = H'(U), \;\; F = F_{H(U)}, \;\; F^* = F^*_{H(U)}, \;\text{ and }\; \partial F = \partial F_{H(U)}.$$

By the assumption $U < N \cap \check{H}$, we have

$$\partial F \cap \mathrm{RF}\,\mathcal{M} = \mathrm{BF}\,\mathcal{M} \cdot \mathrm{C}(H).$$

We will be using the following observation:

Lemma 16.1. *Let x_1L_1 and x_2L_2 be closed orbits where $x_1, x_2 \in \mathrm{RF}\,\mathcal{M}$, $L_1 \in \mathcal{Q}_U$ and $L_2 \in \mathcal{L}_U$. If $x_1L_1 \cap \mathrm{RF}\,\mathcal{M} \subset x_2L_2$, then $L_1 \subset L_2$ and $x_1L_1 \subset x_2L_2$.*

Proof. Since L_2 contains H, we get that $x_1L_1 \cap \mathrm{RF}\,\mathcal{M} \cdot H \subset x_2L_2$. Suppose that $x_1L_1 \cap F^* \neq \emptyset$. We may assume $x_1 \in F^*$. Recall from (3.2) that $F^* \subset \mathrm{RF}\,\mathcal{M} \cdot H$. Hence, we have $x_1L_1 \cap F^* \subset x_2L_2$. Since F^* is open, there exist $g_1, g_2 \in G$ such that $[g_i] = x_i$, and $g_1L_1 \cap \mathcal{O} \subset g_2L_2$ for some open neighborhood $\mathcal{O}$ of g_1. It follows that $L_1 \cap g_1^{-1}\mathcal{O} \subset g_1^{-1}g_2L_2$. Since $e \in g_1^{-1}g_2L_2$, we have $g_1^{-1}g_2L_2 = L_2$. Since L_1 is topologically generated by $L_1 \cap g_1^{-1}\mathcal{O}$, we deduce $L_1 \subset L_2$. Since $x_1L_1 \cap x_2L_2 \neq \emptyset$, it follows that $x_1L_1 \subset x_2L_2$.

Now consider the case when $x_1L_1 \cap F^* = \emptyset$. In this case, $x_1L_1 \cap \mathrm{RF}\,\mathcal{M} \subset \mathrm{RF}\,\mathcal{M} \cap \partial F$. By Theorem 13.1(4), we can assume that $\overline{x_1U} = x_1L_1 \cap \mathrm{RF}_+ \mathcal{M}$. As x_1 is contained in $\mathrm{BF}\,\mathcal{M} \cdot \mathrm{C}(H)$, so is $\overline{x_1U}$. It follows that x_1L_1 is compact and hence is contained in $\mathrm{RF}\,\mathcal{M}$. Hence the hypothesis implies that $x_1L_1 \subset x_2L_2$, which then implies $L_1 \subset L_2$ by the same argument in the previous case. □

Lemma 16.2. *Let y_1L_1 and y_2L_2 be closed orbits where $y_1 \in \mathrm{RF}\,\mathcal{M}$, $y_2 \in \mathrm{RF}_+ \mathcal{M}$, $L_1 \in \mathcal{Q}_U$ and $L_2 \in \mathcal{L}_U$. If $y_1L_1 \subset y_2L_2D$ for some subset $D \subset \mathrm{N}(U)$, then there exists $d \in D$ such that $L_1 \subset d^{-1}L_2d$ and $y_1L_1 \subset y_2L_2d$.*

Proof. By Theorem 13.1(4), we may assume $\overline{y_1U} = y_1L_1 \cap \mathrm{RF}_+ \mathcal{M}$. By the assumption, $y_1 = y_2\ell_2 d$ for some $\ell_2 \in L_2$ and $d \in D$. Since $y_2\ell_2 = y_1d^{-1}$

and $\mathrm{N}(U)$ preserves $\mathrm{RF}_+\mathcal{M}$, $y_2\ell_2 \in \mathrm{RF}_+\mathcal{M}$. Hence we may replace y_2 by $y_2\ell_2$, and hence assume that $y_1 = y_2 d$. Since

$$y_1 L_1 \cap \mathrm{RF}_+\mathcal{M} = \overline{y_2 dU} = \overline{y_2 U} d \subset y_2 L_2 d, \tag{16.1}$$

and $F^* \subset \mathrm{RF}_+\mathcal{M}\cdot H$, we get $y_1 L_1 d^{-1} \cap F^* \subset y_2 L_2$.

If $y_1 L_1 d^{-1} \cap F^* \neq \emptyset$, using the openness of F^*, the conclusion follows as in the first part of the proof of Lemma 16.1. Now consider the case when $y_1 L_1 d^{-1} \cap F^* = \emptyset$. In particular, $y_2 = y_1 d^{-1}$ belongs to

$$\mathrm{RF}_+\mathcal{M} - F^* \subset \mathrm{BF}\,\mathcal{M}\cdot \mathrm{N}(U)$$

by (4.6). It follows from Theorem 11.1 that $\overline{y_2 U} = y_2 L_2'$ for some $L_2' \in \mathcal{Q}_U$ contained in L_2. In view of (16.1), we get $y_1 L_1 \cap \mathrm{RF}_+\mathcal{M} = y_1 d^{-1} L_2' d$. Therefore $d^{-1}L_2' d \subset L_1$. Since $y_1 L_1 \cap \mathrm{RF}_+\mathcal{M}$ is $A(L_1\cap N)$-invariant, it follows that $d^{-1}L_2' d \in \mathcal{L}_U$ and $d^{-1}L_2' d \cap N = L_1 \cap N$. As a result, $(L_1)_{nc} = d^{-1}(L_2')_{nc} d$. By Lemma 5.11, we get that $L_1 = d^{-1}L_2' d \subset d^{-1}L_2 d$ and that $y_1 L_1 = y_2 L_2' d \subset y_2 L_2 d$. □

The following proposition says that the classification of H'-orbit closures yields the classification of H-orbit closures:

Proposition 16.3. *Let $x \in \mathrm{RF}\,\mathcal{M}$, and assume that there exists $U < \tilde U < N$ such that $xH'(\tilde U)$ is closed, and*

$$\overline{xH'} = xH(\tilde U)\cdot \mathrm{C}(H) \cap F.$$

Then there exists a closed subgroup $C < \mathrm{C}(H(\tilde U))$ such that

$$\overline{xH} = xH(\tilde U)C \cap F.$$

Proof. By Proposition 4.9 and Theorem 13.1(2), there exists a closed subgroup $C < \mathrm{C}(H(\tilde U))$ such that $H(\tilde U)C \in \mathcal{L}_U$ and $X := xH(\tilde U)C$ is a closed $H(\tilde U)$-minimal subset. In particular, $\overline{xH} \subset X \cap F$. Now, by Theorem 13.1(3), there exists $y \in X$ such that $\overline{yA} = X \cap \mathrm{RF}\,\mathcal{M}$. Since C is contained in $\mathrm{C}(H)$ and

$$\overline{xH}\cdot \mathrm{C}(H) = \overline{xH'} = xH(\tilde U)\cdot \mathrm{C}(H) \cap F,$$

there exists $c_0 \in \mathrm{C}(H)$ such that $yc_0 \in \overline{xH}$. Since $\overline{yA}c_0 = \overline{yc_0 A} \subset \overline{xH}$ and $c_0 \in \mathrm{C}(H)$, it follows $Xc_0 \cap \mathrm{RF}\,\mathcal{M} \subset \overline{xH} \subset X$. Applying Lemma 16.1, we get $Xc_0 = \overline{xH} = X$. □

In the rest of this section, fix $m \in \mathbb{N}\cup\{0\}$ and assume that

$$1 \le \text{co-dim}_N(U) = m+1.$$

In order to describe the closure of $xH(U)$, in view of Theorem 11.1, we assume that

$$x \in F^* \cap \mathrm{RF}\,\mathcal{M}.$$

By Proposition 16.3, it suffices to show that

$$\overline{xH'} = xL\,\mathrm{C}(H) \cap F \tag{16.2}$$

for some closed orbit xL for some $L \in \mathcal{L}_U$.

In the rest of this section, we set

$$X := \overline{xH'} \text{ and assume that } xH' \text{ is not closed, i.e., } X \neq xH'.$$

We also assume that $(2)_m$ holds in the entire section.

