DYNAMICS FOR DISCRETE SUBGROUPS OF $\mathrm{SL}_2(\mathbb{C})$

HEE OH

Dedicated to Gregory Margulis with affection and admiration

ABSTRACT. Margulis wrote in the preface of his book *Discrete subgroups of semisimple Lie groups* [30]: "A number of important topics have been omitted. The most significant of these is the theory of Kleinian groups and Thurston's theory of 3-dimensional manifolds: these two theories can be united under the common title *Theory of discrete subgroups of* $\mathrm{SL}_2(\mathbb{C})$".

In this article, we will discuss a few recent advances regarding this missing topic from his book, which were influenced by his earlier works.

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1. INTRODUCTION

A discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is called a Kleinian group. In this article, we discuss dynamics of unipotent flows on the homogeneous space $\Gamma\backslash\mathrm{PSL}_2(\mathbb{C})$ for a Kleinian group Γ which is not necessarily a lattice of $\mathrm{PSL}_2(\mathbb{C})$. Unlike the lattice case, the geometry and topology of the associated hyperbolic 3-manifold $M=\Gamma\backslash\mathbb{H}^3$ influence both topological and measure theoretic rigidity properties of unipotent flows.

Around 1984-6, Margulis settled the Oppenheim conjecture by proving that every bounded $\mathrm{SO}(2,1)$-orbit in the space $\mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R})$ is compact ([28], [27]). His proof was topological, using minimal sets and the polynomial divergence property of unipotent flows. With Dani ([11], [12]), he also gave a classification of orbit closures for a certain family of one-parameter unipotent subgroups of $\mathrm{SL}_3(\mathbb{R})$. Based on Margulis' topological approach, Shah [48] obtained a classification of orbit closures

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for the action of any connected closed subgroup generated by unipotent elements in the space $\Gamma\backslash\operatorname{PSL}_2(\mathbb{C})$ when Γ is a lattice. This result in a much greater generality, as conjectured by Raghunathan, was proved by Ratner using her measure rigidity theorem ([43], [44]).

The relation between invariant measures and orbit closures for unipotent flows is not as tight in the infinite volume case as it is in the finite volume case. Meanwhile, the topological approach in the orbit closure classification can be extended to the class of rigid acylindrical hyperbolic 3-manifolds, yielding the complete classification of orbit closures for the action of any connected closed subgroup generated by unipotent elements. This was done jointly with McMullen and Mohammadi ([36], [37]). Much of this article is devoted to explaining these results, although we present slightly different viewpoints in certain parts of the proof. Remarkably, this approach can handle the entire quasi-isometry class of rigid acylindrical hyperbolic 3-manifolds, as far as the action of the subgroup $\operatorname{PSL}_2(\mathbb{R})$ is concerned [38]. An immediate geometric consequence is that for any convex cocompact acylindrical hyperbolic 3-manifold M, any geodesic plane is either closed or dense inside the interior of the convex core of M; thereby producing the first continuous family of locally symmetric manifolds for which such a strong rigidity theorem for geodesic planes holds. This result extends to geometrically finite acylindrical hyperbolic 3-manifolds as shown in joint work with Benoist [4]. We also present a continuous family of quasifuchsian 3-manifolds containing geodesic planes with wild closures [38], which indicates the influence of the topology of the associated 3-manifold in the rigidity problem at hand.

We call a higher dimensional analogue of a rigid acylindrical hyperbolic 3-manifold a convex cocompact hyperbolic d-manifold with Fuchsian ends, following Kerckhoff and Storm [21]. For these manifolds $\Gamma\backslash\mathbb{H}^d$, in joint work with Lee [22], we have established a complete classification of orbit closures in $\Gamma\backslash\operatorname{SO}^\circ(d,1)$ for the action of any connected closed subgroup of $\operatorname{SO}^\circ(d,1)$ generated by unipotent elements. The possibility of accumulation on closed orbits of intermediate subgroups presents new challenges, and the avoidance theorem and the induction arguments involving equidistribution statement are major new ingredients in higher dimensional cases (Theorems 9.10 and 9.11). We note that these manifolds do not admit any non-trivial local deformations for $d \geq 4$ [21].

Acknowledgement This survey article is mostly based on the papers [36], [38], [37], [4], and [22]. I am grateful to my co-authors Curt McMullen, Amir Mohammadi, Yves Benoist and Minju Lee. I would like to thank Yair Minsky and Amir Mohammadi for helpful comments on the preliminary version of this article.

2. Kleinian groups

We give a brief introduction to Kleinian groups, including some basic notions and examples. General references for this section include [43], [26], [24], [33], [47], [17] and [8]. In particular, all theorems stated in this section with no references attached can be found in [26] and [33].

We will use the upper half-space model for hyperbolic 3-space:

$$\mathbb{H}^3 = \{(x_1, x_2, y) : y > 0\}, \quad ds = \frac{\sqrt{dx_1^2 + dx_2^2 + dy^2}}{y}.$$

In this model of $\mathbb{H}^3$, a geodesic is either a vertical line or a vertical semi-circle. The geometric boundary of $\mathbb{H}^3$ is given by the Riemann sphere $\mathbb{S}^2 = \hat{\mathbb{C}}$, when we identify the plane $(x_1, x_2, 0)$ with the complex plane $\mathbb{C}$.

The group $G := \mathrm{PSL}_2(\mathbb{C})$ acts on $\hat{\mathbb{C}}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \quad \text{with } a,b,c,d \in \mathbb{C} \text{ such that } ad-bc = 1.$$

This action of G extends to an isometric action on $\mathbb{H}^3$ as follows: each $g \in G$ can be expressed as a composition $\mathrm{Inv}_{C_1} \circ \cdots \circ \mathrm{Inv}_{C_k}$, where Inv_C denotes the inversion with respect to a circle $C \subset \hat{\mathbb{C}}$.[1] If we set $\Phi(g) = \mathrm{Inv}_{\hat{C}_1} \circ \cdots \circ \mathrm{Inv}_{\hat{C}_k}$ where $\mathrm{Inv}_{\hat{C}}$ denotes the inversion with respect to the sphere $\hat{C}$ in $\mathbb{R}^3$ which is orthogonal to $\mathbb{C}$ and $\hat{C} \cap \mathbb{C} = C$, then $\Phi(g)$ preserves $(\mathbb{H}^3, ds)$. Moreover, the Poincaré extension theorem says that Φ is an isomorphism between the two real Lie groups:

$$\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3),$$

where $\mathrm{PSL}_2(\mathbb{C})$ is regarded as a 6-dimensional real Lie group and $\mathrm{Isom}^+(\mathbb{H}^3)$ denotes the group of all orientation preserving isometries of $\mathbb{H}^3$.

Definition 2.1. *A discrete subgroup Γ of G is called a Kleinian group.*

For a (resp. torsion-free) Kleinian group Γ, the quotient $\Gamma\backslash\mathbb{H}^3$ is a hyperbolic orbifold (resp. manifold). Conversely, any complete hyperbolic 3-manifold M can be presented as a quotient

$$M = \Gamma\backslash\mathbb{H}^3$$

for a torsion-free Kleinian group Γ. The study of hyperbolic manifolds is therefore directly related to the study of Kleinian groups.

Throughout the remainder of the article, we assume that a Kleinian group Γ is non-elementary i.e., Γ does not contain an abelian subgroup of finite index. By Selberg's lemma, every Kleinian group has a torsion-free subgroup of finite index. We will henceforth treat the torsion-free condition loosely.

2.1. **Lattices.** The most well-studied Kleinian groups are lattices of G: a Kleinian group $\Gamma < G$ is a lattice if $M = \Gamma\backslash\mathbb{H}^3$ has finite volume. When M is compact, Γ is called a uniform or cocompact lattice. If $d > 0$ is a square-free integer, then $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-d}])$ is a non-uniform lattice of G. More lattices, including uniform ones, can be constructed by number theoretic methods using the Lie group isomorphism $G \simeq \mathrm{SO}^\circ(3,1)$.

Let $Q(x_1, x_2, x_3, x_4)$ be a quadratic form with coefficients over a totally real number field k of degree n such that Q has signature $(3,1)$ and for any non-trivial embedding $\sigma : k \to \mathbb{R}$, Q^σ has signature $(4,0)$ or $(0,4)$; the orthogonal group $\mathrm{SO}(Q^\sigma)$ is thus compact.

Then for $G = \mathrm{SO}^\circ(Q)$ and for the ring $\mathfrak{o}$ of integers of k, the subgroup

$$\Gamma := G \cap \mathrm{SL}_4(\mathfrak{o}) \tag{2.1}$$

is a lattice in G by a theorem of Borel and Harish-Chandra [6]. Moreover, if Q does not represent 0 over k (which is always the case if the degree of k is bigger than 1), then Γ is a uniform lattice in G by the Godement's criterion. These examples

[1] If $C = \{z : |z - z_0| = r\}$, then $\mathrm{Inv}_C(z)$ is the unique point on the ray $\{tz : t \geq 0\}$, satisfying the equation $|z - z_0| \cdot |\mathrm{Inv}_C(z) - z_0| = r^2$ for all $z \neq z_0$, and $\mathrm{Inv}_C(z_0) = \infty$.

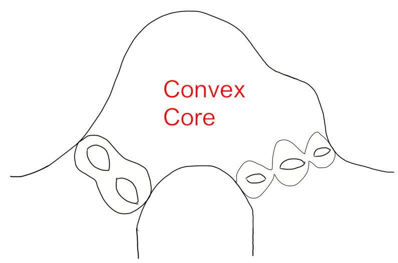

FIGURE 1. The convex core

contain all *arithmetic* lattices (up to a commensurability) which contain cocompact Fuchsian subgroups, that is, uniform lattices of $\mathrm{SO}^\circ(2,1) \simeq \mathrm{PSL}_2(\mathbb{R})$ [24].

Take two arithmetic non-commensurable hyperbolic 3-manifolds N_1 and N_2 which share a common properly imbedded closed geodesic surface S, up to an isometry. We cut each N_i along S, which results in one or two connected components. Let M_i be the metric completion of a component of $N_i - S$, which has geodesic boundary isometric to one or two copies of S. We now glue one or two copies of M_1 and M_2 together along their geodesic boundary and get a a connected finite-volume hyperbolic 3-manifold with no boundary. The resulting 3-manifold is a non-arithmetic hyperbolic 3-manifold, and its fundamental group is an example of the so-called hybrid lattices constructed by Gromov and Piatetski-Schapiro [16].

Mostow rigidity theorem says that any two isomorphic lattices of G are conjugate to each other. Since a lattice is finitely presented, it follows that a conjugacy class of a lattice is determined by its presentation. Hence, despite the presence of non-arithmetic lattices in G, there are only countably many lattices of G up to conjugation, or equivalently, there are only countably many hyperbolic manifolds of finite volume, up to isometry.

2.2. **Finitely generated Kleinian groups.** We will mostly focus on *finitely generated* Kleinian groups. When studying a finitely generated Kleinian group Γ, the associated limit set and the convex core play fundamental roles.

Using the Möbius transformation action of Γ on $\mathbb{S}^2$, we define:

Definition 2.2. *The limit set $\Lambda \subset \mathbb{S}^2$ of Γ is the set of all accumulation points of $\Gamma(z)$ for $z \in \mathbb{H}^3 \cup \mathbb{S}^2$.*

This definition is independent of the choice of $z \in \mathbb{H}^3 \cup \mathbb{S}^2$, and Λ is a minimal Γ-invariant closed subset of $\mathbb{S}^2$.

Definition 2.3. *The convex core of M is the convex submanifold of M given by*

$$\operatorname{core} M := \Gamma\backslash \operatorname{hull}\Lambda \ \subset \ M = \Gamma\backslash\mathbb{H}^3$$

where $\operatorname{hull}\Lambda \subset \mathbb{H}^3$ is the smallest convex subset containing all geodesics connecting two points in Λ.

If $\mathrm{Vol}(M) < \infty$, then $\Lambda = \mathbb{S}^2$ and hence M is equal to its convex core.

Definition 2.4. (1) *A Kleinian group Γ is called geometrically finite if the unit neighborhood of $\operatorname{core} M$ has finite volume.*

(2) *A Kleinian group* Γ *is called convex cocompact if* core M *is compact, or equivalently, if* Γ *is geometrically finite without any parabolic elements.*

An element $g \in G$ is either hyperbolic (if it is conjugate to a diagonal element whose entries have modulus not equal to 1), elliptic (if it is conjugate to a diagonal element whose entries have modulus 1) or parabolic (if it is conjugate to a strictly upper triangular matrix). By discreteness, an element of a torsion-free Kleinian group is either hyperbolic or parabolic.

Geometrically finite (resp. convex cocompact) Kleinian groups are natural generalization of (resp. cocompact) lattices of G. Moreover, the convex core of a geometrically finite hyperbolic manifold admits a thick-thin decomposition: there exists a constant $\varepsilon > 0$ such that core M is the union of a compact subset of injectivity radius at least $\varepsilon > 0$ and finitely many cusps. In the class of geometrically finite groups, lattices are characterized by the property that their limit sets are the whole of $\mathbb{S}^2$, and the limit sets of other geometrically finite groups have Hausdorff dimension strictly smaller than 2 ([52],[53]).

The group $G = \mathrm{PSL}_2(\mathbb{C})$ can be considered as a real algebraic subgroup, more precisely, the group of real points of an algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. A subset $S \subset G$ is called *Zariski dense* if S is not contained in any proper real algebraic subset of G. The Zariski density of a Kleinian group Γ in G is equivalent to the property that its limit set Λ is not contained in any circle of $\mathbb{S}^2$. When Λ is contained in a circle, Γ is conjugate to a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$; such Kleinian groups are referred to as *Fuchsian* groups. Geometrically finite Kleinian groups are always finitely generated, but the converse is not true in general; see (2.6).

2.3. **Examples of geometrically finite groups.** Below we give examples of three different kinds of geometrically finite groups which are relevant to subsequent discussion. Their limit sets are respectively totally disconnected, Jordan curves, and Sierpinski carpets. We note that a geometrically finite non-lattice Zariski dense Kleinian group Γ is determined by its limit set Λ up to commensurability, more precisely, Γ is a subgroup of finite index in the *discrete* subgroup $\mathrm{Stab}(\Lambda) = \{g \in G : g(\Lambda) = \Lambda\}$.

2.3.1. *Schottky groups.* The simplest examples of geometrically finite groups are Schottky groups. A subgroup $\Gamma < G$ is called (classical) Schottky if Γ is generated by hyperbolic elements $g_1, \cdots, g_k \in G$, $k \geq 2$, satisfying that there exist mutually disjoint closed round disks $B_1, \cdots, B_k$ and $B_1', \cdots, B_k'$ in $\mathbb{S}^2$ such that each g_i maps the exterior of B_i onto the interior of B_i'.

If $g_1, \cdots, g_k$ are hyperbolic elements of G whose fixed points in $\mathbb{S}^2$ are mutually disjoint, then $g_1^N, \cdots, g_k^N$ generate a Schottky group for all N large enough. A Schottky group Γ is discrete and free; the common exterior of the hemi-spheres bounded by B_i, B_i' is a fundamental domain F of Γ. Since the limit set of Γ, which is totally disconnected, is contained in the union of interiors of B_i and B_i's, it is easy to see that the intersection of the hull of Λ and the fundamental domain F is a bounded subset of F. Hence Γ is a convex cocompact subgroup. Its convex core is the handle body of genus k; in particular, the boundary of core M is a closed surface of genus k.

Any Kleinian group Γ contains a Schottky subgroup which has the same Zariski closure. If Γ is Zariski dense, take any two hyperbolic elements γ_1 and γ_2 of Γ with disjoint sets of fixed points. Suppose that all of four fixed points lie in a circle,

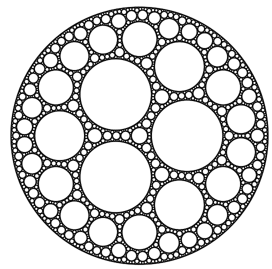

Figure 2. Limit set of a rigid acylindrical group, by McMullen

say, $C \subset \mathbb{S}^2$; note that C is uniquely determined. Since the set of fixed points of hyperbolic elements of Γ forms a dense subset of Λ, there exists a hyperbolic element $\gamma_3 \in \Gamma$ whose fixed points are not contained in C. Now, for any $N \geq 1$, the subgroup generated by $\gamma_1^N, \gamma_2^N, \gamma_3^N$ is Zariski dense, as its limit set cannot be contained in a circle. By taking N large enough, we get a Zariski dense Schottky subgroup of Γ. This in particular implies that any Kleinian group contains a convex cocompact subgroup, which is as large as itself in the algebraic sense.

2.3.2. *Fuchsian groups and deformations: quasifuchsian groups.* An orientation preserving homeomorphism $f : \mathbb{S}^2 \to \mathbb{S}^2$ is called κ-quasiconformal if for any $x \in \mathbb{S}^2$,

$$\limsup_{r\to 0} \frac{\sup\{|f(y) - f(x)| : |y - x| = r\}}{\inf\{|f(y) - f(x)| : |y - x| = r\}} \leq \kappa.$$

The 1-quasiconformal maps are precisely conformal maps [26, Sec.2]. The group $G = \mathrm{PSL}_2(\mathbb{C})$ is precisely the group of all conformal automorphisms of $\mathbb{S}^2$.

A Kleinian group Γ is called *quasifuchsian* if it is a quasiconformal deformation of a (Fuchsian) lattice of $\mathrm{PSL}_2(\mathbb{R})$, i.e., there exists a quasiconformal map f and a lattice $\Delta < \mathrm{PSL}_2(\mathbb{R})$ such that $\Gamma = \{f \circ \delta \circ f^{-1} : \delta \in \Delta\}$. Any quasi-conformal deformation of a geometrically finite group is known to be geometrically finite; so a quasifuchsian group is geometrically finite.

A quasifuchsian group is also characterized as a finitely generated Kleinian group whose limit set Λ is a Jordan curve and which preserves each component of $\mathbb{S}^2 - \Lambda$. If $\Omega_\pm$ are components of $\mathbb{S}^2 - \Lambda$, then $S_\pm := \Gamma\backslash\Omega_\pm$ admits a hyperbolic structure by the uniformization theorem, and the product $\mathrm{Teich}(S_+) \times \mathrm{Teich}(S_-)$ of Teichmuller spaces gives a parameterization of all quasifuchsian groups which are quasiconformal deformations of a fixed lattice of $\mathrm{PSL}_2(\mathbb{R})$.

2.3.3. *Rigid acylindrical groups and their deformations.* A Kleinian group $\Gamma < G$ is called *rigid acylindrical* if the convex core of the associated hyperbolic manifold $M = \Gamma\backslash\mathbb{H}^3$ is a compact manifold with non-empty interior and with totally geodesic boundary. If $\operatorname{core} M$ has empty boundary, then M is compact and hence Γ is a

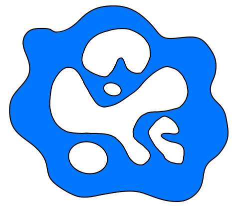

FIGURE 3. Sierpinski carpet

uniform lattice. Rigid acylindrical non-lattice groups are characterized as convex cocompact Kleinian groups whose limit set satisfies that

$$\mathbb{S}^2 - \Lambda = \bigcup B_i$$

where B_i's are round disks with mutually disjoint closures.

If M is a rigid acylindrical hyperbolic 3-manifold of infinite volume then the double of $\operatorname{core} M$ is a closed hyperbolic 3-manifold; hence any rigid acylindrical group is a subgroup of a uniform lattice of G, which contains a co-compact Fuchsian lattice $\pi_1(S)$ for a component S of $\partial \operatorname{core} M$. Conversely, if Γ_0 is a torsion-free uniform lattice of G such that $\Delta := \Gamma_0 \cap \mathrm{PSL}_2(\mathbb{R})$ is a uniform lattice in $\mathrm{PSL}_2(\mathbb{R})$, then $M_0 = \Gamma_0 \backslash \mathbb{H}^3$ is a closed hyperbolic 3-manifold which contains a properly immersed totally geodesic surface $\Delta\backslash\mathbb{H}^2$. By passing to a finite cover of M_0, M_0 contains a properly embedded totally geodesic surface, say S [24, Theorem 5.3.4]. Now the metric completion of a component of $M_0 - S$ is a compact hyperbolic 3-manifold with totally geodesic boundary, and its fundamental group, which injects to $\Gamma_0 = \pi_1(M_0)$, is a rigid acylindrical Kleinian group.

Rigid acylindrical Kleinian groups admit a huge deformation space, comprised of convex cocompact acylindrical groups. We begin with the notion of acylindricality for a compact 3-manifold. Let D^2 denote a closed 2-disk and let $C^2 = S^1 \times [0, 1]$ be a cylinder. A compact 3-manifold N is called *acylindrical*

(1) if ∂N is incompressible, i.e., any continuous map $f : (D^2, \partial D^2) \to (N, \partial N)$ can be deformed into ∂N or equivalently if the inclusion $\pi_1(S) \to \pi_1(N)$ is injective for any component S of ∂N; and
(2) if any essential cylinder of N is boundary parallel, i.e., any continuous map $f : (C^2, \partial C^2) \to (N, \partial N)$, injective on π_1, can be deformed into ∂N.

A convex cocompact hyperbolic 3-manifold M is called *acylindrical* if its convex core is acylindrical. When M has infinite volume, it is also described by the property that its limit set is a Sierpinski carpet: $\mathbb{S}^2 - \Lambda = \bigcup B_i$ is a dense union of Jordan disks B_i's with mutually disjoint closures and with $\operatorname{diam}(B_i) \to 0$. By Whyburn [55], all Sierpinski carpets are known to be homeomorphic to each other. We refer to a recent preprint [57] for a beautiful picture of the limit set of a convex cocompact (non-rigid) acylindrical group.

Any convex cocompact acylindrical Kleinian group Γ is a quasi-conformal deformation of a unique rigid acylindrical Kleinian group Γ_0, and its quasi-conformal class is parametrized by the product $\prod_i \operatorname{Teich}(S_i)$ where S_i's are components of

$\partial\,\mathrm{core}(\Gamma_0\backslash\mathbb{H}^3)$ ([54] and [35]). In terms of a manifold, any convex cocompact acylindrical hyperbolic 3-manifold is quasi-isometric to a unique *rigid* acylindrical hyperbolic 3-manifold M, and its quasi-isometry class is parametrized by $\prod_i \mathrm{Teich}(S_i)$.

The definition of acylindricality can be extended to geometrically finite groups with cusps using the notion of a compact core. If M is a hyperbolic 3-manifold with finitely generated $\pi_1(M)$, then there exists a a compact connected submanifold $\mathcal{C} \subset M$ (with boundary) such that the inclusion $\mathcal{C} \subset M$ induces an isomorphism $\pi_1(\mathcal{C}) \simeq \pi_1(M)$; such $\mathcal{C}$ exists uniquely, up to homeomorphism, and is called the compact core of M. Now a geometrically finite hyperbolic 3-manifold M is called acylindrical if its compact core is an acylindrical compact 3-manifold.

2.4. **Thurston's geometrization theorem.** The complement $\Omega := \mathbb{S}^2 - \Lambda$ is called the set of discontinuity. Let Γ be a finitely generated Kleinian group. Ahlfors finiteness theorem says that $\Gamma\backslash\Omega$ is a union of finitely many closed Riemann surfaces with at most a finite number of punctures. The Kleinian manifold associated to Γ is defined by *adding* $\Gamma\backslash\Omega$ to $\Gamma\backslash\mathbb{H}^3$ on the conformal boundary at infinity:

$$\mathcal{M}(\Gamma) = \Gamma\backslash\mathbb{H}^3 \cup \Omega, \quad \partial\mathcal{M}(\Gamma) = \Gamma\backslash\Omega.$$

The convex cocompactness of Γ is equivalent to the compactness of $\mathcal{M}(\Gamma)$. If Γ is geometrically finite with cusps, then $\mathcal{M}(\Gamma)$ is compact except possibly for a finite number of rank one and rank two cusps. We denote by $\mathcal{M}_0(\Gamma)$ the compact submanifold of $\mathcal{M}(\Gamma)$ obtained by removing the interiors of solid pairing tubes corresponding to rank one cusps and solid cusp tori corresponding to rank two cusps (cf. [26]).

The following is a special case of Thurston's geometrization theorem under the extra non-empty boundary condition (cf. [20]):

Theorem 2.5. *Let N be a compact irreducible*[2] *orientable atoroidal*[3] *3-manifold with non-empty boundary. Then N is homeomorphic to $\mathcal{M}_0(\Gamma)$ for some geometrically finite Kleinian group Γ.*

We remark that if ∂N is incompressible and N does not have any essential cylinders, then Γ is a geometrically finite acylindrical group.

By applying Thurston's theorem to the compact core of $\Gamma\backslash\mathbb{H}^3$, we deduce that every finitely generated Kleinian group Γ is isomorphic to a geometrically finite group.

2.5. **Density of geometrically finite groups.** The density conjecture of Bers, Sullivan and Thurston says that most of Kleinian groups are geometrically finite. This is now a theorem whose proof combines the work of many authors with the proof in full generality due to Namazi-Souto and Ohshika (we refer to [26, Sec. 5.9] for more details and background).

Theorem 2.6 (Density theorem)**.** *The class of geometrically finite Kleinian groups is open and dense in the space of all finitely generated Kleinian groups.*

In order to explain the topology used in the above theorem, let Γ be a finitely generated Kleinian group. By Thurston's geometrization theorem, there exists a geometrically finite Kleinian group Γ_0 and an isomorphism $\rho : \Gamma_0 \to \Gamma$. In fact, a

[2]every 2-sphere bounds a ball

[3]any $\mathbb{Z}^2$ subgroup comes from boundary tori

more refined version gives that ρ is type-preserving, i.e., ρ maps a parabolic element to a parabolic element. Fix a finite generating set $\gamma_1, \cdots, \gamma_k$ of Γ_0. The density theorem says there exists a sequence of geometrically finite groups $\Gamma_n < G$, and isomorphisms $\rho_n : \Gamma_0 \to \Gamma_n$ such that ρ_n converges to ρ as $n \to \infty$, in the sense that $\rho(\gamma_i) = \lim_n \rho_n(\gamma_i)$ for each $i = 1, \cdots, k$.

Here is an alternative way to describe the density theorem: Fix a geometrically finite Kleinian group Γ with a fixed set of generators $\gamma_1, \cdots, \gamma_k$ and relations $\omega_1, \cdots, \omega_r$. Define

$$\mathfrak{R}(\Gamma) := \{\rho : \Gamma \to G \text{ homomorphism}\}/\sim$$

with the equivalence relation given by conjugation by elements of G. The set $\mathfrak{R}(\Gamma)$ can be identified with the algebraic variety $\{(g_1, \cdots, g_k) \in G \times \cdots \times G : \omega_i(g_1, \cdots, g_k) = e \quad \text{for } 1 \le i \le r\}/\sim$ where $\sim$ is given by conjugation by an element of G under the diagonal embedding. This defines a topology on $\mathfrak{R}(\Gamma)$, called the algebraic convergence topology.

The discrete locus is then defined by the subcollection of discrete and faithful representations:

$$AH(\Gamma) := \{\rho \in \mathfrak{R}(\Gamma) : \text{ type preserving isomorphism to a Kleinian group}\}.$$

Then $AH(\Gamma)$ is a closed subset, which parametrizes hyperbolic structures on $\Gamma\backslash\mathbb{H}^3$. The interior of $AH(\Gamma)$ consists of geometrically finite Kleinian groups, and the density theorem says that

$$\overline{\mathrm{Int} AH(\Gamma)} = AH(\Gamma).$$

When Γ is a lattice in G, $AH(\Gamma)$ is a single point by Mostow rigidity theorem. For all other geometrically finite Kleinian groups, $AH(\Gamma)$ is *huge*; the quasiconformal deformation space of Γ given by

$$\mathfrak{T}(\Gamma) = \{\rho \in AH(\Gamma) : \rho \text{ is induced by a quasiconformal deformation of } \Gamma\}$$

is a connected component of the interior of $AH(\Gamma)$ and is a complex analytic manifold of dimension same as the dimension of $\mathrm{Teich}(\Gamma\backslash\Omega)$, i.e., $\sum_{i=1}^m (3g_i + n_i - 3)$ where g_i is the genus of the i-th component of $\Gamma\backslash\Omega = \partial\mathcal{M}(\Gamma)$ and n_i is the number of its punctures [26, Thm. 5.13]. Moreover when Γ is rigid acylindrical, the interior of $AH(\Gamma)$, modulo the orientation (in other words, modulo the conjugation by elements of $\mathrm{Isom}(\mathbb{H}^3)$, rather than by elements of $G = \mathrm{Isom}^+(\mathbb{H}^3)$), is connected, and hence is equal to $\mathfrak{T}(\Gamma)$; this can be deduced from [9], as explained to us by Y. Minsky. Therefore $\mathrm{Int}\, AH(\Gamma)/\pm = \mathfrak{T}(\Gamma) = \mathrm{Teich}(\Gamma\backslash\Omega)$.