Lemma 16.4 (Moving from $\mathcal{Q}_U$ to $\mathcal{L}_U$)**.** *If $x_0L \cap \mathrm{RF}_+\mathcal{M} \subset X$ for some closed orbit x_0L with $x_0 \in \mathrm{RF}\,\mathcal{M}$, and $L \in \mathcal{Q}_U - \mathcal{L}_U$, then*

$$x_1\widehat{L} \cap \mathrm{RF}_+\mathcal{M} \subset X$$

for some closed orbit $x_1\widehat{L}$ with $x_1 \in \mathrm{RF}\,\mathcal{M}$, and $\widehat{L} \in \mathcal{L}_U$ with $\dim(\widehat{L} \cap N) > \dim(L \cap N)$. Moreover, x_1 can be taken to be any element of the set $\limsup_{t\to+\infty} x_0ua_{-t}$ for any $u \in U$.

Proof. By Lemma 5.10, we can write $L = v^{-1}\widehat{L}v$ for some $\widehat{L} \in \mathcal{L}_U$ and $v \in (\widehat{L} \cap N)^{\perp}$. As $L \notin \mathcal{L}_U$, we have $v \neq e$. Set $\widehat{U} := \widehat{L} \cap N$. Note that $x_0v^{-1}\widehat{U}Av \subset x_0L \cap \mathrm{RF}_+\mathcal{M}$, as $\widehat{U}A < \widehat{L}$. Since X is A-invariant, $x_0v^{-1}\widehat{U}AvA \subset X$. Let V^+ be the unipotent one-parameter subsemigroup contained in AvA, and let V be the one-parameter subgroup containing V^+. Then $x_0v^{-1}V^+\widehat{U} \subset X$. Since $x_0A \subset \mathrm{RF}\,\mathcal{M}$ and $\mathrm{RF}\,\mathcal{M}$ is compact, $\limsup_{t\to+\infty} x_0a_{-t}$ is not empty. Now let x_1 be any limit of $x_0ua_{-t_n}$ for some sequence $t_n \to \infty$ and $u \in U$. Since $v^{-1}V^+$ is an open neighborhood of e in V, $\liminf_{n\to\infty} a_{t_n}v^{-1}V^+a_{-t_n} = V$. Note that as $u \in \widehat{U}$,

$$x_0ua_{-t_n}(a_{t_n}v^{-1}\widehat{U}V^+a_{-t_n}) = x_0v^{-1}\widehat{U}V^+a_{-t_n} \subset X.$$

As a result, we obtain that $x_1\widehat{U}V \subset X$ and hence $x_1\widehat{U}VA \subset X$. Since $\text{co-dim}_N(\widehat{U}V) \leq m$, the claim follows from by $(2.a)_m$. ☐

Proposition 16.5. *If $R := X \cap F^* \cap \mathrm{RF}\,\mathcal{M}$ accumulates on ∂F, i.e., there exists $x_n \in R$ converging to a point in ∂F, then*

$$X \supset x_0L \cap \mathrm{RF}_+\mathcal{M}$$

for some closed orbit x_0L with $x_0 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and $L \in \mathcal{L}_U$ such that $\dim(L \cap N) > \dim U$.

Proof. There exists $x_n \in R$ which converges to some $z \in \mathrm{BF}\,\mathcal{M}\cdot\mathrm{C}(H)$ as $n \to \infty$. We may assume $z \in \mathrm{BF}\,\mathcal{M}$ without loss of generality, since R is $\mathrm{C}(H)$-invariant. We claim that $X \cap R$ contains z_1v where $z_1 \in \mathrm{BF}\,\mathcal{M}$ and $v \in \check{V} - \{e\}$. Write $x_n = zh_nr_n$ for some $h_n \in \check{H}$ and $r_n \in \exp\check{\mathfrak{h}}^{\perp}$, where $\check{\mathfrak{h}}^{\perp}$ denotes the $\mathrm{Ad}(\check{H})$-complementary subspace to $\mathrm{Lie}(\check{H})$ in $\mathfrak{g}$. Since $x_n \in F^*$ and $z \in \mathrm{BF}\,\mathcal{M}$, it follows that $r_n \notin \mathrm{C}(H)$ for all large n. By (3.2) and (3.5), we have

$$\mathrm{N}(U) \cap \exp(\check{\mathfrak{h}}^{\perp} \cap \mathcal{O}) \subset \check{V}\,\mathrm{C}(H)$$

for a small neighborhood $\mathcal{O}$ of 0 in $\mathfrak{g}$. Therefore, if $r_n \in \mathrm{N}(U)$ for some n, then the $\check{V}$-component of r_n should be non-trivial. Hence by Theorem 11.1, $X \supset \overline{zh_nU}r_n = zh_nLr_n$ for some $L \in \mathcal{Q}_U$ contained in $\check{H}$. Note that $x_n = zh_nr_n \in F^*$ and that $r_n^{-1}Lr_n \in \mathcal{Q}_U - \mathcal{L}_U$, since $r_n \in \check{V} - \{e\}$. Hence the claim follows from Lemma 16.4.

Now suppose that $r_n \notin \mathrm{N}(U)$ for all n. Then there exists a one-parameter subgroup $U_0 = \{u_t\} < U$ such that $r_n \notin \mathrm{N}(U_0)$. Applying Lemma 9.3, with a sequence of k-thick subsets

$$\mathsf{T}(x_n) := \{t \in \mathbb{R} : x_n u_t \in \mathrm{RF}\,\mathcal{M}\},$$

we get a sequence $t_n \in \mathsf{T}(x_n)$ such that $u_{t_n}^{-1} r_n u_{t_n}$ converges to non-trivial element $v \in \check{V}$. Since $zh_n u_{t_n} \in z\check{H}$ and $z\check{H}$ is compact, the sequence $zh_n u_{t_n}$ converges to some $z_1 \in z\check{H}$, after passing to a subsequence. Then

$$z_1 v = \lim (zh_n u_{t_n})(u_{t_n}^{-1} r_n u_{t_n}) \in X \cap \mathrm{RF}\,\mathcal{M}. \tag{16.3}$$

Since $z_1 \in \mathrm{BF}\,\mathcal{M}$ and $v \in \check{V} - \{e\}$, $z_1 v \in \mathrm{RF}\,\mathcal{M}$ implies that $z_1 v \in F^*$, and hence $z_1 v \in R$. This proves the claim.

Now by Theorem 11.1, $\overline{z_1 U} = z_1 L$ for some $L \in \mathcal{Q}_U$ contained in $\check{H}$, and hence

$$X \supset \overline{z_1 v U} = \overline{z_1 U} v = (z_1 v)(v^{-1} L v).$$

Since $v \in \check{V} - \{e\}$, $v^{-1} L v \notin \mathcal{L}_U$. Therefore, by Lemma 16.4, it suffices to prove that there exists $u \in U$ such that

$$(F^* \cap \mathrm{RF}\,\mathcal{M}) \cap \limsup_{t \to +\infty} z_1 u v a_{-t} \neq \emptyset. \tag{16.4}$$

Let $g_1 \in G$ be such that $z_1 = [g_1]$, and set $A_{(-\infty,-t]} := \{a_{-s} : s \geq t\}$ for $t > 0$. Since $z_1 v \in F^* \cap \mathrm{RF}\,\mathcal{M}$, the sphere $(gvU)^- \cup g^+$ intersects $\Lambda - \bigcup_i \overline{B_i}$ non-trivially. Let $u \in U$ be an element such that $(gvu)^- \in \Lambda - \bigcup_i \overline{B_i}$. As $z_1 v u \in \mathrm{RF}\,\mathcal{M}$, $\pi(zuvA) \subset \operatorname{core} \mathcal{M}$. Take $\varepsilon > 0$ small enough so that the ε-neighborhoods of $\operatorname{hull} B_j$'s are mutually disjoint. If (16.4) does not hold for $z_1 uv$, then there exists $t > 1$ such that the geodesic ray $\pi(z_1 v u A_{(-\infty,-t]})$ is contained in the ε-neighborhood of $\partial \operatorname{core} \mathcal{M}$ (cf. proof of Lemma 8.8). As $\pi(g_1 uv A_{(-\infty,-t]})$ is connected, there exists B_j such that $\pi(g_1 uv A_{(-\infty,-t]})$ is contained in the ε-neighborhood of $\operatorname{hull} B_j$. This implies that $(g_1 uv)^- \in \partial B_j$, yielding a contradiction. This proves (16.4). ☐

Proposition 16.6. *The orbit xH' is not closed in F^*.*

Proof. Suppose that xH' is closed in F^*. Since we are assuming that xH' is not closed in F, $\overline{xH'}$ contains some point $y \in \partial F$. Since $\partial F = \mathrm{BF}\,\mathcal{M}\check{V}^+\,\mathrm{C}(H)$, we may assume $y \in \mathrm{BF}\,\mathcal{M} \cdot \check{V}^+$. Write $y = zv$ where $z \in \mathrm{BF}\,\mathcal{M}$ and $v \in \check{V}^+$. If $v \neq e$, $\overline{zvH'}$ intersects $\mathrm{BF}\,\mathcal{M}$ by Theorem 11.5. Therefore $\overline{xH'}$ always contains a point of $\mathrm{BF}\,\mathcal{M}$, say z. Let $x_n \in xH'$ be a sequence converging to a point z. Since $xH' \subset F^*$, there exist $k_n \in H \cap K$ converging to some $k \in H \cap K$ such that $x_n k_n \in xH' \cap \mathrm{RF}_+\,\mathcal{M}$ and $x_n k_n \to zk$. Then $zk \in \mathrm{BF}\,\mathcal{M} \cdot H' = \mathrm{BF}\,\mathcal{M}\,\mathrm{C}(H)$. Since $x_n k_n \in \mathrm{RF}\,\mathcal{M} \cdot U$ by Lemma 4.5, there exists $u_n \in U$ such that $x_n k_n u_n$ belongs to $\mathrm{RF}\,\mathcal{M}$ and converges to a point in ∂F by Lemma 8.6. Hence $X \cap F^* \cap \mathrm{RF}\,\mathcal{M}$ accumulates on ∂F. Now the claim follows from Proposition 16.5. ☐

This proposition implies that

$$(X - xH') \cap (F^* \cap \mathrm{RF}\,\mathcal{M}) \neq \emptyset. \tag{16.5}$$

Roughly speaking, our strategy in proving $(1)_{m+1}$ is first to find a closed L-orbit x_0L such that $x_0L \cap F$ is contained in X for some $L \in \mathcal{L}_U$. If $X \neq x_0L\,\mathrm{C}(H) \cap F$, then we enlarge x_0L to a bigger closed orbit $x_1\widehat{L}$ for some $\widehat{L} \in \mathcal{L}_{\widehat{U}}$ for some $\widehat{U}$ properly containing U, such that $x_1\widehat{L} \cap F$ is contained in X.

It is in the enlargement step where Proposition 15.1 (Additional invariance I) is a crucial ingredient of the arguments. In order to find a sequence x_i accumulating on a generic point of x_0L satisfying the hypothesis of the proposition, we find a closed orbit x_0L with a base point x_0 *in* $F^* \cap \mathrm{RF}\,\mathcal{M}$, and enlarge it to a bigger closed orbit, again based at a point *in* $F^* \cap \mathrm{RF}\,\mathcal{M}$. The advantage of having a closed orbit xL with $x \in F^* \cap \mathrm{RF}\,\mathcal{M}$ is that any U_0-generic point in $xL \cap \mathrm{RF}\,\mathcal{M}$ can be approximated by a sequence of $\mathrm{RF}\,\mathcal{M}$-points in $F^* \cap xL$ by Lemma 8.3. The enlargement process must end after finitely many steps because of dimension reason.

Finding a closed orbit of $L \in \mathcal{L}_U$ in X.

Proposition 16.7. *There exists a closed orbit x_0L with $x_0 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and $L \in \mathcal{L}_U$ such that*

$$x_0L \cap \mathrm{RF}_+\,\mathcal{M} \subset X.$$

Proof. Let $R := X \cap F^* \cap \mathrm{RF}\,\mathcal{M}$. If R is non-compact, the claim follows from Proposition 16.5. Now suppose that R is compact. By $(2.a)_m$, it is enough to show that X contains an orbit $z\widehat{U}$, and hence $z\widehat{U}A$, for some $\widehat{U} < N$ properly containing U and $z \in R$. By Proposition 10.9, it suffices to find a U-minimal subset $Y \subset X$ with respect to R and a point $y \in Y \cap R$ such that $X - yH'$ is not closed.

If xH' is not locally closed, then take any U-minimal subset Y of X with respect to R. If $Y \cap R \subset xH'$, then choose any $y \in Y \cap R$. Then $X - yH' = X - xH'$ cannot be closed, as xH' is not locally closed. If $Y \cap R \not\subset xH'$, then choose $y \in (Y \cap R) - xH'$. Then $X - yH'$ contains xH' and hence cannot be closed.

If xH' is locally closed, then $X - xH'$ is a closed H'-invariant subset which intersects R non-trivially. So we can take a U-minimal subset $Y \subset X - xH'$ with respect to R. Take any $y \in Y \cap R$. Then $X - yH'$ is not closed. □

Enlarging a closed orbit of $L \in \mathcal{L}_U$ in X.

Proposition 16.8. *Assume that $(3)_m$ holds as well. Suppose that there exists a closed orbit x_0L for some $x_0 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and $L \in \mathcal{L}_U$ such that*

$$x_0L \cap \mathrm{RF}_+\,\mathcal{M} \subset X \text{ and } X \neq x_0L \cdot \mathrm{C}(H) \cap F. \tag{16.6}$$

Then there exists a closed orbit $x_1\widehat{L}$ *for some* $x_1 \in F^* \cap \mathrm{RF}\,\mathcal{M}$, *and* $\widehat{L} \in \mathcal{L}_{\widehat{U}}$ *for some* $\widehat{U} < N$ *with* $\dim \widehat{U} > \dim(L\cap N)$ *such that*

$$x_1\widehat{L} \cap \mathrm{RF}_+\,\mathcal{M} \subset X.$$

Proof. Note that if $X \subset x_0 L \cdot \mathrm{C}(H)$, then $X = x_0 L\cdot \mathrm{C}(H) \cap F$. Therefore we assume that $X \not\subset x_0 L \cdot \mathrm{C}(H)$. First note that the hypothesis implies that $L \neq G$, and hence $\text{co-dim}_{L\cap N}(U) \le m$. Let $U_-^{(1)}, \cdots, U_-^{(\ell)}$ be one-parameter subgroups generating U. Similarly, let $U_+^{(1)}, \cdots, U_+^{(\ell)}$ be one-parameter subgroups generating U^+. By Theorem 13.1, $\bigcap_{i=1}^{\ell} \mathscr{G}(U_\pm^{(i)}, x_0L) \neq \emptyset$. Therefore without loss of generality, we can assume

$$x_0 \in \bigcap_{i=1}^{\ell} \mathscr{G}(U_\pm^{(i)}, x_0 L). \tag{16.7}$$

Let us write $L = H(\tilde{U})C$ for some $\tilde{U} < N$ and a closed subgroup C of $\mathrm{C}(H(\tilde{U}))$. Note from the hypothesis that we have

$$(x_0L \cap \mathrm{RF}_+\,\mathcal{M})\cdot H' \subset X.$$

Observe that (16.6) implies that $x \notin x_0L\cdot H' = x_0L\cdot \mathrm{C}(H)$. Since $C < \mathrm{C}(H)$, we have $x \notin x_0 H(\tilde{U})$. Now choose a sequence $w_i \in H'$ such that $xw_i \to x_0$, as $i \to \infty$. Write $xw_i = x_0 g_i$ where $g_i \to e$ in $G - LH'$. Let us write $g_i = \ell_i r_i$ where $\ell_i \in L$, and $r_i \in \exp \mathfrak{l}^\perp$. In particular, $r_i \notin \mathrm{C}(H)$. Let $x_i = x_0\ell_i$, so that $x_i r_i \in X$.

We claim that we can assume that $x_i \in \mathrm{RF}\,\mathcal{M} \cap x_0 L$, $r_i \notin \mathrm{C}(H)$, and $x_i r_i \in X$. Since $x_0 \in F^*$, by Lemma 8.3, we can find $w_i' \to w' \in H$ such that $x_0\ell_i w_i' \in \mathrm{RF}\,\mathcal{M}$, and $x_0 w' \in \bigcap_{i=1}^{\ell} \mathscr{G}(U_\pm^{(i)}, x_0L)$; hence

$$\overline{x_0 w' U} = x_0 L \cap \mathrm{RF}_+\,\mathcal{M}.$$

Writing $x_i' = x_0 \ell_i w_i'$ and $r_i' = w_i'^{-1} r_i w_i'$, we have

$$x_i' r_i' = x w_i w_i' \in X,$$

where $x_i' \to x_0 w'$ in $x_0 L \cap \mathrm{RF}\,\mathcal{M}$, and $r_i' \to e$ in $\exp \mathfrak{l}^\perp$. Since F^* is H'-invariant, we have $x_0 w' \in F^*$. Since F^* is open and $x_0 w' \in F^*$, it follows that $x_i' \in X \cap \mathrm{RF}\,\mathcal{M} \cap F^*$ for sufficiently large i. Note that $r_i' \notin \mathrm{C}(H)$, as $r_i \notin \mathrm{C}(H)$. This proves the claim.