2.6. **Examples of geometrically infinite groups.** Not every finitely generated Kleinian group is geometrically finite. An important class of finitely generated geometrically infinite Kleinian groups is given by the fundamental groups of $\mathbb{Z}$-covers of closed hyperbolic 3-manifolds. The virtual fibering theorem, proved by Agol, building upon the previous work of Wise, says that every closed hyperbolic 3-manifold is a surface bundle over a circle, after passing to a finite cover [26, Sec 6.4]. This implies that, up to passing to a subgroup of finite index, any uniform lattice Γ_0 of G contains a normal subgroup Δ such that $\Gamma_0/\Delta \simeq \mathbb{Z}$ and Δ is a surface subgroup, i.e., isomorphic to the fundamental group of a closed hyperbolic surface. Note that Δ is finitely generated (being a surface subgroup) but geometrically infinite; as no normal subgroup of a geometrically finite group of infinite index is geometrically finite. In fact, any finitely generated, geometrically infinite, subgroup

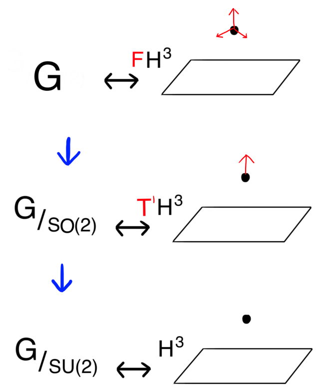

FIGURE 4. Frame bundle of $\mathbb{H}^3$

of a uniform lattice of G arises in this way, up passing to a subgroup of finite index (cf. [8]). These manifolds give examples of degenerate hyperbolic 3-manifolds with $\Lambda = \mathbb{S}^2$. We mention that there are also degenerate hyperbolic manifolds with $\Lambda \neq \mathbb{S}^2$.

3. Mixing and classification of N-orbit closures

Let $\Gamma < G = \mathrm{PSL}_2(\mathbb{C})$ be a Zariski dense geometrically finite Kleinian group, and $M := \Gamma\backslash\mathbb{H}^3$ the associated hyperbolic 3-manifold. We denote by

$$\pi : \mathbb{H}^3 \to M = \Gamma\backslash\mathbb{H}^3$$

the quotient map.

We fix $o \in \mathbb{H}^3$ and a unit tangent vector $v_o \in \mathrm{T}_o(\mathbb{H}^3)$ so that $K = \mathrm{SU}(2)$ and $M_0 = \{\mathrm{diag}(e^{i\theta}, e^{-i\theta}) : \theta \in \mathbb{R}\}$ are respectively the stabilizer subgroups of o and v_o. The action of G on $\mathbb{H}^3$ induces identifications $G/K \simeq \mathbb{H}^3$, $G/M_0 \simeq \mathrm{T}^1(\mathbb{H}^3)$, and $G \simeq \mathrm{F}(\mathbb{H}^3)$, where $\mathrm{T}^1(\mathbb{H}^3)$ and $\mathrm{F}(\mathbb{H}^3)$ denote respectively the unit tangent bundle and the oriented frame bundle over $\mathbb{H}^3$.

Thus we may understand the oriented frame bundle $\mathrm{F}\,M$ as the homogeneous space $\Gamma\backslash G$. Denote by

$$p : \Gamma\backslash G \to M$$

the base-point projection map.

Unless Γ is a lattice, the G-invariant measure on $\Gamma\backslash G$ is infinite, and dissipative for natural geometric flows such as the geodesic flow and horospherical flow. Two locally finite measures on $\Gamma\backslash G$, called the Bowen-Margulis-Sullivan measure, and the Burger-Roblin measure, play important roles, and they are defined using the Patterson-Sullivan density on the limit set of Γ.

3.1. **Patterson-Sullivan density.** We denote by δ the critical exponent of Γ, i.e., the infimum over all $s \geq 0$ such that the Poincare series $\sum_{\gamma\in\Gamma} e^{-sd(o,\gamma(o))}$ converges. As Γ is geometrically finite, δ is equal to the Hausdorff dimension of Λ [52].

Bishop and Jones proved that δ is strictly bigger than 1, unless Λ is totally disconnected or contained in a circle [5]. As Γ is assumed to be Zariski dense, we have:

Theorem 3.1. *If Λ is connected, then $\delta > 1$.*

Recall that for $x, y \in \mathbb{H}^3$ and $\xi \in \mathbb{S}^2$, the Busemann function $\beta_\xi(x, y)$ is given by $\lim_{t\to\infty} d(x, \xi_t) - d(y, \xi_t)$ where ξ_t is a geodesic ray toward ξ.

Definition 3.2. A Γ-invariant conformal density of dimension $s \geq 0$ is a family $\{\mu_x : x \in \mathbb{H}^3\}$ of finite measures on $\mathbb{S}^2$ satisfying:

(1) For any $\gamma \in \Gamma$ and $x \in \mathbb{H}^3$, $\gamma_* \mu_x = \mu_{\gamma(x)}$;
(2) For all $x, y \in \mathbb{H}^3$ and $\xi \in \mathbb{S}^2$, $\frac{d\mu_x}{d\mu_y}(\xi) = e^{s\beta_\xi(y,x)}$.

Theorem 3.3 (Patterson-Sullivan)**.** *There exists a Γ-invariant conformal density $\{\nu_x : x \in \mathbb{H}^3\}$ of dimension δ, unique up to a scalar multiple.*

We call this Patterson-Sullivan density. Denoting by Δ the hyperbolic Laplacian on $\mathbb{H}^3$, the Patterson-Sullivan density is closely related to the bottom of the spectrum of Δ for its action on smooth functions on $\Gamma\backslash\mathbb{H}^3$. The function ϕ_0 defined by

$$\phi_0(x) := |\nu_x|$$

for each $x \in \mathbb{H}^3$ is Γ-invariant, and hence we may regard ϕ_0 as a function on the manifold $\Gamma\backslash\mathbb{H}^3$. It is the unique function (up to a constant multiple) satisfying $\Delta\phi_0 = \delta(2 - \delta)\phi_0$; so we call ϕ_0 the base eigenfunction.

Set $\nu := \nu_o$ and call it the Patterson-Sullivan measure (viewed from o). When Γ is convex-cocompact, the Patterson-Sullivan measure ν_o is simply proportional to the δ-dimensional Hausdorff measure on Λ in the spherical metric of $\mathbb{S}^2$.

3.2. **Mixing of the BMS measure.** Consider the following one-parameter subgroup of G:

$$A := \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}.$$

The right translation action of A on $\mathrm{F}\,\mathbb{H}^3 = G$ induces the frame flow: if $g = (e_1, e_2, e_3)$, then ga_t for $t > 0$ is the frame given by translation in direction of e_1 by hyperbolic distance t. Let $v_o^\pm \in \mathbb{S}^2$ denote the forward and backward end points of the geodesic given by v_o respectively. In the upper half space model of $\mathbb{H}^3$, choosing v_o to be the upward normal vector at $o = (0, 0, 1)$, we have $v_o^+ = \infty$ and $v_o^- = 0$.

For $g \in G$, we define

$$g^+ = g(v_o^+) \in \mathbb{S}^2 \quad \text{and} \quad g^- = g(v_o^-) \in \mathbb{S}^2.$$

The map $g \mapsto (g^+, g^-, s = \beta_{g^-}(o, g))$ induces a homeomorphism between $\mathrm{T}^1(\mathbb{H}^3)$ and $(\mathbb{S}^2 \times \mathbb{S}^2 - \text{diagonal}) \times \mathbb{R}$; called the Hopf-parametrization.

We define a locally finite measure $\tilde{m}^{\mathrm{BMS}}$ on $\mathrm{T}^1(\mathbb{H}^3) = G/M_0$ as follows:

$$d\tilde{m}^{\mathrm{BMS}}(g) = e^{\delta\beta_{g^+}(o,g)}\, e^{\delta\beta_{g^-}(o,g)}\, d\nu(g^+) d\nu(g^-) ds$$

where ds is the Lebesgue measure on $\mathbb{R}$.

Denote by m^{BMS} the unique M_0-invariant measure on $\Gamma\backslash G$ which is induced by $\tilde{m}^{\mathrm{BMS}}$; we call this the Bowen-Margulis-Sullivan measure (or the BMS measure for short).

Sullivan showed that m^{BMS} is a *finite* A-invariant measure. The following is due to Babillot [2] for M_0-invariant functions and to Winter [56] for general functions:

Theorem 3.4. *The frame flow on* $(\Gamma\backslash G, m^{\mathrm{BMS}})$ *is mixing, that is, for any* $\psi_1, \psi_2 \in L^2(\Gamma\backslash G, m^{\mathrm{BMS}})$,

$$\lim_{t\to\infty}\int_{\Gamma\backslash G}\psi_1(ga_t)\psi_2(g)\, dm^{\mathrm{BMS}}(g) = \frac{1}{|m^{\mathrm{BMS}}|} m^{\mathrm{BMS}}(\psi_1)\cdot m^{\mathrm{BMS}}(\psi_2).$$

We define the renormalized frame bundle of M as:

$$\mathrm{RF}\, M = \{[g] \in \Gamma\backslash G : g^{\pm} \in \Lambda\}.$$

This is a closed A-invariant subset of $\Gamma\backslash G$ which is precisely the support of m^{BMS}, and an immediate consequence of Theorem 3.4 is the topological mixing of the A-action on $\mathrm{RF}\, M$: for any two open subsets $\mathcal{O}_1, \mathcal{O}_2$ intersecting $\mathrm{RF}\, M$, $\mathcal{O}_1 a_t \cap \mathcal{O}_2 \neq \emptyset$ for all sufficiently large $|t|$.

3.3. **Essential unique-ergodicity of the BR measure.** We denote by $N := \{g \in G : a_{-t} g a_t \to e \text{ as } t \to +\infty\}$ the contracting horospherical subgroup for the action of A, which is explicitly given as

$$N = \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C} \right\}.$$

The projection $\pi(gN)$ in $\mathbb{H}^3$ is a Euclidean sphere tangent to $\mathbb{S}^2$ at g^+ and gN consists of frames (e_1, e_2, e_3) whose last two vectors e_2, e_3 are tangent to $\pi(gN)$. That N is a contracting horospherical subgroup means geometrically that $\pi(gNa_t)$ for $t > 0$ is a Euclidean sphere based at g^+ but shrunk toward g^+ by the hyperbolic distance t.

We define $\tilde{m}^{\mathrm{BR}}$ on $G/M_0 = \mathrm{T}^1(\mathbb{H}^3)$ as follows:

$$d\tilde{m}^{\mathrm{BR}}(g) = e^{\delta\beta_{g^+}(o,g)}\, e^{2\beta_{g^-}(o,g)}\, d\nu(g^+) dg^- ds$$

where dg^- is the Lebesgue measures on $\mathbb{S}^2$. We denote by m^{BR} the unique M_0-invariant measure on $\Gamma\backslash G$ which is induced by $\tilde{m}^{\mathrm{BR}}$. We call this measure the Burger-Roblin measure (or the BR measure for short). If Γ is a lattice, m^{BR} is simply the G-invariant measure. Otherwise m^{BR} is an infinite, but locally finite, Borel N-invariant measure whose support is given by

$$\mathrm{RF}_+ M := \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\} = \mathrm{RF}\, M \cdot N.$$

The projection of the BR measure to M is an absolutely continuous measure on M with Radon-Nikodym derivative given by ϕ_o: if $f \in C_c(\Gamma\backslash G)$ is K-invariant, then

$$m^{\mathrm{BR}}(f) = \int_{\Gamma\backslash G} f(x)\phi_o(x)\, dx$$

where dx is a G-invariant measure on $\Gamma\backslash G$. Using Theorem 3.4, Roblin and Winter showed the following measure classification of N-invariant locally finite measures, extending an earlier work of Burger [7]:

Theorem 3.5. ([46], [56]) *Any locally finite N-ergodic invariant measure on $\Gamma\backslash G$ is either supported on a closed N-orbit or proportional to m^{BR}.*

3.4. **Closures of N-orbits.** If $x \notin \mathrm{RF}_+ M$, then xN is a proper immersion of N to $\Gamma\backslash G$ via the map $n \mapsto xn$, and hence xN is closed. In understanding the topological behavior of xN for $x = [g] \in \mathrm{RF}_+ M$, the relative location of g^+ in the limit set becomes relevant. The hypothesis that Γ is geometrically finite implies that any $\xi \in \Lambda$ is either radial (any geodesic ray $\xi_t \in M$ converging to ξ accumulates on a compact subset) or parabolic (it is fixed by some parabolic element of Γ). Since this property is Γ-invariant, we will say that x^+ is radial (resp. parabolic) if g^+ is for $x = [g]$. When Γ is convex cocompact, Λ consists only of radial limit points.

The topological mixing of the A-action on $\mathrm{RF}\, M$ implies the following dichotomy for the closure of an N-orbit:

Theorem 3.6. *([15], [56]) For $x \in \mathrm{RF}_+ M$, xN is closed (if x^+ is parabolic) or dense in $\mathrm{RF}_+ M$ (if x^+ is radial).*

4. Almost all results on orbit closures

Let $\Gamma < G = \mathrm{PSL}_2(\mathbb{C})$ be a Zariski dense geometrically finite Kleinian group, and $M := \Gamma\backslash\mathbb{H}^3$ the associated hyperbolic 3-manifold.

We are mainly interested in the action of the following two subgroups on $\Gamma\backslash G$:

$$H := \mathrm{PSL}_2(\mathbb{R}) \text{ and}$$

$$U := \{u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\}. \tag{4.1}$$

Any one-parameter unipotent subgroup of G is conjugate to U, and any connected closed subgroup of G generated by unipotent one-parameter subgroups is conjugate to either N, H or U. We also note that the subgroups N, H and U are normalized by the subgroup A, which is an important point for the following discussion, as the measures m^{BMS} and m^{BR} are invariant and quasi-invariant under A respectively.

The first question is whether there exist almost all results for the closures of these orbits for appropriate measures.

We recall:

Theorem 4.1 (Moore's ergodicity theorem)**.** *Let $\Gamma < G$ be a lattice. For any unbounded subgroup W of G, xW is dense in $\Gamma\backslash G$ for almost all $x \in \Gamma\backslash G$.*

When Γ is geometrically finite but not a lattice in G, no orbit of a proper connected subgroup W is dense in $\Gamma\backslash G$. Moreover, it is easy to verify that if $\partial(gW) \subset \mathbb{S}^2$ does not intersect Λ, then the map $W \to [g]W \subset \Gamma\backslash G$ given by $w \mapsto [g]w$ is a proper map, and hence $[g]W$ is closed [4].

Hence if W has the property that $\partial(gW) = (gW)^+$, for instance, if $W = H$ or U, then the non-trivial dynamics of the action of W on $\Gamma\backslash G$ exists only inside the closure of $\mathrm{RF}_+ M \cdot W$.