We may assume $r_i \notin \mathrm{N}(U)$ for all i, up to switching the roles of U and U^+, by Lemma 3.5. Note that $x_i \to x_0$ in $\mathrm{RF}\,\mathcal{M} \cap x_0 L$ and x_0 satisfies (16.7). As we are assuming $(2)_m$, and $(3)_m$, we may now apply Proposition 15.1 to the sequence $x_0 \ell_i r_i \to x_0$ to obtain a non-trivial element $v \in \tilde{U}^\perp$ such that

$$x_0 L v \cap \mathrm{RF}_+\,\mathcal{M} \subset X.$$

Since $x_0 \in F^* \cap \mathrm{RF}\,\mathcal{M}$, it follows from Lemma 8.12 that there exist $x_2 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and a connected closed subgroup $\widehat{U} < N$ properly containing

$L \cap N$ such that

$$x_2 \widehat{U} A \subset X.$$

Since co-dim$_N(\widehat{U}) \leq m$, it remains to apply $(2.a)_m$ to finish the proof of the proposition. □

Proof of $(1)_{m+1}$. Combining Propositions 16.7 and 16.8, we now prove:

Theorem 16.9. *If $(2)_m$ and $(3)_m$ are true, then $(1)_{m+1}$ is true.*

Proof. Recall that we only need to consider the case $X = \overline{xH'}$ where $x \in F^*$ and xH' is not closed in F^*. By Proposition 16.7, there exists $x_0 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and $L \in \mathcal{L}_U$ such that $x_0 L$ is closed and

$$x_0 L \cap \mathrm{RF}_+ \mathcal{M} \subset X.$$

Since X is H'-invariant, it follows

$$(x_0 L \cap \mathrm{RF}_+ \mathcal{M}) \cdot H' \subset X. \tag{16.8}$$

Note that $(x_0 L \cap \mathrm{RF}_+ \mathcal{M}) \cdot H' = x_0 L \cdot \mathrm{C}(H) \cap F$ is a closed set. We may assume the inclusion in (16.8) is proper, otherwise we have nothing further to prove. Then by Proposition 16.8, there exists $\widehat{L} \in \mathcal{L}_{\widehat{U}}$ for some $\widehat{U} < N$ properly containing $L \cap N$, and a closed orbit $x_1 \widehat{L}$ with $x_1 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ such that $x_1 \widehat{L} \cap \mathrm{RF}_+ \mathcal{M} \subset X$. If

$$(x_1 \widehat{L} \cap \mathrm{RF}_+ \mathcal{M}) \cdot \mathrm{C}(H) \neq X,$$

then we can apply Proposition 16.8 on

$$x_1 \widehat{L} \cap \mathrm{RF}_+ \mathcal{M} \subset X,$$

as $\mathcal{L}_{\widehat{U}} \subset \mathcal{L}_U$. Continuing in this fashion, the process terminates in a finite step for a dimension reason, and hence

$$X = (x_1 \tilde{L} \cap \mathrm{RF}_+ \mathcal{M}) \cdot H' = x_1 \tilde{L} \cdot \mathrm{C}(H) \cap F$$

for some $\tilde{L} \in \mathcal{L}_U$, completing the proof. □

17. U and AU-orbit closures: proof of $(2)_{m+1}$

In this section, we fix a closed orbit $x_0 \widehat{L}$ for $x_0 \in F^*$ and $\widehat{L} \in \mathcal{L}_U$. Let $U < \widehat{L} \cap N$ be a connected closed subgroup with co-dim$_{\widehat{L} \cap N} U \leq m+1$. By replacing U and $\widehat{L}$ by their conjugates using an element $m \in M$, we may assume that

$$U \subset \widehat{L} \cap \check{H} \cap N.$$

We keep the same notation $H, F, \partial F, F^*$ etc from section 16. If $x \in \mathrm{RF}_+ \mathcal{M} \cap \partial F$ (resp. if $x \in \mathrm{RF}\,\mathcal{M} \cap \partial F$), then $(2.a)$ (resp. $(2.b)$) follows from Theorem 11.1.

We fix $x \in \mathrm{RF}\,\mathcal{M} \cap x_0 \widehat{L} \cap F^*$, and set

$$X := \overline{xU} \text{ and assume that } X \neq x_0 \widehat{L} \cap \mathrm{RF}_+ \mathcal{M}. \tag{17.1}$$

This assumption implies that U is a proper connected closed subgroup of $\widehat{L} \cap N$ and hence $\dim(\widehat{L} \cap N) > \dim U \geq 1$.

By Proposition 5.16, either $x_0\widehat{L}$ is compact or $\mathscr{S}(U, x_0\widehat{L})$ contains a compact orbit zL_0 with $L_0 \in \mathcal{L}_U$. If $x_0\widehat{L}$ is compact, then $(2)_{m+1}$ follows from Theorem 11.1. Therefore we assume in the rest of the section that

$$\mathscr{S}(U, x_0\widehat{L}) \text{ contains a compact orbit } zL_0 \text{ with } L_0 \in \mathcal{L}_U. \tag{17.2}$$

Lemma 17.1. *Assume that* $(1)_{m+1}$ *and* $(2)_m$ *hold. Then*

$$\overline{xAU} \cap \mathscr{S}(U, x_0\widehat{L}) \neq \emptyset.$$

Proof. Since $(1)_{m+1}$ is true, we have

$$\overline{xH} = xQ \cap F$$

for some $Q \in \mathcal{L}_U$ such that xQ is closed. By Lemma 16.1, $Q < \widehat{L}$. It follows from Lemma 5.11 that either $Q = \widehat{L}$ or $\dim(Q \cap N) < \dim(\widehat{L} \cap N)$. Suppose that $Q = \widehat{L}$. By (17.2), there exists a compact orbit $zL_0 \subset \mathscr{S}(U, x_0\widehat{L})$ for some $L_0 \in \mathcal{L}_U$. On the other hand, $x_0\widehat{L} \cap F = \overline{xH} = \overline{xAU}(K \cap H)$. Hence for some $k \in K \cap H$, $zk \in \overline{xAU}$. Since $H \subset L_0$, $zk \in zL_0$. So $\overline{xAU}$ intersects zL_0, proving the claim. If $\dim(Q \cap N) < \dim(\widehat{L} \cap N)$. then $\overline{xAU} \subset xQ \subset \mathscr{S}(U, x_0\widehat{L})$. $\square$

Lemma 17.2. *Assume that* $(1)_{m+1}$ *and* $(2)_m$ *hold. Then*

$$\overline{xU} \cap \mathscr{S}(U, x_0\widehat{L}) \neq \emptyset.$$

Proof. Since

$$\left(x_0\widehat{L} \cap \mathrm{RF}_+ \mathcal{M}\right) - F^* \subset \mathscr{S}(U, x_0\widehat{L}), \tag{17.3}$$

it suffices to consider the case when $X := \overline{xU} \subset F^*$. Let $Y \subset X$ be a U-minimal set with respect to $\mathrm{RF}\,\mathcal{M}$. Since $Y \subset F^*$, by Proposition 10.6, there exists an unbounded one-parameter subsemigroup S inside $AU^{\perp}\,\mathrm{C}_2(U) \cap \widehat{L}$ such that $YS \subset Y$. In view of Lemma 3.3, we could remove $\mathrm{C}_2(U)$-component of S so that S is either of the following

- $v^{-1}A^+v$ for a one-parameter semigroup $A^+ \subset A$ and $v \in U^{\perp} \cap \widehat{L}$;
- V^+ for a one-parameter semigroup $V^+ \subset U^{\perp} \cap \widehat{L}$,

and

$$YS \subset X(\mathrm{C}_2(U) \cap \widehat{L}).$$

Since $\mathscr{S}(U, x_0\widehat{L})$ is invariant by $N\,\mathrm{C}_2(U) \cap \widehat{L}$, it suffices to show that

$$X(N\,\mathrm{C}_2(U) \cap \widehat{L}) \cap \mathscr{S}(U, x_0\widehat{L}) \neq \emptyset.$$

If $S = v^{-1}A^+v$, then $Yv^{-1}A^+ \subset Xv^{-1}(\mathrm{C}_2(U) \cap \widehat{L})$. Choose $y \in Y$. We may assume that $yv^{-1} \in F^*$ by (17.3). Then, replacing y with an element in yU if necessary, we may assume $yv^{-1} \in \mathrm{RF}\,\mathcal{M} \cap F^*$. Choose a sequence

$a_n \to \infty$ in A^+. Then $yv^{-1}a_n$ converges to some $y_0 \in \operatorname{RF}\mathcal{M}$ by passing to a subsequence. Since $\liminf a_n^{-1}A^+ = A$, and

$$(yv^{-1}a_n)(a_n^{-1}A^+) \subset Xv^{-1}(\mathrm{C}_2(U) \cap \widehat{L}),$$

we obtain that

$$y_0 A \subset Xv^{-1}(\mathrm{C}_2(U) \cap \widehat{L}).$$

Since $\overline{y_0AU} \subset Xv^{-1}(\mathrm{C}_2(U) \cap \widehat{L})$ and $\overline{y_0AU}$ meets $\mathscr{S}(U, x_0\widehat{L})$ by Lemma 17.1, the claim follows.

Next, assume that $S = V^+$, so that $YV^+ \subset \ X\,\mathrm{C}_2(U) \cap \widehat{L}$. Let $v_n \to \infty$ be a sequence in V^+. We have $Yv_n \subset X \subset F^*$. Together with the fact Yv_n is U-invariant, this implies Yv_n meets $\operatorname{RF}\mathcal{M}$. Note that

$$Yv_n(v_n^{-1}V^+) \subset X(\mathrm{C}_2(U) \cap \widehat{L}).$$

Choose $y_n \in Yv_n \cap \operatorname{RF}\mathcal{M}$. As $\operatorname{RF}\mathcal{M}$ is compact, y_n converges to some $y_0 \in \operatorname{RF}\mathcal{M}$, by passing to a subsequence, and hence

$$y_0UV \subset X(\mathrm{C}_2(U) \cap \widehat{L}).$$

Since $\text{co-dim}_N(UV) \le m$, the conclusion follows from $(2)_m$. □

Lemma 17.3. *Assume that* $(1)_{m+1}$ *and* $(2)_m$ *hold. Then*

$$\overline{xU} \cap \mathscr{S}(U, x_0\widehat{L}) \cap F^* \neq \emptyset.$$

Proof. By Lemma 17.2, there exists $y \in \overline{xU} \cap \mathscr{S}(U, x_0\widehat{L})$. Hence by $(2)_m$,

$$\overline{yU} = yL \cap \operatorname{RF}_+\mathcal{M} \subset \overline{xU}$$

for some $L \in \mathcal{Q}_U$ properly contained in $\widehat{L}$. Consider the collection of all subgroups $L \in \mathcal{Q}_U$ such that $yL \subset \overline{xU}$ for some $y \in \operatorname{RF}_+\mathcal{M}$. Choose L from this collection so that $L \cap N$ has maximal dimension. If $yL \cap F^* \neq \emptyset$, then the claim follows.

Now suppose that $yL \subset \partial F$. As $y \in \operatorname{RF}_+\mathcal{M} \cap \partial F$, we have

$$y = zv_0c_0$$

for some $z \in \operatorname{BF}\mathcal{M}$, $v_0 \in \check{V}^+$ and $c_0 \in \mathrm{C}(H)$. Since $y \in \overline{xU}$, there exists $u_i \in U$ such that xu_i converges to y as $n \to \infty$. Set

$$z_i := xu_ic_0^{-1}v_0^{-1} \in \overline{xU}c_0^{-1}v_0^{-1}$$

so, $z_i \to z$. As $v_0 \in \check{V}^+$ and hence $v_0^{-1} \in \check{V}^-$ and $xu_i \in F^*$, we have $z_i \in F^* \cap \operatorname{RF}_+\mathcal{M} \subset \operatorname{RF}\mathcal{M} \cdot U$. By Lemma 8.6, we may modify z_i by elements of U so that $z_i \in \operatorname{RF}\mathcal{M}$ and z_i converges to some $z_0 \in z\check{H}$. Write $z_i = z_0\ell_ir_i$ for some $\ell_i \in \check{H}$ and $r_i \in \exp\check{\mathfrak{h}}^\perp$ converging to e. Since $z_i \in F^*$ and $z_0\ell_i \in \partial F$, we have $r_i \neq e$. By Theorem 11.1, we have $\overline{z_0\ell_iU} = z_0\ell_iL_i$ for some $L_i \in \mathcal{Q}_U$ contained in $\check{H}$.

Case 1: $r_i \in \mathrm{N}(U)$ **for some** i. Then

$$\overline{xU} = \overline{z_0\ell_ir_iv_0c_0U} = \overline{z_0\ell_iU}(r_iv_0c_0) = z_0\ell_iL_i(r_iv_0c_0).$$

As $\overline{xU} \neq x_0\widehat{L}$ by the hypothesis, it follows that $x \in \mathscr{S}(U, x_0\widehat{L}) \cap F^*$, proving the claim.

Case 2: $r_i \notin \mathrm{N}(U)$ **for all** i. Then there exists a one-parameter subgroup $U_0 < U$ such that $r_i \notin \mathrm{N}(U_0)$ for all i, by passing to a subsequence.

By Lemma 9.3, we can find $u_{t_i} \to \infty$ in U_0 so that $z_i u_{t_i} \in \mathrm{RF}\,\mathcal{M}$ and $u_{t_i}^{-1} r_i u_{t_i}$ converges to a non-trivial element $v \in \check{V}$, whose size is strictly bigger than $\|v_0\|$. As $z_0\ell_i u_{t_i}$ is contained in the compact subset $z_0\check{H}$, we may assume that $z_0\ell_i u_{t_i}$ converges to some $z' \in z_0\check{H}$. Hence

$$z_i u_{t_i} = z_0\ell_i u_{t_i}(u_{t_i}^{-1} r_i u_{t_i}) \to z'v \in \mathrm{RF}\,\mathcal{M} \cap \overline{xU} c_0^{-1} v_0^{-1}.$$

Since $z' \in \mathrm{BF}\,\mathcal{M}$ and $z'v \in \mathrm{RF}\,\mathcal{M}$, we have $v \in \check{V}^-$.

By Theorem 11.1, $\overline{z'U} = z'Q_1$ for some $Q_1 \in \mathcal{Q}_U$. Since $z'vv_0c_0 \in \overline{xU}$, we get

$$\overline{xU} \supset z'Q_1(vv_0)c_0.$$

Since the size of v is larger than the size of v_0, then vv_0 is a non-trivial element of $\check{V}^-$. Since $z'Q_1 \subset \mathrm{BF}\,\mathcal{M}$, the closed orbit $z'Q_1(vv_0)c_0$ meets F^*. Hence the claim follows. $\square$

Theorem 17.4. *Assume that* $(1)_{m+1}$, $(2)_m$, *and* $(3)_m$ *are true. Then* $(2)_{m+1}$ *is true.*

Proof. We first show $(2.a)_{m+1}$ holds for $X = \overline{xU}$. By Lemma 17.3 and $(2)_m$, there exists a closed orbit yL with $y \in F^*$ and $L \in \mathcal{Q}_U$ such that

$$\overline{xU} \supset yL \cap \mathrm{RF}_+\,\mathcal{M}$$

and $L \cap N \neq \widehat{L} \cap N$. We choose $L \in \mathcal{Q}_U$ so that $\dim(L \cap N)$ is maximal. Note that $\text{co-dim}_{L\cap N} U \leq m$. By Theorem 13.1, we can assume that

$$y \in \bigcap_{i=1}^{\ell} \mathscr{G}(U^{(i)}, yL) \cap F^* \cap \mathrm{RF}\,\mathcal{M} \tag{17.4}$$

where $U^{(1)}, \cdots, U^{(\ell)}$ are one-parameter subgroups generating U. As $y \in \overline{xU}$, there exists $u_i \in U$ such that $xu_i \to y$ as $i \to \infty$. Since $y \in F^*$, we can assume $xu_i \in \mathrm{RF}\,\mathcal{M}$ after possibly modifying u_i by Lemma 8.6. We will write $xu_i = y\ell_i r_i$ where $\ell_i \in L$ and $r_i \in \exp \mathfrak{l}^\perp \cap \widehat{L}$.

Case 1: $r_i \in \mathrm{N}(U)$ **for some** i. Then $y\ell_i \in \mathrm{RF}_+\,\mathcal{M}$ and $X = \overline{xu_iU} = \overline{y\ell_iU}r_i$. Since $y\ell_iU \subset yL$, and $\text{co-dim}_{L\cap N}(U) \leq m$, we have

$$X = \overline{y\ell_i U r_i} = y\ell_i L' r_i \cap \mathrm{RF}_+\,\mathcal{M}$$

for some $L' \in \mathcal{Q}_U$, proving the claim.