We will see that $\mathrm{RF}_+ M \cdot H$ is always closed; it is useful to understand the geometric description of $\mathrm{RF}_+ M \cdot H$ in order to understand its closedness.

4.1. **Geodesic planes and almost all H-orbits.** A geodesic plane in $\mathbb{H}^3$ is a totally geodesic imbedding of $\mathbb{H}^2$, which is simply either a vertical plane or a vertical hemisphere in the upper half space model.

[4]For a subset $S \subset G$, we use the notation ∂S to denote $\overline{\pi(S)} \cap \mathbb{S}^2$ under the projection $\pi : \mathrm{F}\,\mathbb{H}^3 \to \mathbb{H}^3 \cup \mathbb{S}^2$.

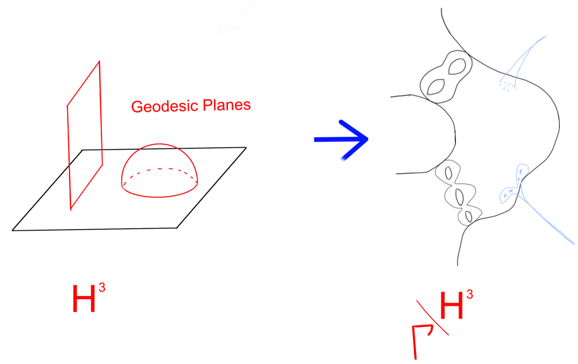

FIGURE 5. Geodesic planes in M

Let $\mathcal{P}$ denote the set of all oriented geodesic planes of $\mathbb{H}^3$, and $\mathcal{C}$ the set of all oriented circles in $\mathbb{S}^2$. The map $P \mapsto \partial P$ gives an isomorphism between $\mathcal{P}$ and $\mathcal{C}$.

On the other hand, the map

$$gH \mapsto P_g := \pi(gH)$$

gives an isomorphism between the quotient space G/H and the set $\mathcal{P}$, whose inverse can be described as follows: for $P \in \mathcal{P}$, the set of frames (e_1, e_2, e_3) based in P such that e_1 and e_2 are tangent to P and e_3 is given by the orientation of P is precisely a single H-orbit. Consequently, the map

$$gH \to C_g := \partial P_g$$

gives an isomorphism between G/H and $\mathcal{C}$.

Definition 4.2. *An oriented geodesic plane $P \subset M$ is a totally geodesic immersion of an oriented hyperbolic plane $\mathbb{H}^2$ in M, or equivalently, P is the image of an oriented geodesic plane of $\mathbb{H}^3$ under π.*

In this paper, geodesic planes and circles are always considered to be oriented. Note that any geodesic plane $P \subset M$ is of the form:

$$P = p(gH) \quad \text{for some } g \in G.$$

Therefore the study of H-orbits on $\Gamma\backslash G$ has a direct implication on the behavior of geodesic planes in the manifold $\Gamma\backslash\mathbb{H}^3$.

We set

$$F_\Lambda := \mathrm{RF}_+ M \cdot H \quad \text{and} \quad \mathcal{C}_\Lambda := \{C \in \mathcal{C} : C \cap \Lambda \neq \emptyset\}. \tag{4.2}$$

It follows from the compactness of Λ that $\mathcal{C}_\Lambda$ is a closed subset of $\mathcal{C} = G/H$. As

$$F_\Lambda/H = \Gamma\backslash\mathcal{C}_\Lambda,$$

we deduce:

Lemma 4.3. *The set F_Λ is a closed H-invariant subset of $\Gamma\backslash G$.*

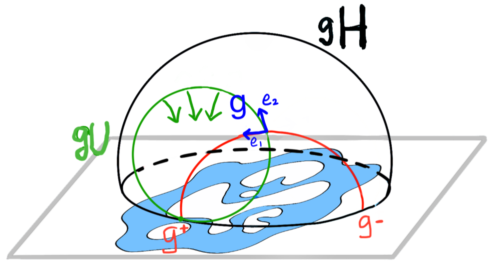

FIGURE 6. Orbits under A, U and H

Proposition 4.4. *For m^{BMS}-a.e. $x \in \mathrm{RF}\, M$,*

$$\overline{xH} = F_\Lambda;$$

in particular, the geodesic plane $p(xH)$ is dense in M.

Proof. We have $\overline{\mathrm{RF}\, M \cdot U} = \mathrm{RF}_+ M$ [34] and hence $\overline{\mathrm{RF}\, M \cdot H} = F_\Lambda$. Theorem 3.4 implies that m^{BMS} is ergodic, and hence by the Birkhoff ergodic theorem, for almost all x, xA is dense in $\mathrm{RF}\, M$. Since $A \subset H$, we deduce

$$\overline{xH} \supset \overline{\mathrm{RF}\, M} \cdot H = F_\Lambda.$$

$\square$

4.2. **Horocycles and almost all U-orbits.** A horocycle in $\mathbb{H}^3$ is a Euclidean circle tangent to $\mathbb{S}^2$.

Definition 4.5. *A horocycle χ in M is an isometrically immersed copy of $\mathbb{R}$ with zero torsion and geodesic curvature 1, or equivalently, χ is the image of a horocycle of $\mathbb{H}^3$ under π.*

The right translation action of U on $\Gamma\backslash G$ is the horocyclic action: if $g = (e_1, e_2, e_3)$, then gu_t for $t > 0$ is the frame given by translation in the direction of e_2 by Euclidean distance t. In fact, any horocycle $\chi \subset M$ is of the form

$$\chi = p(gU) \quad \text{for some } g \in G.$$

Note that both gA and gU have their trajectories inside the plane $P_g = \pi(gH)$. In particular, $\pi(gU)$ is a Euclidean circle lying on P_g tangent to $\mathbb{S}^2$ at g^+.

We now discuss the almost all results for U-orbits in terms of the Burger-Roblin measure. It turns out that the size of the critical exponent δ matters in this question. The following was proved in joint work with Mohammadi for Γ convex cocompact [39] and by Maucourant and Schapira [34] for geometrically finite groups.

Theorem 4.6. *If $\delta > 1$, m^{BR} is U-ergodic and conservative.*

Proposition 4.7. *Let $\delta > 1$ (e.g., Λ is connected). Then for m^{BR}-a.e. $x \in \mathrm{RF}_+ M$,*

$$\overline{xU} = \mathrm{RF}_+ M;$$

In particular, the horocycle $p(xU)$ is dense in M.

Proof. Since m^{BR} is an infinite measure, unless Γ is a lattice, the Birkhoff ergodic theorem does not apply. Instead we use the Hopf ratio theorem which applies by Theorem 4.6, and hence the claim follows. □

In [34], it was proved that if $\delta < 1$, m^{BR} is totally U-dissipative and hence almost all U-orbits are divergent (cf. [14]). Whether m^{BR} is ergodic or not at $\delta = 1$ remains an open question.

4.3. **Orbit closure theorem for lattices.** The almost all results on orbit closures in Propositions 4.4 and 4.7 do not describe the topological behavior of a given individual orbit. In the lattice case, we have the following remarkable classification of all possible orbit closures, due to Ratner [45] and Shah [48] independently:

Theorem 4.8. *Let $\Gamma < G$ be a lattice, and $x \in \Gamma\backslash G$.*

(1) *The closure $\overline{xH}$ is either xH or $\Gamma\backslash G$.*
(2) *The closure $\overline{xU}$ is either xU, $xv^{-1}Hv$, for some $v \in N$, or $\Gamma\backslash G$.*

This theorem immediately implies the first part of the following theorem; the rest follows from the results in the same paper loc.cit.

Theorem 4.9. *If M has finite volume, the closures of a geodesic plane and a horocycle are properly immersed submanifolds of M. Moreover,*

(1) *any properly immersed geodesic plane has finite area;*
(2) *there are at most countably many properly immersed geodesic planes in M;*
(3) *any infinite sequence of properly immersed geodesic planes P_i becomes dense in M, i.e, $\lim_{i\to\infty} P_i = M$* [5].

The density statement (3) above, which is a topological version of Mozes-Shah theorem [41], implies that every properly immersed geodesic plane P is topologically isolated, in the sense that there exists an open neighborhood of P which does not contain any other properly immersed geodesic plane in its entirety.

4.4. **Topological obstructions to orbit closure theorem.** In this section, we describe a family of quasi-fuchsian manifolds some of whose geodesic planes have fractal closures, in particular, they have non-integral dimensions. These geodesic planes pass through the interior of the convex core of M but their boundaries meet the limit set Λ only at two points.

These examples can be seen easily for Fuchsian manifolds, and by performing a small bending deformation along a simple closed geodesic far away from our fractal closures of a fixed plane, we will obtain quasi-fuchsian manifolds keeping the fractal closure intact.

4.4.1. *Fuchsian* 3*-manifolds.* Consider a Fuchsian 3-manifold M which can be expressed as

$$M = S \times \mathbb{R}$$

in cylindrical coordinates where S is a closed hyperbolic surface of genus at least 2. Or equivalently, take a torsion-free uniform lattice $\Gamma < \mathrm{PSL}_2(\mathbb{R})$, and consider Γ as a subgroup of G, so that $M = \Gamma\backslash\mathbb{H}^3 = \left(\Gamma\backslash\mathbb{H}^2\right) \times \mathbb{R}$. We have $\operatorname{core} M = S$.

[5] For a sequence of closed subsets Y_i of a topological space X, we write $\lim_{i\to\infty} Y_i = Y$ if $\limsup_{i\to\infty} Y_i = \liminf_{i\to\infty} Y_i = Y$

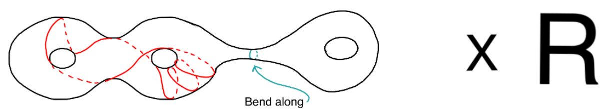

FIGURE 7. Bending deformation

It is well-known that geodesics on a closed hyperbolic surface S can behave as wild as we wish for, in particular, for any $\beta \geq 1$, there exists a geodesic whose closure has Hausdorff dimension precisely β.

(1) The closure of a geodesic plane needs not be a submanifold: if $\gamma \subset S$ is a geodesic and P is a geodesic plane orthogonal to S with $P \cap S = \gamma$, then
$$\overline{P} \simeq \overline{\gamma} \times \mathbb{R}.$$
Therefore if we take a geodesic $\gamma \subset S$ whose closure $\overline{\gamma}$ is wild, then $\overline{P}$ is very far from being a submanifold.

(2) There are uncountably many properly immersed geodesic planes intersecting $\operatorname{core} M$; if $\gamma \subset S$ is a closed geodesic and P is a geodesic plane with $P \cap S = \gamma$, then P is a properly immersed geodesic plane. By varying angles between P and S, we obtain a continuous family of such P.

We can now use a small bending deformation of M to obtain quasifuchsian manifolds in which the same phenomenon persists.

4.4.2. *Quasifuchsian hyperbolic* 3*-manifolds.* Let $\gamma_0 \in \Gamma$ be a primitive hyperbolic element representing a separating simple closed geodesic β in S. Without loss of generality, we assume $\gamma_0 \in A$, up to conjugation. If S_1 and S_2 are components of $S - \beta$, then each $\Gamma_i := \pi_1(S_i)$ is a subgroup of Γ and Γ can be presented as the amalgamated free product
$$\Gamma = \Gamma_1 *_{\langle \gamma_0 \rangle} \Gamma_2.$$
Setting $m_\theta = \operatorname{diag}(e^{i\theta}, e^{-i\theta})$, note that m_θ centralizes γ_0. For each non-trivial m_θ, we have $\Gamma_1 \cap m_\theta^{-1}\Gamma_2 m_\theta = \langle \gamma_0 \rangle$ and the map which maps γ to γ if $\gamma \in \Gamma_1$ and to $m_\theta^{-1}\gamma m_\theta$ if $\gamma \in \Gamma_2$ extends to an isomorphism $\Gamma \to \Gamma_\theta$ where
$$\Gamma_\theta := \Gamma_1 *_{\langle \gamma_0 \rangle} m_\theta^{-1}\Gamma_2 m_\theta.$$
If θ is sufficiently small, then

- Γ_θ is a discrete subgroup of G;
- $M_\theta := \Gamma_\theta \backslash \mathbb{H}^3$ is a quasifuchsian manifold and
- there is a path isometric embedding $j_\theta : S \to \partial \operatorname{core} M_\theta$ such that its image S_θ is bent with a dihedral angle of θ along the image of β and otherwise totally geodesic.

Fix $\varepsilon > 0$ sufficiently small that β has an embedded annular collar neighborhood in S of width 2ε. Let $\gamma \subset S_1$ be a geodesic whose closure $\overline{\gamma}$ is disjoint from a 2ε-neighborhood $\mathcal{O}(\beta, 2\varepsilon)$ of β. Now if we set $S_1(\varepsilon) := S_1 - \mathcal{O}(\beta, 2\varepsilon)$, then there is a unique orientation-preserving isometric immersion

$$J_\theta : S_1(\varepsilon) \times \mathbb{R} \to M_\theta$$

which extends $j_\theta|_{S_1(\varepsilon)}$ and sends geodesics normal to $S_1(\varepsilon)$ to geodesics normal to $j_\theta(S_1(\varepsilon))$. Now, if θ is small enough (relative to ε), then

$$J_\theta \text{ is a proper isometric embedding.}$$

This can be proved using the following observation. Let $\alpha = [a, b_1] \cup [b_1, b_2] \cup \cdots \cup [b_{n-1}, b_n] \cup [b_n, c]$ be a broken geodesic in $\mathbb{H}^3$, which is a union of geodesic segments and which bends by angle $0 \le \theta < \pi/2$ at each b_i's. Suppose the first and the last segments have length at least $\varepsilon > 0$ and the rest have length at least 2ε. Let P_i denote the geodesic plane orthogonal to $[b_i, b_{i+1}]$ at b_i. If $\theta = 0$, then the distance among P_i's are at least ε. Now if θ is small enough so that $\sin(\theta/2) < \tanh \varepsilon$, then the planes P_i remain a positive distance apart, giving a nested sequence of half-planes in $\mathbb{H}^3$. This implies that J_θ is a proper imbedding.

It now follows that for the plane $P := \gamma \times \mathbb{R} \subset S_1(\varepsilon) \times \mathbb{R}$, its image $P_\theta := J_\theta(P) \subset M_\theta$ is an immersed geodesic plane whose closure $\overline{P}_\theta$ is isometric to $\overline{P} \simeq \overline{\gamma} \times \mathbb{R}$. Therefore by choosing γ whose closure is wild, we can obtain a geodesic plane P_θ of M_θ with wild closure (cf. [36] for more details).

This example demonstrates that the presence of an essential cylinder in M gives an obstruction to the topological rigidity of geodesic planes. For the behavior of an individual geodesic plane P, it also indicates that the *finite* intersection $\partial P \cap \Lambda$ can be an obstruction.