Case 2: $r_i \notin \mathrm{N}(U)$ **for all** i. By (17.4), we can apply Proposition 15.2 to the sequence $xu_i \to y$ and obtain a sequence $v_j \to \infty$ in $(L \cap N)^\perp$ such that

$$yLv_j \cap \mathrm{RF}_+\,\mathcal{M} \subset X.$$

Since $y \in F^*$, by Lemma 8.10, there exists a one-parameter subgroup $V \subset (L \cap N)^\perp$ such that $y_1(L \cap N)V \subset X$ for some $y_1 \in F^* \cap \mathrm{RF}\,\mathcal{M}$. Hence, by $(2)_m$, we get a contradiction to the maximality of $L \cap N$; this proves $(2.a)_{m+1}$.

Now we show $(2.b)_{m+1}$ for the closure $\overline{xAU}$. By $(1)_{m+1}$, we have $\overline{xH} = xL \cap F$ for some $L \in \mathcal{L}_U$ contained in $\widehat{L}$. Hence $\overline{xAU} \subset xL \cap \mathrm{RF}_+ \mathcal{M}$. It suffices to show that

$$\overline{xAU} = xL \cap \mathrm{RF}_+ \mathcal{M}. \tag{17.5}$$

If $U = L \cap N$, then $\overline{xU} = xL \cap \mathrm{RF}_+ \mathcal{M}$ by Theorem 13.1, which implies (17.5). So, suppose that U is a proper closed subgroup of $L \cap N$. Since $\overline{xAU}(K \cap H) = \overline{xH} = xL \cap F$, it follows from Lemma 5.15 that we can choose $y \in \overline{xAU} \cap \mathscr{G}(U, xL)$. By $(2.a)_{m+1}$ and Lemma 14.4, we have $\overline{yU} = xL \cap \mathrm{RF}_+ \mathcal{M}$, finishing the proof. □

18. Topological equidistribution: proof of $(3)_{m+1}$

In this section, we prove $(3)_{m+1}$. Let $U < N$ be a non-trivial connected closed subgroup. Let $x_0\widehat{L}$ be a closed orbit for $x_0 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and $\widehat{L} \in \mathcal{L}_U$ such that $\text{co-dim}_{\widehat{L}\cap N}(U) = m + 1$. As before we may assume that $U \subset \widehat{L} \cap \check{H} \cap N$.

Let $x_iL_i \subset x_0\widehat{L}$ be a sequence of closed orbits intersecting $\mathrm{RF}\,\mathcal{M}$ where $x_i \in \mathrm{RF}_+ \mathcal{M}$, $L_i \in \mathcal{Q}_U$. We write x_iL_i as $y_iL_iv_i$ where $y_i \in \mathrm{RF}_+ \mathcal{M}$, $L_i \in \mathcal{L}_U$, and $v_i \in (L_i \cap N)^\perp \cap \widehat{L}$. Assume that no infinite subsequence of $y_iL_iv_i$ is contained in a subset of the form $y_0L_0D \subset \mathscr{S}(U, x_0\widehat{L})$ where y_0L_0 is a closed orbit for some $L_0 \in \mathcal{L}_U$ and $D \subset \mathrm{N}(U)$ is a compact subset. Let

$$E = \limsup_{i\to\infty} \; (y_iL_iv_i \cap \mathrm{RF}_+ \mathcal{M}).$$

Note that $\liminf_{i\to\infty}(y_iL_iv_i \cap \mathrm{RF}_+ \mathcal{M})$ coincides with the intersection of the subsets $\limsup(y_{i_k}L_{i_k}v_{i_k} \cap \mathrm{RF}_+ \mathcal{M})$ for all infinite subsequences $\{i_k : k \in \mathbb{N}\}$ of $\mathbb{N}$. If the hypothesis of $(3)_{m+1}$ holds for a given sequence $y_iL_iv_i$, then it also holds for all subsequences. Hence to prove $(3)_{m+1}$, it suffices to show that

$$E = \mathrm{RF}_+ \mathcal{M} \cap x_0\widehat{L}.$$

We note that by $(3)_m$, we may assume that

$$L_i \cap N = U \quad \text{for all } i.$$

This in particular implies that each $y_iL_iv_i \cap \mathrm{RF}_+ \mathcal{M}$ is U-minimal by Theorem 13.1.

Lemma 18.1. *Assume that $(1)_{m+1}$, $(2)_{m+1}$ and $(3)_m$ are true. Then there exist $y \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and $L \in \mathcal{Q}_U$ with $\dim(L \cap N) > \dim U$ such that yL is closed and*

$$E \supset yL \cap \mathrm{RF}_+ \mathcal{M}.$$

Proof. By $(2)_m$, it suffices to show that there exist $y_0 \in F^* \cap \mathrm{RF}\,\mathcal{M}$ and $\widehat{U} < N$ properly containing U such that

$$E \supset y_0\widehat{U}.$$

Suppose that $y_iL_iv_i \subset \partial F$ for infinitely many i. Since $y_iL_iv_i \cap \mathrm{RF}\,\mathcal{M} \neq \emptyset$, we may assume $y_iv_i \in z_i\check{H}\,\mathrm{C}(H)$ for some $z_i \in \mathrm{BF}\,\mathcal{M}$ by (4.6). Since $L_i \cap N = U$, we get $y_iL_iv_i = \overline{y_iU} \subset z_i\check{H}\,\mathrm{C}(H)$ by Theorem 11.1. This contradicts the hypothesis on $y_iL_iv_i$'s.

Therefore by passing to a subsequence, for all i,

$$y_iL_iv_i \cap \mathrm{RF}_+\,\mathcal{M} \cap F^* \neq \emptyset.$$

Since $AU < L_i$ for all i, it follows that

$$E = \limsup_{i\to\infty} (y_iL_iv_i \cap \mathrm{RF}_+\,\mathcal{M})(v_i^{-1}AUv_i)$$

By Lemma 8.9, there exists $y_0 \in \limsup_i (y_iL_iv_i \cap \mathrm{RF}_+\,\mathcal{M}) \cap F^*$. Hence

$$y_0 \limsup_{i\to\infty} (v_i^{-1}AUv_i) \subset E, \tag{18.1}$$

after passing to a subsequence.

If $v_i \to \infty$, then $\limsup_i (v_i^{-1}AUv_i)$ contains $A\widehat{U}$ for some $\widehat{U}$ properly containing U by Lemma 3.4. Therefore, we get the conclusion $y_0\widehat{U} \subset E$ from (18.1). Now suppose that, by passing to a subsequence, v_i converges to some $v \in N \cap \widehat{L}$. Then (18.1) gives

$$y_0v^{-1}AUv \subset E.$$

Then by $(2)_{m+1}$, $\overline{y_0v^{-1}AU}$ is of the form $y_0v^{-1}L_0 \cap \mathrm{RF}_+\,\mathcal{M}$ for some $L_0 \in \mathcal{L}_U$. Hence

$$E \supset y_0L \cap \mathrm{RF}_+\,\mathcal{M} \tag{18.2}$$

where $L := v^{-1}L_0v$. If $L \cap N$ contains U properly, this proves the claim. So we suppose that $L \cap N = U$. By Theorem 13.1, we can assume that $y_0 \in \bigcap_{i=1}^{\ell} \mathscr{G}(U^{(i)}, y_0L) \cap F^* \cap \mathrm{RF}\,\mathcal{M}$, where $U^{(1)}, \cdots, U^{(\ell)}$ are one-parameter subgroups generating U. By replacing y_i by an element of $y_iL \cap \mathrm{RF}_+\,\mathcal{M}$, we may assume that $y_iv_i \to y_0$. Furthermore, as $y_0 \in F^* \cap \mathrm{RF}\,\mathcal{M}$, for all i sufficiently large, $y_iv_i \in F^* \cap \mathrm{RF}_+\,\mathcal{M} \subset \mathrm{RF}\,\mathcal{M} \cdot U$ (as F^* is open). Hence we can also assume $y_iv_i \in \mathrm{RF}\,\mathcal{M}$ by Lemma 8.7. Therefore we may write

$$y_iv_i = y_0\ell_ir_i$$

for some $\ell_i \to e$ in L and non-trivial $r_i \to e$ in $\exp \mathfrak{l}^{\perp}$.

Suppose that r_i belongs to $\mathrm{N}(U)$ for infinitely many i. Then

$$y_iL_iv_i \cap \mathrm{RF}_+\,\mathcal{M} = \overline{y_iv_iU} = \overline{y_0\ell_iU}r_i = y_0Lr_i \cap \mathrm{RF}_+\,\mathcal{M}.$$

Hence $y_iL_iv_ir_i^{-1} \cap \mathrm{RF}_+\,\mathcal{M} = y_0L \cap \mathrm{RF}_+\,\mathcal{M}$. In particular, $y_iL_iv_ir_i^{-1} \cap \mathrm{RF}\,\mathcal{M}$ is non-empty (as it contains y_0) and is contained in y_0L. By Lemma 16.1, this implies that $y_iL_iv_i \subset y_0Lr_i$. As $r_i \to e$, this contradicts the hypothesis on $y_iL_iv_i$'s.

Therefore $r_i \notin \mathrm{N}(U)$ for all i but finitely many. We may now apply Proposition 15.2 and Lemma 8.10 to deduce that E contains an orbit $z_0\widehat{U}$ for some $\widehat{U} < \widehat{L} \cap N$ containing U properly and for some $z_0 \in \mathrm{RF}_+\,\mathcal{M} \cap F^*$. This proves the claim. □

Theorem 18.2. *If* $(1)_{m+1}$*,* $(2)_{m+1}$*, and* $(3)_m$ *are true, then* $(3)_{m+1}$ *is true.*

Proof. We claim that

$$x_0\widehat{L} \cap \mathrm{RF}_+ \mathcal{M} = E. \tag{18.3}$$

By Lemmas 18.1, we can take a maximal $\widehat{U}$ such that $E \supset y\widehat{U}$ for some $y \in F^* \cap \mathrm{RF}\,\mathcal{M}$. By $(2)_m$, we get a closed orbit yL for some $L \in \mathcal{Q}_{\widehat{U}}$ such that

$$yL \cap \mathrm{RF}_+ \mathcal{M} \subset E. \tag{18.4}$$

If $L = \widehat{L}$, then the claim (18.3) is clear. Now suppose that L is a proper subgroup of $\widehat{L}$. This implies that $L \cap N$ is a proper subgroup of $\widehat{L} \cap N$, since $\widehat{L} \cap N$ acts minimally on $x_0\widehat{L} \cap \mathrm{RF}_+ \mathcal{M}$ as $\widehat{L} \in \mathcal{L}_U$. By Theorem 13.1, we can assume that $y \in \bigcap_{i=1}^{\ell} \mathscr{G}(U^{(i)}, yL) \cap F^* \cap \mathrm{RF}\,\mathcal{M}$, where $U^{(1)}, \cdots, U^{(\ell)}$ are one-parameter subgroups generating U. As $y \in E$, there exists a sequence $x_i \in y_iL_iv_i \cap \mathrm{RF}_+ \mathcal{M}$ converging to y, by passing to a subsequence. Since $U = v_i^{-1}L_iv_i \cap N$, we have $x_i \in \mathrm{RF}\,\mathcal{M} \cdot U$. By Lemma 8.7, by replacing x_i with x_iu_i for some $u_i \to e$ in U, we may assume $x_i \in \mathrm{RF}\,\mathcal{M}$.

We claim that

$$x_i \notin yL\,\mathrm{N}(U).$$

Suppose not, i.e., $x_i = y\ell_ir_i$ for some $\ell_i \in L$ and $r_i \in \mathrm{N}(U)$. Then

$$y_iL_iv_i \cap \mathrm{RF}_+ \mathcal{M} = \overline{x_iU} = \overline{y\ell_iU}r_i \subset yLr_i.$$

By the assumption on $y_iL_iv_i$'s, this cannot happen as r_i's are bounded.

On the other hand, $\dim(L_i \cap N)$ is strictly smaller than $\dim(L \cap N)$, since $L_i \cap N = U$ and $\widehat{U} < L \cap N$, yielding a contradiction. Hence $x_i \notin yL\,\mathrm{N}(U)$.

We can now apply Proposition 15.2 and Lemma 8.10 and deduce that E contains $y_1\widehat{U}V$ for some $y_1 \in F^* \cap \mathrm{RF}\,\mathcal{M}$. This is a contradiction to the maximality assumption on $\dim \widehat{U}$. □

Proof of Theorem 1.7. We explain how to deduce this theorem from Theorem 14.1(3). For (1), we may first assume that P_i have all same dimension so that for some fixed connected closed subgroup $U < N$, $P_i = \pi(x_iH'(U))$ where $x_iH'(U)$ is a closed orbit of some $x_i \in \mathrm{RF}\,\mathcal{M}$. Then there exists $L_i \in \mathcal{L}_U$ such that x_iL_i is closed and $P_i = \pi(x_iL_i)$ by Proposition 4.9. We claim that the sequence x_iL_i satisfies the hypothesis of Theorem 14.1(3). Suppose not. Then there exists a closed orbit y_0L_0 with $L_0 \in \mathcal{L}_U$, $L_0 \neq G$ and a compact subset $D \subset \mathrm{N}(U)$ such that $x_iL_i \subset y_0L_0D$ for infinitely many i. By Lemma 16.2, this can happen only when $L_i \subset d_i^{-1}L_0d_i$ and $x_iL_i \subset y_0L_0d_i$ for some $d_i \in D$. Since $D \subset \mathrm{N}(U) \subset L_0(L_0 \cap N)^{\perp}M$, we may assume that $d_i \in (L_0 \cap N)^{\perp}M$. Since $A \subset L_i \subset d_i^{-1}L_0d_i$, we have $d_i \in M$. This implies that $P_i = \pi(x_iL_i) \subset \pi(y_0L_0d_i) = \pi(y_0L_0)$. By the maximality assumption on P_i's, it follows that P_i is a constant sequence, yielding a contradiction. Hence by Theorem 14.1(3), $\lim(x_iL_i \cap \mathrm{RF}_+ \mathcal{M}) = \mathrm{RF}_+ \mathcal{M}$. Since $\pi(\mathrm{RF}_+ \mathcal{M}) = \Gamma\backslash\mathbb{H}^d$, the claim follows. (2) follows from Corollary 5.8.

For (3), if there are infinitely many bounded properly immersed P_i's, then $\lim P_i = M$ by (1). On the other hand, $P_i \subset \operatorname{core} \mathcal{M}$; because any bounded $H'(U)$ orbit should be inside $\mathrm{RF}\,\mathcal{M}$. Since $\operatorname{core} \mathcal{M}$ is a proper closed subset of M, as $\mathrm{Vol}(\mathcal{M}) = \infty$, this gives a contradiction.

Remark 18.3. In fact, when $\mathcal{M}$ is any convex cocompact hyperbolic manifold of infinite volume, there are only finitely many bounded maximal closed $H'(U)$-orbits, and hence only finitely many maximal properly immersed bounded geodesic planes. The reason is that if not, we will be having infinitely many maximal closed orbits $x_i L_i$ *contained in* $\mathrm{RF}\,\mathcal{M}$ for some $L_i \in \mathcal{L}_U$, and for any U-invariant subset E contained in $\mathrm{RF}\,\mathcal{M}$, the 1-thickness for points in E holds for any one-parameter subgroup of U for the trivial reason, which makes our proof of Theorem 14.1 work with little modification (in fact, much simpler) for a general $\mathcal{M}$.

19. Appendix: Orbit closures for $\Gamma\backslash G$ compact case

In this section we give an outline of the proof of the orbit closure theorem for the actions of $H(U)$ and U, assuming that $\Gamma\backslash G$ is compact and there exists at least one closed orbit of $\mathrm{SO}^\circ(d-1,1)$. We hope that giving an outline of the proof of Theorem 14.1 in this special case will help readers understand the whole scheme of the proof better and see the differences with the infinite volume case more clearly.

Note that in the case at hand,

$$\mathrm{RF}\,\mathcal{M} = F^*_{H(U)} = \mathrm{RF}_+\,\mathcal{M} = \Gamma\backslash G.$$

Without loss of generality, we assume that $U \subset \mathrm{SO}^\circ(d-1,1) \cap N$.

Theorem 19.1. *Let $x \in \Gamma\backslash G$.*

(1) *There exists $L \in \mathcal{L}_U$ such that*

$$\overline{xH(U)} = xL.$$

(2) *There exists $L \in \mathcal{Q}_U$ such that*

$$\overline{xU} = xL.$$

The base case $(2)_0$ follows from a special case of Theorem 13.1. For $m \geq 0$, we will show that $(2)_m$ implies $(1)_{m+1}$, and that $(1)_{m+1}$ and $(2)_m$ together imply $(2)_{m+1}$.

We note that when $\Gamma\backslash G$ is compact, we don't need the topological equidistribution statement, which is Theorem 14.1(3) to run the induction argument, thanks to (2.6). In order to prove $(1)_{m+1}$, it suffices to use $(2)_m$ only when the ambient space is $\Gamma\backslash G$; in the proof of Theorem 14.1, we needed to use $(2)_m$ whenever $\text{co-dim}_{N\cap\hat{L}}\, U \leq m$ for any closed orbit $x_0\widehat{L}$ containing xU (this was needed in order to use results in section 15).