5. Unipotent blowup and renormalizations

The distinguished property of a unipotent flow on the homogeneous space $\Gamma\backslash G$ is the polynomial divergence of nearby points. Given a sequence $zg_n \in \Gamma\backslash G$ where $g_n \to e$ in G, the transversal divergence between two orbits zg_nU and zU can be understood by studying the double coset Ug_nU in view of the equality:

$$zg_nu_t = zu_s(u_s^{-1}g_nu_t)$$

and the behavior of rational maps $t \mapsto u_{\alpha_n(t)}g_nu_t$ for certain reparametrizations $\alpha_n : \mathbb{R} \to \mathbb{R}$ so that $\limsup_{n\to\infty}\{u_{\alpha_n(t)}g_nu_t : t \in \mathbb{R}\}$ contains a non-trivial element of $G - U$.[6]

We denote by V the transversal subgroup

$$V = \{u_{it} : t \in \mathbb{R}\}$$

to U inside N, so that $N = UV$. Note that the normalizer $\mathrm{N}(U)$ of U is equal to AN, and the centralizer $\mathrm{C}(U)$ of U is equal to N.

The following unipotent blowup lemma (though stated in the setting of $\mathrm{SL}_3(\mathbb{R})$) was first observed by Margulis [27, Lemma 5], in his proof of Oppenheim conjecture.

[6] If Q_n is a sequence of subsets of G, $q \in \limsup_{n\to\infty} Q_n$ if and only if every neighborhood of q meets infinitely many Q_n, and $q \in \liminf_{n\to\infty} Q_n$ if and only if every neighborhood of q meets all but finitely many Q_n. If $\limsup_n Q_n = Q_\infty = \liminf Q_n$, then Q_n is said to be convergent and Q_∞ is the limit of Q_n [19].

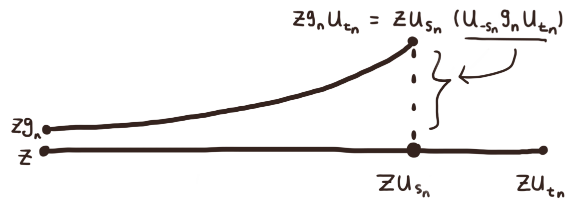

FIGURE 8. Divergence of U-orbits of two nearby points

Lemma 5.1. (1) *If $g_n \to e$ in $G - AN$, then $\limsup_{n\to\infty} Ug_nU$ contains a sequence of elements of $AV - \{e\}$ tending to e.*
(2) *If $g_n \to e$ in $G - VH$, then $\limsup_{n\to\infty} Ug_nH$ contains a one-parameter semigroup* [7] *of V.*

5.1. **Use of unipotent blowup in the compact $\Gamma\backslash G$ case.** In order to demonstrate the significance of this lemma, we present a proof of the following orbit closure theorem, which uses the notion of U-minimal subsets. A closed U-invariant subset $Y \subset \Gamma\backslash G$ is called U-minimal if every U orbit in Y is dense in Y. By Zorn's lemma, any compact U-invariant subset of $\Gamma\backslash G$ contains a U-minimal subset.

Theorem 5.2. *Let $\Gamma < G$ be a uniform lattice. For any $x \in \Gamma\backslash G$, xH is either closed or dense.*

Proof. Set $X := \overline{xH}$. Suppose that $X \neq xH$. By the minimality of the N-action on $\Gamma\backslash G$ (Corollary 3.6), it suffices to show that X contains an orbit of V.
Step 1: For any U-minimal subset $Y \subset X$,

$$YL = Y \quad \text{for a one-parameter subgroup } L < AV.$$

It suffices to show that $Yq_n = Y$ for some sequence $q_n \to e$ in AV. Fix $y_0 \in Y$. As Y is U-minimal, there exists $t_n \to \infty$ such that $y_0 u_{t_n} \to y_0$. Write $y_0 u_{t_n} = y_0 g_n$ for $g_n \in G$. Then $g_n \to e$ in $G - U$, because if g_n belonged to U, the orbit y_0U would be periodic, which is a contradiction to the assumption that Γ is a uniform lattice and hence contains no parabolic elements. If $g_n = a_n v_n u_n \in AN = AUV$, then we may take $q_n = a_n v_n$. If $g_n \notin AN$, then by Lemma 5.1, $\limsup_{n\to\infty} Ug_nU$ contains a one-parameter semigroup L of AV. Hence for any $q \in L$, there exist $t_n, s_n \in \mathbb{R}$ such that $q = \lim u_{t_n} g_n u_{s_n}$.

(5.1) Since Y is compact, $y_0 u_{-t_n}$ converges to some $y_1 \in Y$,

[7]A one-parameter semigroup of V is given by $\{\exp t\xi : t \geq 0\}$ for some non-zero $\xi \in \mathrm{Lie}(V)$

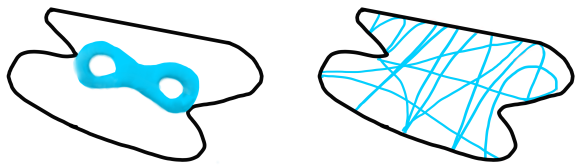

FIGURE 9. Closed or dense

by passing to a subsequence. Therefore $y_0 g_n u_{s_n} = y_0 u_{-t_n}(u_{t_n} g_n u_{s_n})$ converges to $y_1 q \in Y$. Since $q \in \mathrm{N}(U)$ and Y is U-minimal, we have

$$\overline{y_1 qU} = \overline{y_1 U} q = Yq = Y.$$

This proves the claim.

Step 2: There exists a U-minimal subset $Y \subset X$ such that $X - y_0 H$ is not closed for some $y_0 \in Y$.

If xH is not locally closed, i.e., $X - xH$ is not closed, then let Y be any U-minimal subset of X. If $Y \subset xH$, then for any $y_0 \in Y$, $X - y_0 H = X - xH$ is not closed. If $Y \not\subset xH$, then choose $y_0 \in Y - xH$. If xH is locally closed, then let Y be a U-minimal subset of $X - xH$. Then $X - y_0 H$ is not closed for any $y_0 \in Y$.

Step 3: For Y from Step (2), we have

$$Yv \subset X \quad \text{ for some non-trivial } v \in V.$$

By Step (2), we have $y_0 g_n \in X$ for some $y_0 \in Y$ and a sequence $g_n \to e$ in $G - H$. If $g_n \in VH$ for some n, then the claim follows. If $g_n \notin VH$ for all n, then by Lemma 5.1(2), $\limsup_{n\to\infty} U g_n H$ contains a non-trivial element $v \in V$. Since $v = \lim u_{t_n} g_n h_n$ for some $t_n \in \mathbb{R}$ and $h_n \in H$, we deduce $Yv \subset X$ as in Step (1).

Step 4: X contains a V-orbit.

It suffices to show that X contains $x_0 V_+$ for a one-parameter semigroup V_+ of V; because if $v_n \to \infty$ in V_+ and $x_0 v_n \to x_1$, then

$$x_1 V = x_1 \cdot \limsup_n (v_n^{-1} V_+) \subset \overline{x_0 V_+} \subset X.$$

Let $Y \subset X$ be a U-minimal subset from Step 2. By Step 1, $YL \subset Y$ where L is either V or $v_0 A v_0^{-1}$ for some $v_0 \in V$. If $L = V$, this finishes the proof. If $L = A$, then by Step 3, we get $Yv = Yv(v^{-1}Av) \subset X$. Hence we get $X \supset x_0 v^{-1} A v A$ for some $x_0 \in X$ and a non-trivial $v \in V$. Since $v^{-1}AvA$ contains a one-parameter semigroup of V, this finishes the proof. $\square$

We highlight the importance of (5.1) from the above proof: if q belongs to the set $\limsup_{n\to\infty} U g_n U$ in Lemma 5.1, i.e., $q = \lim_{n\to\infty} u_{t_n} g_n u_{s_n}$ for some $t_n, s_n \in \mathbb{R}$, then the size of t_n and s_n are essentially determined by the sequence $g_n \to e$, up to multiplicative constants. On the other hand, we need the convergence of the sequence $y_0 u_{-t_n}$ in order to derive $Yq \subset Y$. That is, if $y_0 u_{-t_n}$ diverges, which will be typical when $\Gamma\backslash G$ has infinite volume, Lemma 5.1, whose proof depends on the polynomial property of unipotent action, does not lead anywhere in the study of orbit closure problem.

5.2. **Unipotent blowup and renormalizations of the return time.** Loosely speaking, for a given $y_0 \in \Gamma\backslash G$, we now would like to understand the set

$$\limsup_{n\to\infty} \mathsf{T} g_n U$$

where T is the recurrence time of $y_0 U$ into a fixed compact subset of $\Gamma\backslash G$. Most of time, $\limsup \mathsf{T} g_n U$ may be empty. In order to make sure that this set is non-trivial enough for our purpose, we need a certain polynomial $\phi(t)$ (cf. proof of Lemma 5.5) not to vanish on the renormalized set $\limsup \lambda_n^{-1}\mathsf{T}$ where $\lambda_n > 0$ is a sequence whose size is dictated by the speed of convergence of the sequence $g_n \to e$. Since we do not have a control on g_n in general, the following condition on T, or more generally on a sequence T_n, is necessary for an arbitrary sequence $\lambda_n \to \infty$.

Definition 5.3. *We say that a sequence $\mathsf{T}_n \subset \mathbb{R}$ has accumulating renormalizations if for any sequence $\lambda_n \to \infty$,*

$$\mathsf{T}_\infty := \limsup_{n\to\infty} \lambda_n^{-1}\mathsf{T}_n$$

accumulates both at 0 *and* ∞*.*

That is, T_∞ contains a sequence tending to 0, as well as a sequence tending to ∞. We allow a constant sequence T_n in this definition.

The following lemma is immediate:

Lemma 5.4. *If there exists $\kappa > 1$ such that each T_n is κ-thick in the sense that for all $r > 0$, $\mathsf{T}_n \cap \pm[r, \kappa r] \neq \emptyset$, then the sequence T_n has accumulating renormalizations.*

We now present a refined version of Lemma 5.1, which will be a main tool in the study of U-orbits in the infinite volume homogeneous space: via the map $t \mapsto u_t$, we identify $\mathbb{R} \simeq U$.

We write $\mathfrak{g} = \mathfrak{h}^\perp \oplus \mathfrak{h}$ where $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{R})$ is the Lie algebra of H and $\mathfrak{h}^\perp = i\mathfrak{sl}_2(\mathbb{R})$; note that $\mathfrak{h}^\perp$ is H-invariant under conjugation.

Lemma 5.5 (Unipotent blowup)**.** *Let $\mathsf{T}_n \subset U$ be a sequence with accumulating renormalizations.*

(1) *For any $g_n \to e$ in $G - AN$, the subset $AV \cap \left(\limsup_{n\to\infty} \mathsf{T}_n g_n U\right)$ accumulates at e and ∞.*
(2) *For any $g_n \to e$ in $G - VH$, the subset $V \cap \left(\limsup_{n\to\infty} \mathsf{T}_n g_n H\right)$ accumulates at e and ∞.*
(3) *For any $g_n \to e$ in $\exp \mathfrak{h}^\perp - V$, the subset $V \cap \left(\limsup_{n\to\infty}\{u_t g_n u_{-t} : t \in \mathsf{T}_n\}\right)$ accumulates at e and ∞.*

Proof. For (1), we will find a sequence $\lambda_n \to \infty$ (depending on g_n) and a rational map $\psi : \mathbb{R} \to AV$ such that for $\mathsf{T}_\infty := \limsup_{n\to\infty} \lambda_n^{-1}\mathsf{T}_n$,

- $\psi(\mathsf{T}_\infty) \subset \limsup_{n\to\infty} \mathsf{T}_n g_n U$;
- $\psi(\mathsf{T}_\infty)$ accumulates at e and ∞.

The construction of ψ follows the arguments of Margulis and Tomanov [32]. Since U is a real algebraic subgroup of G, by Chevalley's theorem, there exists an $\mathbb{R}$-regular representation $G \to \mathrm{GL}(W)$ with a distinguished point $p \in W$ such that $U = \mathrm{Stab}_G(p)$. Then pG is locally closed, and

$$\mathrm{N}(U) = \{g \in G : pgu = pg \text{ for all } u \in U\}. \tag{5.2}$$

Set $\mathcal{L} := VAMN^+$ where N^+ is the transpose of N. Then $U\mathcal{L}$ is a Zariski dense open subset of G and $p\mathcal{L}$ is a Zariski open neighborhood of p in the Zariski closure of pG. We choose a norm on W so that $B(p,1) \cap \overline{pG} \subset p\mathcal{L}$, where $B(p,1) \subset W$ denotes the closed ball of radius 1 centered at p.

Without loss of generality, we may assume $g_n \in U\mathcal{L}$ for all n. For each n, define $\tilde{\phi}_n : \mathbb{R} \to W$ by

$$\tilde{\phi}_n(t) = pg_n u_t,$$

which is a polynomial of degree uniformly bounded for all n. Define $\lambda_n \geq 0$ by

$$\lambda_n := \sup\{\lambda \geq 0 : \tilde{\phi}_n[-\lambda, \lambda] \subset B(p,1)\}.$$

As $g_n \notin \mathrm{N}(U) = AN$, $\tilde{\phi}_n$ is non-constant, and hence $\lambda_n < \infty$. As $\tilde{\phi}_n(0) = pg_n \to p$, we have $\lambda_n \to \infty$. We reparametrize $\tilde{\phi}_n$ using λ_n:

$$\phi_n(t) := \tilde{\phi}_n(\lambda_n t).$$

Then for all n,

$$\phi_n[-1,1] \subset B(p,1).$$

Therefore the sequence ϕ_n forms an equicontinuous family of polynomials, and hence, after passing to a subsequence, ϕ_n converges to a polynomial

$$\phi : \mathbb{R} \to \overline{pG} \subset W$$

uniformly on every compact subset of $\mathbb{R}$. Note that ϕ is non-constant, since $\phi(0) = p$ and $\max \|\phi(\pm 1)\| = 1$. As the map $\rho : \mathcal{L} \to p\mathcal{L}$ defined by $\ell \mapsto p\ell$ is a regular isomorphism, and $p\mathcal{L}$ is a Zariski open neighborhood of p in the Zariski closure of pG, we now get a rational map $\psi : \mathbb{R} \to \mathcal{L}$ given by

$$\psi(t) = \rho^{-1}(\phi(t)).$$

If we define $\psi_n(t)$ as the unique $\mathcal{L}$-component of $g_n u_t$ in the $U\mathcal{L}$ decomposition, that is, $g_n u_t = u_{s_n}\psi_n(t)$ for some $s_n \in \mathbb{R}$, then

$$\psi(t) = \lim_{n\to\infty} \psi_n(\lambda_n t)$$

where the convergence is uniform on compact subsets of $\mathbb{R}$. It is easy to check that $\operatorname{Im}\psi \subset \mathrm{N}(U) \cap \mathcal{L} = AV$ using (5.2).

Set

$$\mathsf{T}_\infty := \limsup_{n\to\infty} \lambda_n^{-1}\mathsf{T}_n.$$

By the hypothesis on T_n, T_∞ accumulates at 0 and ∞. Since $\psi : \mathbb{R} \to AV$ is a non-constant rational map with $\psi(0) = e$, $\psi(\mathsf{T}_\infty)$ accumulates at e and ∞.

Letting $t \in \mathsf{T}_\infty$, choose a sequence $t_n \in \mathsf{T}_n$ such that $\lim_{n\to\infty} \lambda_n^{-1} t_n = t$. Since $\psi_n \circ \lambda_n \to \psi$ uniformly on compact subsets,

$$\psi(t) = \lim_{n\to\infty} (\psi_n \circ \lambda_n)\left(\lambda_n^{-1} t_n\right) = \lim_{n\to\infty} \psi_n(t_n) = \lim_{n\to\infty} u_{s_n} g_n u_{t_n}.$$

for some sequence $s_n \in \mathbb{R}$. Hence

$$\psi(\mathsf{T}_\infty) \subset \limsup_{n\to\infty} U g_n \mathsf{T}_n.$$

By applying this argument to g_n^{-1}, we may switch the position of U and T_n, and hence finish the proof of (1).

To prove (2), by modifying g_n using an element of H, we may assume that $g_n = \exp q_n \in \exp \mathfrak{h}^\perp - V$. Hence (2) follows from (3). We define a polynomial $\psi_n : \mathbb{R} \to \mathfrak{h}^\perp$ by

$$\psi_n(t) = u_t q_n u_{-t} \quad \text{ for all } t \in \mathbb{R}.$$

Since $g_n \notin V$ and hence does not commute with U, ψ_n is a nonconstant polynomial. Define

$$\lambda_n := \sup\{\lambda \geq 0 : \psi_n([-\lambda, \lambda]) \subset B(0,1)\}$$

where $B(0,1)$ is the closed unit ball around 0 in $\mathfrak{h}^\perp$. Then $0 < \lambda_n < \infty$ and $\lambda_n \to \infty$.

Now the rescaled polynomials $\phi_n = \psi_n \circ \lambda_n : \mathbb{R} \to \mathfrak{h}^\perp$ form an equicontinuous family of polynomials of uniformly bounded degree and $\lim_{n\to\infty} \phi_n(0) = 0$. Therefore ϕ_n converges to a non-constant polynomial

$$\phi : \mathbb{R} \to \mathfrak{h}^\perp$$

uniformly on compact subsets.

We claim that $\mathrm{Im}(\phi) \subset \mathrm{Lie}(V)$. For any fixed $s, t \in \mathbb{R}$,

$$\begin{aligned} u_s \phi(t) u_{-s} &= \lim_{n\to\infty} u_{\lambda_n t + s} q_n u_{-\lambda_n t - s} \\ &= \lim_{n\to\infty} u_{\lambda_n (t + \lambda_n^{-1} s)} q_n u_{-\lambda_n (t+\lambda_n^{-1} s)} \\ &= \lim_{n\to\infty} u_{\lambda_n t} q_n u_{-\lambda_n t} = \phi(t). \end{aligned}$$

Hence $\phi(t)$ commutes with U. Since the centralizer of U in $\mathfrak{h}^\perp$ is equal to $\mathrm{Lie}\, V$, the claim follows. Define $\psi : \mathbb{R} \to V$ by $\psi(t) = \exp(\phi(t))$, noting that $\exp : \mathrm{Lie}\, V \to V$ is an isomorphism. Setting

$$\mathsf{T}_\infty := \limsup_{n\to\infty} \lambda_n^{-1} \mathsf{T}_n,$$

we deduce that $\psi(\mathsf{T}_\infty)$ accumulates at e and ∞. For any $t \in \mathsf{T}_\infty$, we choose $t_n \in \mathsf{T}_n$ so that $t = \lim \lambda_n^{-1} t_n$. Then

$$\psi(t) = \lim_{n\to\infty} u_{t_n} g_n u_{-t_n}.$$

as $\phi_n(t) \to \phi(t)$ uniformly on compact subsets. Hence,

$$\psi(\mathsf{T}_\infty) \subset V \cap \big(\limsup_{n\to\infty} \{u_t g_n u_{-t} : t \in \mathsf{T}_n\}\big).$$

This completes the proof of (3). $\square$

5.3. **Relative minimal sets and additional invariance.** Let $\Gamma < G$ be a discrete subgroup. Let $X \subset \Gamma\backslash G$ be a closed H-invariant subset with no periodic U-orbits[8]. Let $W \subset \Gamma\backslash G$ be a compact subset such that $X \cap W \neq \emptyset$. We suppose that for any $y \in X \cap W$,

(5.3) $\qquad \mathsf{T}(y) := \{t \in \mathbb{R} : yu_t \in W\}$ has accumulating renormalizations.

Under this hypothesis, we can obtain analogous steps to Step (1) and (3) in the proof of Theorem 5.2 for relative U-minimal subsets of X. Since X is not compact in general, a U-minimal subset of X may not exist. Hence we consider a relative U-minimal subset of X instead.

[8]The case when X contains a periodic U-orbit turns out to be more manageable; see [4, Prop. 4.2]

Definition 5.6. *A closed U-invariant subset $Y \subset X$ is called U-minimal with respect to W, if $Y \cap W \neq \emptyset$ and yU is dense in Y for every $y \in Y \cap W$.*

As W is compact, it follows from Zorn's lemma that X always contains a U-minimal subset with respect to W.

Lemma 5.7 (Translates of Y inside of Y)**.** *Let $Y \subset X$ be a U-minimal subset with respect to W. Then*

$$YL \subset Y$$

for some one-parameter semigroup $L < AV$.

Proof. It suffices to find a sequence $q_n \to e$ in AV such that $Yq_n \subset Y$.

Fix $y_0 \in Y \cap W$. We claim that there exists $g_n \to e$ in $G - U$ such that $y_0 g_n \in Y \cap W$. By the minimality assumption on Y, there exists $t_n \to \infty$ in $\mathsf{T}(y_0)$ so that $y_0 u_{t_n}$ converges to $y_0 \in Y \cap W$ (cf. [4, Lemma 8.2]). Hence there exists $g_n \to e$ such that

$$y_0 u_{t_n} = y_0 g_n.$$

Then $g_n \notin U$, because if g_n belonged to U, $y_0 U$ would be periodic, contradicting the assumption that X contains no periodic U-orbit.
Case (1): $g_n \in AN$. By modifying g_n with elements of U, we may assume that $g_n \in AV$. Since $g_n \in \mathrm{N}(U)$ and $y_0 \in Y \cap W$, we get $y_0 U g_n = y_0 g_n U \subset Y$ and hence $\overline{y_0 U} g_n = Y g_n \subset Y$.
Case (2): $g_n \notin AN$. By Lemma 5.5, for any neighborhood $\mathcal{O}$ of e, there exist $t_n \in \mathsf{T}(y_0)$ and $s_n \in \mathbb{R}$ such that $u_{-t_n} g_n u_{s_n}$ converges to some $q \in (AV - \{e\}) \cap \mathcal{O}$. Since $y_0 u_{t_n} \in W$ and W is compact, $y_0 u_{t_n}$ converges to some $y_1 \in Y \cap W$, by passing to a subsequence. Therefore as $n \to \infty$,

$$y_0 g_n u_{-s_n} = (y_0 u_{t_n})(u_{-t_n} g_n u_{s_n}) \to y_1 q \in Y.$$

As $y_1 \in Y \cap W$ and $q \in \mathrm{N}(U)$, it follows $Yq \subset Y$. Since such q can be found in any neighborhood of e, this finishes the proof.

□

Lemma 5.8 (One translate of Y inside of X)**.** *Let $Y \subset X$ be a U-minimal subset with respect to W such that $X - y_0 H$ is not closed for some $y_0 \in Y \cap W$. Then*

$$Yv \subset X \quad \textit{for some non-trivial } v \in V.$$

Proof. By the hypothesis, there exists $g_n \to e$ in $G - H$ such that $y_0 g_n \in X$.

If $g_n \in VH$ for some n, the claim is immediate as X is H-invariant. If $g_n \notin VH$ for all n, by Lemma 5.5, there exist $t_n \in \mathsf{T}(y_0)$ and $h_n \in H$ so that $u_{t_n}^{-1} g_n h_n$ converges to some non-trivial $v \in V$. Since $y_0 u_{t_n}$ belongs to the compact subset W, by passing to a subsequence $y_0 u_{t_n}$ converges to some $y_1 \in Y \cap W$. Hence $y_0 g_n h_n = y_0 u_{t_n} (u_{t_n}^{-1} g_n h_n)$ converges to $y_1 v$. By the minimality of Y with respect to W, we get $Yv \subset Y$, as desired. □

For a subset $I \subset \mathbb{R}$, we write $V_I = \{u_{it} : t \in I\}$. When the conditions for Lemmas 5.7 and 5.8 are met, we can deduce that X contains some interval of an V-orbit:

Lemma 5.9. *Let X be a closed H-invariant subset of $\Gamma \backslash G$ containing a compact A-invariant subset W. Let $Y \subset X$ be a U-minimal subset with respect to W. Suppose*

(1) *$YL \subset Y$ for some one-parameter semigroup $L < AV$;*
(2) *$Yv \subset X$ for some non-trivial $v \in V$.*

Then X contains $x_0 V_I$ for some $x_0 \in W$ and an interval $0 \in I$.

Proof. Any one-parameter semigroup $L < AV$ is either a one-parameter semigroup $V_+ < V$ or $v_0 A_+ v_0^{-1}$ for some $v_0 \in V$ and a one-parameter semigroup $A_+ < A$.

Case (a). If $L = V_+$, we are done.

Case (b). If $L = v_0 A_+ v_0^{-1}$ for a non-trivial $v_0 \in V$, then

$$X \supset Y(v_0 A_+ v_0^{-1})A.$$

Since $v_0 A_+ v_0^{-1} A$ contains V_I for some interval $0 \in I$, the claim follows.

Case (c). If $L = A_+$, we first note that $YA \subset Y$; take any sequence $a_n \to \infty$ in A_+, and $y_0 \in Y \cap W$. Then $y_0 a_n \in Y \cap W$ converges to some $y_1 \in Y \cap W$. Now $\limsup_{n\to\infty} a_n^{-1} A_+ = A$. Therefore $Y \supset y_1 A$. Since $\overline{y_1 U} = Y$, we get $Y \supset YA$. Since AvA contains a semigroup V_+ of V, we deduce

$$X \supset YvA \supset YAvA \supset YV_+.$$

□

In the next section, we discuss the significance of the conclusion that X contains a segment $x_0 V_I$, depending on the relative location of x_0 to $\partial \operatorname{core} M$.

6. Interior frames and boundary frames

Let $\Gamma < G$ be a Zariski dense geometrically finite group, and let $M = \Gamma \backslash \mathbb{H}^3$. When M has infinite volume, its convex core has a non-empty boundary, which makes the dynamical behavior of a frame under geometric flows different depending on its relative position to $\partial \operatorname{core} M$.

Recall the notation F_Λ from (4.2). We denote by F^* the interior of F_Λ, and ∂F the boundary of F_Λ. We explain that in order to show that a given closed H-invariant subset $X \subset F_\Lambda$ with no periodic U-orbits is equal to F_Λ, it suffices to show that X contains $x_0 V_I$ for some $x_0 \in F^* \cap \mathrm{RF}_+ M$ and an interval $0 \in I$ (Lemma 6.1). It is important to get $x_0 \in F^*$, as the similar statement is not true if $x_0 \in \partial F$. For example, in the rigid acylindrical case, if $x_0 \in \partial F$, then $x_0 H V_+ H$ is a closed H-invariant subset of ∂F for a certain semigroup $V_+ < V$ (cf. Theorem 7.1 below), and hence if V_I belongs to V_+, we cannot use $x_0 V_I$ to obtain useful information on $X \cap F^*$.

6.1. Interior frames.

In this section, we assume that Λ is connected. Under this hypothesis, the closed H-invariant set $F_\Lambda = \mathrm{RF}_+ M \cdot H$ has non-empty interior which can be described as follows:

$$\begin{aligned} F^* &= \{[g] \in \Gamma\backslash G : \pi(P_g) \cap M^* \neq \emptyset\} \\ &= \bigcup \{xH \subset \Gamma \backslash G : p(xH) \cap M^* \neq \emptyset\} \end{aligned}$$

where M^* denotes the interior of $\operatorname{core} M$.

The condition $\pi(P_g) \cap M^* \neq \emptyset$ is equivalent to the condition that the circle $C_g = \partial P_g$ separates the limit set Λ, that is, both components of $\mathbb{S}^2 - C_g$ intersects Λ non-trivially. If we set

$$\mathcal{C}^* := \{C \in \mathcal{C} : C \text{ separates } \Lambda\},$$

we have

$$F^*/H = \Gamma\backslash\mathcal{C}^*.$$

We observe that the connectedness of Λ implies the following two equivalent statements:

(1) For any $C \in \mathcal{C}^*$, $\#C \cap \Lambda \geq 2$;
(2) $F^* \cap \mathrm{RF}_+ M \subset \mathrm{RF}\, M \cdot U$

By the openness of F^* and (2) above, for any $x \in F^* \cap \mathrm{RF}_+ M$, there exists a neighborhood $\mathcal{O}$ of e in G such that

$$x\mathcal{O} \cap \mathrm{RF}_+ M \subset \mathrm{RF}\, M \cdot U. \tag{6.1}$$

Thanks to this stability, we have the following lemma:

Lemma 6.1. *Let $X \subset F_\Lambda$ be a closed H-invariant subset intersecting* RF M *and with no periodic U-orbits. If X contains $x_0 V_I$ for some $x_0 \in F^* \cap \mathrm{RF}_+ M$ and an interval $0 \in I$, then*

$$X = F_\Lambda.$$

Proof. It suffices to find $z_0 V$ inside X for some $z_0 \in \mathrm{RF}\, M$ by Theorem 3.6. Without loss of generality, we may assume $I = [0, s]$ for some $s > 0$. We write $v_t := u_{it}$. Since $x_0 \in F^* \cap \mathrm{RF}_+ M$, there exists $0 < \varepsilon < s$ such that $x_0 v_\varepsilon \in X \cap \mathrm{RF}\, M \cdot U$ by (6.1). Hence there exists $x_1 \in x_0 v_\varepsilon U \cap \mathrm{RF}\, M \cap X$; so $x_1 v_\varepsilon^{-1} V_I = x_1 V_{[-\varepsilon, s-\varepsilon]} \subset X$. Since X has no periodic U-orbit, x_1^+ is a radial limit point of Λ, and hence there exists $t_n \to +\infty$ such that $x_1 a_{t_n}$ converges to some $z_0 \in \mathrm{RF}\, M$. Since

$$\limsup_{n\to\infty} a_{t_n}^{-1} V_{[-\varepsilon, s-\varepsilon]} a_{t_n} = V$$

and $x_1 V_{[-\varepsilon, s-\varepsilon]} a_{t_n} = x_1 a_{t_n} (a_{t_n}^{-1} V_{[-\varepsilon, s-\varepsilon]} a_{t_n}) \subset X$, we obtain $z_0 V \subset X$ as desired. $\square$

6.2. **Boundary frames.** The geometric structure of the boundary $\partial F = F_\Lambda - F^*$ plays an important (rather decisive) role in the rigidity study. For instance, unless xH is bounded, xH is expected to accumulate on ∂F. In the most dramatic situation, all the accumulation of xH may fall into the boundary ∂F so that $\overline{xH} \subset xH \cup \partial F$. Unless we have some analysis on what possible closed H-invariant subsets of ∂F are, there isn't too much more we can say on such situation.