Remark 19.2. Theorem 19.1 is proved by Shah [44] by topological arguments. Our proof presented in this appendix is somewhat different from Shah's in that we prove that $(1)_m$ implies $(2)_m$ using the existence of a closed $\mathrm{SO}^\circ(d-1,1)$-orbit, while he shows that $(2)_m$ implies $(1)_m$.

Proof of $(1)_{m+1}$**.** We assume that $1 \leq \text{co-dim}_N\, U = m+1$. By Proposition 16.3, it suffices to show that $X := \overline{xH'(U)} = xL\,\mathrm{C}(H(U))$ for some $L \in \mathcal{L}_U$. Assume that $xH'(U)$ is not closed in the following.

Step 1: Find a closed orbit inside X**.** We claim that X contains a U-minimal subset Y such that $X - yH'$ is not closed for some $y \in Y$ (cf. the case when R is compact in the proof of Proposition 16.7). If $xH'(U)$ is not locally closed, then any U-minimal subset $Y \subset X$ does the job. If $xH'(U)$ is locally closed, then any U-minimal subset Y of $X - xH'(U)$ does the job; note that the set $X - xH'(U)$ is a compact $H'(U)$-invariant subset and hence contains a U-minimal subset.

Hence, by Lemma 10.9, X contains an orbit $x_0\widehat{U}$ with $\dim \widehat{U} > \dim U$. By $(2)_m$ and Lemma 16.4, X contains a closed orbit zL for some $L \in \mathcal{L}_U$. We may assume that $X \neq zL\,\mathrm{C}(H(U))$; otherwise, we are done.

Step 2: Enlarge a closed orbit inside X. Since zL is compact, by Theorem 13.1, we can assume that $zU_{\pm}^{(i)}$ is dense in zL where $U_{\pm}^{(1)}, \cdots, U_{\pm}^{(k)}$ are one-parameter subgroups of $U^{\pm}$ generating $U^{\pm}$. Note that there exists $g_i \to e$ in $G - L\,\mathrm{C}(H(U))$ such that $zg_i \in X$. We can write $g_i = \ell_i r_i$ where $r_i \in \exp \mathfrak{l}^{\perp}$ and $\ell_i \in L$. Then $r_i \notin \mathrm{C}(H(U))$. Since $\bigcap_{i=1}^k (\mathrm{N}(U_+^{(i)}) \cap \mathrm{N}(U_-^{(i)})) \cap \exp \mathfrak{l}^{\perp}$ is locally contained in $\mathrm{C}(H(U))$, we have $r_i \notin \mathrm{N}(U_0)$ where U_0 is one of the subgroups $U_{\pm}^{(i)}$. If $U_0 \in \{U_+^{(i)}\}$, then replace U by U^+.

Fix any $k > 1$. Applying (2.6) to the sequence $z_i := z\ell_i \to z$, the set

$$\mathsf{T}(z_i) := \{t \in \mathbb{R} : z_i u_t \in \Gamma\backslash G - \bigcup_{j=1}^{i} \mathcal{O}_j\} \tag{19.1}$$

is a k-thick subset (take $0 < \varepsilon < 1 - 1/k$). By Lemma 9.3, there exists $t_i \in \mathsf{T}(z_i)$ such that $u_{t_i}^{-1} r_i u_{t_i}$ converges to a non-trivial element $v \in (L \cap N)^{\perp}$. Now the sequence $z_i u_{t_i}$ converges to $z_0 \in \mathscr{G}(U_0, zL)$. Since $zg_i u_{t_i}$ converges to $z_0 v$, we deduce

$$zLv = \overline{z_0 v U_0} \subset X \text{ and hence } zLV^+ \subset zL(AvA) \subset X$$

where V^+ is the one-parameter unipotent subsemigroup contained in AvA. Take any sequence $v_i \to \infty$ in V^+ such that zv_i converges to some x_0. Then $x_0 V \subset \limsup (zv_i)(v_i^{-1}V^+) \subset X$ and hence X contains $x_0(L \cap N)V$. By the induction hypothesis $(2)_m$ and Lemma 16.4, X contains a closed orbit of $\widehat{L}$ for some $\widehat{L} \in \mathcal{L}_{\widehat{U}}$. This process of enlargement must end after finitely many steps.

Proof of $(2)_{m+1}$. Set $X := \overline{xU}$. We assume that $X \neq \Gamma\backslash G$. Since the co-dimension of U in N is at least 1, we may assume without loss of generality that $U < N \cap \mathrm{SO}^{\circ}(d-1,1)$ using conjugation by an element of M.

Step 1: Find a closed orbit inside X. By the hypothesis on the existence of a closed $L_0 := \mathrm{SO}^{\circ}(d-1,1)$-orbit, $\mathscr{S}(U) \neq \emptyset$. It follows from $(1)_{m+1}$, $(2)_m$, and the cocompactness of AU in $H'(U)$ that any AU-orbit closure intersects $\mathscr{S}(U)$ (cf. proof of Lemma 17.1).

We claim that X intersects $\mathscr{S}(U)$. Since $\mathscr{S}(U)$ is $N\,\mathrm{C}_2(U)$-invariant, it suffices to show that $XN\,\mathrm{C}_2(U)$ intersects $\mathscr{S}(U)$. Let $Y \subset X$ be a U-minimal subset. Then there exists a one-parameter subgroup $S < AU^{\perp}\,\mathrm{C}_2(U)$ such that $Yg = Y$ for all $g \in S$ by Lemma 10.6. Strictly speaking, the cited lemma gives $Yg \subset Y$ for g in a semigroup S, but in the case at hand, $Yg \subset Y$ implies $Yg = Y$, since Yg is U-minimal again, and hence $Yg^{-1} = Y$ as well. In view of Lemma 3.3, we get $YA \subset XN\,\mathrm{C}_2(U)$ or $YvA \subset XN\,\mathrm{C}_2(U)$ for some $v \in N$. In either case, $XN\,\mathrm{C}_2(U)$ contains an AU-orbit and hence intersects $\mathscr{S}(U)$. So the claim follows. Since X intersects $\mathscr{S}(U)$, by applying $(2)_m$, X contains a closed orbit zL for some $L \in \mathcal{Q}_U$.

Step 2: Enlarge a closed orbit inside X**.** Suppose $L \neq G$ and $X \neq zL$. It suffices to show that X contains a closed orbit of $\widehat{L}$ for some $\widehat{L} \in \mathcal{L}_{\widehat{U}}$ for some $\widehat{U}$ properly containing $L \cap N$. We may assume $X \not\subset zL\,\mathrm{C}(H(U))$; otherwise, the claim follows from $(2)_m$. We may assume $z \in \bigcap_{i=1}^{\ell} \mathscr{G}(U^{(i)}, yL)$ where $U^{(i)}$'s are one-parameter generating subgroups of U. Take a sequence $xu_i \to z$ where $u_i \in U$, and write $xu_i = z\ell_i r_i$ where $\ell_i \in L$ and $r_i \in \exp(\mathfrak{l}^{\perp})$. The case of $r_i \in \mathrm{N}(U)$ for some i follows from $(2)_m$ (cf. proof of Lemma 17.4). Hence we may assume $r_i \notin \mathrm{N}(U)$, and by passing to a subsequence, $r_i \notin \mathrm{N}(U_0)$ for some $U_0 \in \{U^{(i)}\}$.

Fix any $k > 1$. Then $\mathsf{T}(z_i)$ as in (19.1) is a k-thick subset. We now repeat the same argument of a step in the proof of $(1)_{m+1}$. By Lemma 9.3, there exists $t_i \in \mathsf{T}(z_i)$ such that $u_{t_i}^{-1} r_i u_{t_i}$ converges to a non-trivial element $v \in U^{\perp}$. Now the sequence $z_i u_{t_i}$ converges to $z_0 \in \mathscr{G}(U_0, zL)$. Hence $X \supset \overline{z_0(L\cap N)}v = zLv$. Moreover, by Lemma 9.3, such v can be made of arbitrarily large size, so we get $X \supset zLv_j$ for a sequence $v_j \in (L \cap N)^{\perp}$ tending to ∞. The set $\limsup_{j\to\infty} v_j^{-1} A v_j$ contains a one-parameter subgroup $V \subset (L\cap N)^{\perp}$ by Lemma 3.4. Passing to a subsequence, there exists $y \in \liminf zLv_j$ and hence

$$X \supset \limsup_{j\to\infty}(zLv_j) \supset y(L\cap N)\limsup_{j\to\infty}(v_j^{-1}Av_j) \supset y(L\cap N)V.$$

Hence X contains $y(L\cap N)V$, and hence the claim follows from $(2)_m$.

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