A geodesic plane $P \subset \mathbb{H}^3$ is called a supporting plane if it intersects hull(Λ) and one component of $\mathbb{H}^3 - P$ is disjoint from hull(Λ), or equivalently, the circle $C = \partial P$ is a supporting circle in the sense that $\#C \cap \Lambda \geq 2$ and C does not separate Λ.

For $C \in \mathcal{C}$, we denote by Γ^C the stabilizer of C in Γ. The theory of bending laminations yields:

Theorem 6.2. [38, Theorem 5.1] *For any supporting circle $C \in \mathcal{C}$,*

(1) *Γ^C is a finitely generated Fuchsian group;*
(2) *there exists a finite subset $\Lambda_0 \subset C \cap \Lambda$ such that*

$$C \cap \Lambda = \Lambda(\Gamma^C) \cup \Gamma^C \Lambda_0$$

where $\Lambda(\Gamma^C)$ denotes the limit set of Γ^C.

Definition 6.3. *We call $x \in \partial F$ a boundary frame, and call $x = [g] \in \partial F$ a thick boundary frame if there exists a supporting circle C with non-elementary stabilizer Γ^C such that $C_g = C$ or C_g is tangent to C at $g^+ \in \Lambda(\Gamma^C)$.*

Theorem 6.4. *If $x \in \partial F$ is a thick boundary frame such that xU is not closed, then $\overline{xU} \supset xvAv^{-1}$ for some $v \in V$. If $x \in \mathrm{RF}\, M$ in addition, then $\overline{xU} \supset xA$.*

Proof. Choose $g \in G$ so that $[g] = x$. By the hypothesis on x, there exists a supporting circle C with Γ^C non-elementary and $g^+ \in \Lambda(\Gamma^C)$. The circle C_g is equal to C or tangent to C at g^+. It follows that there exists $v \in V$ such that $C = C_{gv}$. By Theorem 6.2, the stabilizer Γ^C is finitely generated and non-elementary. It now follows from a theorem Dalbo [10] that xvU is either periodic (if $g^+ = (gv)^+$ is a parabolic fixed point of Γ^C) or $\overline{xvU}$ contains $xvH \cap \mathrm{RF}_+ M \supset xvA$. Since v commutes with U, the first claim follows. If $x \in \mathrm{RF}\, M$ in addition, then C_g must be equal to C, and hence $v = e$. □

Lemma 6.5. *Let $X \subset F_\Lambda$ be a closed H-invariant subset intersecting $\mathrm{RF}\, M$ and with no periodic U-orbits. If $X \cap F^*$ contains zv_0 for some thick boundary frame $z \in \partial F \cap \mathrm{RF}\, M$ and $v_0 \in V - \{e\}$, then*

$$X = F_\Lambda.$$

Proof. By Lemma 6.1, it suffices to find $x_0 V_I$ inside X for some $x_0 \in F^*$ and an interval $0 \in I$. By Theorem 6.4, we have $\overline{zU} \supset zA$. Therefore

$$X \supset \overline{zv_0UA} = \overline{zUv_0A} \supset \overline{zAv_0A} \supset zV_+$$

where V_+ is the one-parameter semigroup contained in $V \cap Av_0A$. Since $zv_0 \in zV_+ \cap F^* \neq \emptyset$, and F^* is open, $zv_0V_I \subset zV_+ \cap F^*$ for some interval $0 \in I$, as desired. □

7. Rigid acylindrical groups and circular slices of Λ

Let $\Gamma < G$ be a rigid acylindrical Kleinian group, and $M := \Gamma\backslash\mathbb{H}^3$ the associated hyperbolic 3-manifold. We assume $\mathrm{Vol}(M) = \infty$.

7.1. **Boundary frames for rigid acylindrical groups.**

In this case, we have a complete understanding of the orbit closures in the boundary ∂F; which makes it possible to give a complete classification for *all* orbit closures in F.

When Γ is rigid acylindrical, *every* supporting circle C is contained in the limit set, so that $C \cap \Lambda = C$. It follows that Γ^C is a uniform lattice of G^C and the orbit $xH = p(gH)$ is compact whenever C_g is a supporting circle. This implies:

Theorem 7.1. [36] *Let Γ be rigid acylindrical, and let $x \in \partial F$ be a boundary frame.*

(1) *If $x \in \mathrm{RF}_+ M$, then*
$$\overline{xU} = xvHv^{-1} \quad \text{for some } v \in V.$$
(2) *If $x \in \mathrm{RF}\, M$, then xH is compact.*
(3) *If $x \in \mathrm{RF}_+ M - \mathrm{RF}\, M$, there exist a one-parameter semigroup V_+ of V and a boundary frame $x_0 \in \partial F$ with x_0H compact such that*
$$\overline{xH} = x_0HV_+H.$$

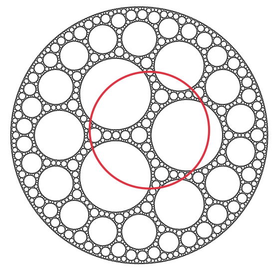

Figure 10. Circular slice of Λ

7.2. **Circular slices of Λ.** Circular slices of the limit set Λ control the recurrence time of U-orbits into the compact subset $\mathrm{RF}\, M$. For $x \in \mathrm{RF}\, M$, set

$$\mathsf{T}(x) := \{t \in \mathbb{R} : xu_t \in \mathrm{RF}\, M\}.$$

If $x = [g]$, then $(gu_t)^+ = g^+ \in C_g \cap \Lambda$ and hence

$$t \in \mathsf{T}(x) \text{ if and only if } (gu_t)^- \in C_g \cap \Lambda.$$

We will use the following geometric fact for a rigid acylindrical manifold M: if we write $S^2 - \Lambda = \bigcup B_i$ where B_i's are round open disks, then

$$\inf_{i \neq j} d(\mathrm{hull}(B_i), \mathrm{hull}(B_j)) \geq \varepsilon_0 \tag{7.1}$$

where $2\varepsilon_0$ is the systol of the double of the convex core of M. This follows because a geodesic in $\mathbb{H}^3$ which realizes the distance $d(\mathrm{hull}(B_i), \mathrm{hull}(B_j))$ is either a closed geodesic in M or the half of a closed geodesic in the double of $\mathrm{core}(M)$.

Proposition 7.2. *Let Γ be rigid acylindrical. There exists $\kappa > 1$ such that for all $x \in \mathrm{RF}\, M$, $\mathsf{T}(x)$ is κ-thick. In particular, for any sequence $x_i \in \mathrm{RF}\, M$, $\mathsf{T}(x_i)$ has accumulating renormalizations.*

Proof. For $\varepsilon_0 > 0$ given by (7.1), consider the upper-half plane model of $\mathbb{H}^2 = \{(x_1, 0, y) : x_1 \in \mathbb{R}, y > 0\}$. For $a < b$, $\mathrm{hull}_{\mathbb{H}^2}(a, b) \subset \mathbb{H}^2$ denotes the convex hull of the interval connecting $(a, 0, 0)$ and $(b, 0, 0)$. Define $\kappa > 1$ by the equation

$$d_{\mathbb{H}^2}(\mathrm{hull}(-\kappa, -1), \mathrm{hull}(1, \kappa)) = \varepsilon_0/2;$$

since $\lim_{s\to\infty} d_{\mathbb{H}^2}(\mathrm{hull}(-s, -1), \mathrm{hull}(1, s)) = 0$, such $\kappa > 1$ exists.

Since $z \mapsto tz$ is a hyperbolic isometry in $\mathbb{H}^2$ for any $t > 0$, we have

$$d_{\mathbb{H}^2}(\mathrm{hull}(-\kappa t, -t), \mathrm{hull}(t, \kappa t)) = \varepsilon_0/2.$$

We now show that $\mathsf{T}(x)$ is κ-thick for $x \in \mathrm{RF}\, M$. It suffices to show the claim for $x = [g]$ where $g = (e_1, e_2, e_3)$ is based at $(0, 0, 1)$ with e_2 in the direction of the positive real axis and $g^+ = \infty, g^- = 0$. Note that $gu_t \in \mathrm{RF}\, M$ if and only if $t = (gu_t)^- \in \Lambda$ and hence

$$\mathsf{T}(x) = \mathbb{R} \cap \Lambda.$$

Suppose that $\mathsf{T}(x)$ is not κ-thick. Then for some $t > 0$, $\mathsf{T}(x)$ does not intersect $[-\kappa t, -t] \cup [t, \kappa t]$, that is, $[-\kappa t, -t] \cup [t, \kappa t] \subset \bigcup B_i$. Since B_i's are convex and $0 \in \Lambda$, there exist $i \neq j$ such that $[-\kappa t, -t] \subset B_i$ and $[t, \kappa t] \subset B_j$. Hence

$$d(\mathrm{hull}(-\kappa t, -t), \mathrm{hull}(t, \kappa t)) = \varepsilon_0/2 \geq d(\mathrm{hull}(B_i), \mathrm{hull}(B_j)) \geq \varepsilon_0,$$

yielding contradiction. The second claim follows from Lemma 5.4. □

7.3. **Closed or dense dichotomy for H-orbits.** In Theorem 7.1, we have described all possible orbit closures for H and U-action inside ∂F. It remains to consider orbits of $x \in F^*$.

Theorem 7.3. [36] *For any $x \in F^*$, xH is either closed or dense in F_Λ.*

Proof. Set $X := \overline{xH}$, and assume that $X \neq xH$. We then need to show $X = F_\Lambda$. Since $F^* \cap \mathrm{RF}_+ M \subset \mathrm{RF}\, M \cdot U$, and $xH \subset F^* \cap \mathrm{RF}_+ M \cdot H$, we may assume without loss of generality that $x = [g] \in \mathrm{RF}\, M$.

Set $W := X \cap F^* \cap \mathrm{RF}\, M$.

Case 1: *W* **is not compact**. In this case, there exists $x_n \in W$ converging to some $z \in \partial F \cap \mathrm{RF}\, M$. Write $x_n = z g_n$ with $g_n \to e$ in $G - H$.

Suppose that $g_n = h_n v_n \in HV$ for some n. Since $z h_n \in zH \subset \partial F \cap \mathrm{RF}\, M$ and $(z h_n) v_n \in F^* \cap \mathrm{RF}\, M$, the claim follows from Lemma 6.5.

Now suppose that $g_n \notin HV$ for all n. By Lemma 5.5, there exist $t_n \in \mathsf{T}(x_n)$ and $h_n \in H$ such that $h_n g_n u_{t_n}$ converges to some $v \in V - \{e\}$. Since zH is compact, $z h_n^{-1}$ converges to some $z_0 \in \partial F \cap \mathrm{RF}\, M$ by passing to a subsequence. Hence, as $n \to \infty$,

$$x_n u_{t_n} = z h_n^{-1}(h_n g_n u_{t_n}) \to z_0 v.$$

Since $z_0 \in \partial F \cap \mathrm{RF}\, M$ and $z_0 v \in \mathrm{RF}\, M$, we get $z_0 v \in F^*$; hence the claim follows by Lemma 6.5.

Case 2: *W* **is compact**. It follows from the definition of W that for any $x \in W$,

$$\mathsf{T}(x) = \{t : x u_t \in W\}.$$

We claim that X contains a U-minimal subset Y with respect to W such that $X - y_0 H$ is not closed for some $y_0 \in Y \cap W$. We divide our proof into two cases:

Case (a). Suppose that xH is not locally closed, i.e., $X - xH$ is not closed. In this case, any U-minimal subset $Y \subset X$ with respect to W works. First, if $Y \cap W \subset xH$, then choose any $y_0 \in Y \cap W$. Observe that $\overline{xH} - y_0 H = \overline{xH} - xH$ is not closed, which implies the claim. If $Y \cap W \not\subset xH$, choose $y_0 \in (Y \cap W) - xH$. Then $\overline{xH} - y_0 H$ contains xH, and hence cannot be closed.

Case (b). Suppose that xH is locally closed, and $X - xH$ intersects W non-trivially. Therefore $X - xH$ contains a U-minimal subset Y with respect to W. Then any $y_0 \in Y \cap W$ has the desired property; since $y_0 \in X - xH$, there exists $h_n \in H$ such that $x h_n \to y$. If we write $x h_n = y g_n$, then $g_n \to e$ in $G - H$, since $y \notin xH$.

By Lemmas 5.7, 5.8 and 5.9, X contains $x_0 V_I$ for some $x_0 \in W$ and for an interval $0 \in I$; since $x_0 \in F^*$, this finishes the proof by Lemma 6.1. □

7.4. **Topological rigidity of geodesic planes.** In ([36], [37]), the following theorem was also obtained:

Theorem 7.4. *Let M be a rigid acylindrical hyperbolic 3-manifold. Then*

(1) *any geodesic plane P intersecting* $\operatorname{core} M$ *is either properly immersed or dense;*
(2) *the fundamental group of a properly immersed P intersecting* $\operatorname{core} M$ *is a non-elementary Fuchsian subgroup;*
(3) *there are at most countably many properly immersed geodesic planes in M intersecting* $\operatorname{core} M$*;*
(4) *any infinite sequence of geodesic planes P_i intersecting* $\operatorname{core} M$ *becomes dense in M, i.e,* $\lim P_i = M$.

Remark 7.5.
(1) There exists a closed arithmetic hyperbolic 3-manifold $\Gamma\backslash\mathbb{H}^3$ without any properly immersed geodesic plane, as shown by Maclachlan-Reid [25].
(2) When M has finite volume and has at least one properly immersed geodesic plane, then M is arithmetic if and only if there are infinitely many properly immersed geodesic planes ([31], [3]).
(3) A natural question is whether a rigid acylindrical hyperbolic 3-manifold M necessarily covers an arithmetic hyperbolic 3-manifold if there exists infinitely many properly immersed (unbounded) geodesic planes intersecting its core. The reason for the word "unbounded" in the parenthesis is that in any geometrically finite hyperbolic 3-manifold of infinite volume, there can be only finitely many bounded geodesic planes ([36], [4]). In view of the proofs given in ([31], [3]), the measure-theoretic equidistribution of infinitely many closed H-orbits needs to be understood first.

7.5. **Classification of U-orbit closures.** In the rigid acylindrical case, the complete classification of the U-orbit closures inside ∂F given in Theorem 7.1 can be extended to the whole space $\mathrm{RF}_+ M$:

Theorem 7.6. [37] *For any $x \in \mathrm{RF}_+ M$,*

$$\overline{xU} = xL \cap \mathrm{RF}_+ M$$

where L is either $v^{-1}Hv$ for some $v \in N$, or G.

There are two main features of a rigid acylindrical group which our proof is based on. The first property is that

there exists a compact H-orbit in $\mathrm{RF}_+ M$,

namely those $[g]H$ whose corresponding plane P_g is a supporting plane. This very important feature of M is a crucial ingredient of our proof. In particular, the following *singular* set is non-empty:

$$\mathscr{S}(U) = \bigcup zHV \cap \mathrm{RF}_+ M.$$

where the union is taken over all closed H orbits zH.

We set

$$\mathscr{G}(U) := \mathrm{RF}_+ M - \mathscr{S}(U)$$

and call it the generic set. Note that

$$\mathscr{G}(U) \subset F^*.$$

The second property is the following control on the pre-limiting behavior of $\mathrm{RF}\, M$-points, whose proof is based on the totally geodesic nature of $\partial \operatorname{core} M$.

Lemma 7.7. [37, Lemma 4.2] *If $x_n \in F^*$ converges to some $y \in \mathrm{RF}\, M$, then there exists a sequence $x_n' \in x_n U \cap \mathrm{RF}\, M$ converging to y, or converging to some boundary frame $y' \in \partial F \cap \mathrm{RF}\, M$.*

For $x \in \mathscr{S}(U)$, Theorem 7.6 follows from a theorem of Hedlund [18] and Dalbo [10] on the minimality of horocyclic action on the Fuchsian case.

Proposition 7.8. [37] *If $x \in \mathscr{G}(U)$, then*

$$\overline{xU} = \mathrm{RF}_+ M.$$

Proof. Setting $X := \overline{xU}$, we first claim that

$$X \cap \mathscr{S}(U) \neq \emptyset. \tag{7.2}$$

If $X \cap \partial F \neq \emptyset$, the claim follows from Theorem 7.1(1). Hence we assume that $X \subset F^*$. Let Y be a U-minimal subset of X with respect to $\mathrm{RF}\, M$. By Lemma 5.7, $YL \subset Y$ where $L < AV$ is a one-parameter semigroup. If L is a semigroup of V, say, V_+, then take a sequence $v_n \to \infty$ in V_+. Since $YV^+ \subset Y \subset F^* \subset \mathrm{RF}\, M \cdot U$, up to passing to a subsequence, there exists $y_n \in Y$ such that $y_n v_n$ converges to an $\mathrm{RF}\, M$-point, say y_0. Then

$$y_0 V = \lim_{n\to\infty} (y_n v_n) \cdot \limsup_n (v_n^{-1} V_+) \subset \overline{YV_+} \subset Y.$$

Hence $Y = X = \mathrm{RF}_+ M$, proving the claim. If $L = vA_+v^{-1}$ for some semigroup A_+ of A, since $\mathscr{S}(U)$ is V-invariant, we may assume that $L = A_+$. Take a sequence $a_n \to \infty$ in A_+. Then for any $y \in Y$, ya_n converges to an $\mathrm{RF}\, M$-point, say $y_0 \in Y$, by passing to a subsequence. So

$$y_0 A = \lim_{n\to\infty} (ya_n) \cdot \limsup_n (a_n^{-1} A_+) \subset \overline{YA_+} \subset Y.$$

On the other hand, either $y_0 \in \mathscr{S}(U)$ or $\overline{y_0 H} = F_\Lambda$ (Theorem 7.3). In the latter case, $\overline{y_0 H}$ contains a compact H-orbit zH. Since $\overline{y_0 AUM_0} = \overline{y_0 H}$, it follows that $\overline{y_0 AU} \cap zH \neq \emptyset$, proving the claim (7.2).

Therefore X contains $\overline{yU} = yvHv^{-1} \cap \mathrm{RF}_+ M$ for some $y \in \mathscr{S}(U)$. Without loss of generality, we may assume $X \supset yH \cap \mathrm{RF}_+ M$ by replacing x with xv. Set $Y := yH \cap \mathrm{RF}_+ M$, which is a U-minimal subset. There exists $s_n \in \mathbb{R}$ such that $y = \lim_{n\to\infty} xu_{s_n}$. In view of Lemma 7.7, we may assume that $xu_{s_n} \in \mathrm{RF}\, M$ for all n. Write $xu_{s_n} = yg_n$ for some sequence $g_n \to e$ in G. Since $y \in \mathscr{S}(U)$ and $x \in \mathscr{G}(U)$, it follows that $g_n \notin HV$ for all n. Hence by Lemma 5.5, there exist $t_n \in \mathsf{T}(yg_n)$ and $h_n \in H$ such that $h_n g_n u_{t_n}$ converges to some $v \in V$; moreover v can be taken arbitrarily large. By passing to a subsequence, $yg_n u_{t_n}$ converges to some $y_0 \in \mathrm{RF}\, M$, and hence yh_n^{-1} converges to $y_1 := y_0 v^{-1} \in yH \cap \mathrm{RF}_+ M = Y$. Therefore $X \supset y_0 = y_1 v$ and hence $X \supset Yv$. As $y_1 v \in \mathrm{RF}\, M$, $Yv \cap \mathrm{RF}\, M \neq \emptyset$. As v can be taken arbitrarily large, there exists a sequence $v_n \to \infty$ in V such that X contains Yv_n. Choose $y_n \in Y$ so that $y_n v_n \in \mathrm{RF}\, M$ converges to some $x_0 \in \mathrm{RF}\, M$, by passing to a subsequence. Since Y is A-invariant and $\limsup_{n\to\infty} v_n^{-1} A v_n \supset V$, we deduce

$$X \supset \lim_{n\to\infty} (y_n v_n) \cdot \limsup_{n\to\infty} (v_n^{-1} A v_n) \supset x_0 V.$$

Therefore $X = \mathrm{RF}_+ M$. $\square$

As an immediate corollary, we deduce:

Corollary 7.9. [37] *Let M be a rigid acylindrical hyperbolic 3-manifold. Then the closure of any horocycle is either a properly immersed surface, parallel to a geodesic plane, or equal to M.*

7.6. **Measure rigidity?** If there exists a closed orbit xH for $x \in \mathrm{RF}_+ M$, then the stabilizer of x in H is a non-elementary convex cocompact fuchsian subgroup and there exists a unique U-invariant ergodic measure supported on $xH \cap \mathrm{RF}_+ M$, called the Burger-Roblin measure m^{BR}_{xH} on xH.

Question: Let M be a rigid acylindrical hyperbolic 3-manifold. Is any locally finite U-ergodic measure on $\mathrm{RF}_+ M$ either m^{BR} or m^{BR}_{xH} for some closed H orbit xH, up to a translation by the centralizer of U?

Theorem 7.6 implies the positive answer to this question at least in terms of the support of the measure: the support of any locally finite U-ergodic measure on $\mathrm{RF}_+ M$ is either $\mathrm{RF}_+ M$ or $\mathrm{RF}_+ M \cap xHv$ for some closed orbit xH and $v \in N$.

8. Geometrically finite acylindrical hyperbolic 3-manifolds

Let $\Gamma < G$ be a Zariski dense geometrically finite group, and let $M = \Gamma\backslash\mathbb{H}^3$. We assume $\mathrm{Vol}(M) = \infty$. In the rigid acylindrical case, we were able to give a complete classfication of all possible closures of a geodesic plane in M; this is largely due to the *rigid* structure of the boundary of $\operatorname{core} M$. In particular, the intersection of a geodesic plane and the convex core of M is either closed or dense in $\operatorname{core} M$.

In general, the convex core of M is not such a natural ambient space to study the topological behavior of a geodesic plane, because of its non-homogeneity property. Instead, its interior, which we denote by M^*, is a better space to work with; first of all, M^* is a hyperbolic 3-manifold with no boundary (although incomplete), which is diffeomorphic to M, and a geodesic plane P which does not intersect M^* cannot come arbitrarily close to M^*, as $\overline{P}$ must be contained in the ends $M - M^*$.

Definition 8.1. *A geodesic plane P^* in M^* is defined to be a non-empty intersection $P \cap M^*$ for a geodesic plane P of M.*

Let $P = \pi(\tilde{P})$ for a geodesic plane $\tilde{P} \subset \mathbb{H}^3$, and set $S = \mathrm{Stab}_\Gamma(\tilde{P})\backslash\tilde{P}$. Then the natural map $f : S \to P \subset M$ is an immersion (which is generically injective), $S^* := f^{-1}(M^*)$ is a non-empty convex subsurface of S with $\pi_1(S) = \pi_1(S^*)$ and P^* is given as the image of the restriction of f to S^*. The group $\pi_1(S^*)$ will be referred to as the fundamental group of P^*. We note that a geodesic plane P^* is always connected as P^* is covered by the convex subset $\tilde{P} \cap \mathrm{Interior}(\mathrm{hull}\,\Lambda)$.

8.1. **Rigidity of geodesic planes in M^*.** An analogous topological rigidity of planes to Theorem 7.4 continues to hold inside M^*, provided M is a geometrically finite *acylindrical* hyperbolic 3-manifold.

The following rigidity theorem was proved jointly with McMullen and Mohammadi for convex cocompact cases in [37], and extended to geometrically finite cases jointly with Benoist [4]:

Theorem 8.2. *Let M be a geometrically finite acylindrical hyperbolic 3-manifold. Then geodesic planes in M^* are topologically rigid in the following sense:*

(1) *any geodesic plane P^* in M^* is either properly immersed or dense;*

(2) *the fundamental group of a properly immersed P^* is a non-elementary geometrically finite Fuchsian subgroup;*
(3) *there are at most countably many properly immersed geodesic planes in M^*;*
(4) *any infinite sequence of geodesic planes P_i^* becomes dense in M^*, i.e, $\lim P_i^* = M^*$.*

This theorem is deduced from following results on H-orbits in F^*:

Theorem 8.3. *(*[37], [4]*) Let M be a geometrically finite acylindrical hyperbolic 3-manifold. Then*

(1) *any H-orbit in F^* is either closed or dense;*
(2) *if xH is closed in F^*, then $\mathrm{Stab}_H(x)$ is Zariski dense in H;*
(3) *there are at most countably many closed H-orbits in F^*;*
(4) *any infinite sequence of closed H-orbits x_iH becomes dense in F^*, i.e, $\lim x_iH = F^*$.*

8.2. **Closed or dense dichotomy for acylindrical groups.** In this section, we discuss the proof of the following closed or dense dichotomy:

Theorem 8.4. *Let M be a geometrically finite acylindrical hyperbolic 3-manifold. Then any H-orbit in F^* is either closed or dense.*

Indeed, the proof of Theorem 7.3 for the rigid acylindrical case can be modified to prove the following proposition.

Proposition 8.5 (Main proposition)**.** *Let Γ be a Zariski dense convex cocompact subgroup of G with connected limit set. Let $\mathcal{R}$ be a closed A-invariant subset of $\mathrm{RF}\, M$ satisfying that for any $x \in \mathcal{R}$, $\mathsf{T}(x) := \{t : xu_t \in \mathcal{R}\}$ has accumulating renormalizations. Then for any $x \in \mathcal{R} \cap F^*$, xH is either locally closed or dense in F. When xH is locally closed, it is closed in $\mathcal{R}H \cap F^*$.*

Proof. Let $X := \overline{xH}$ for $x \in \mathcal{R} \cap F^*$. Set $W := X \cap \mathcal{R} \cap F^*$. Suppose that either xH is not locally closed or $(X - xH) \cap W \neq \emptyset$. We claim that $X = F_\Lambda$.
Case 1: W is not compact. By repeating verbatim the proof of Theorem 7.3, we obtain $zv \in X \cap \mathcal{R}$ for some $z \in \partial F \cap \mathcal{R}$ and non-trivial $v \in V$. As $z = [g] \in \mathcal{R}$, Γ^{C_g} is non-elementary and hence z is a *thick* boundary frame. Since $zv \in F^*$, the claim follows from Lemma 6.5.
Case 2: W is compact. By repeating verbatim the proof of Theorem 7.3, we show that X contains a U-minimal subset Y with respect to W such that $X - y_0H$ is not closed for some $y_0 \in Y \cap W$. Hence by applying Lemmas 5.7, 5.8, 5.9 and 6.1, we get $X = F_\Lambda$. □

When Γ is rigid acylindrical, note that $\mathcal{R} = \mathrm{RF}\, M$ satisfies the hypothesis of Proposition 8.5. In view of this proposition, Theorem 8.4 for a convex cocompact case now follows from the following theorem, and a general geometrically finite case can be proved by an appropriately modified version, taking account of closed horoballs, which is responsible for the non-compactness of $\mathrm{RF}\, M$.

Theorem 8.6. *Let M be a geometrically finite acylindrical hyperbolic 3-manifold. Then there exists a closed A-invariant subset $\mathcal{R} \subset \mathrm{RF}\, M$ such that for any $x \in \mathcal{R}$, $\mathsf{T}(x) := \{t : xu_t \in \mathcal{R}\}$ accumulating renormalizations. Moreover $F^* \subset \mathcal{R}H$.*

We use the notion of a conformal modulus in order to find a closed subset $\mathcal{R}$ satisfying the hypothesis of Theorem 8.6. An annulus $\mathcal{A} \subset \mathbb{S}^2$ is an open region whose compliment consists of two components. If neither component is a single point, $\mathcal{A}$ is conformally equivalent to a unique round annulus of the form $\{z \in \mathbb{C} : 1 < |z| < R\}$. The modulus $\mathrm{mod}(\mathcal{A})$ is then defined to be $\log R$. If P is a compact set of a circle C such that its complement $C - P = \bigcup I_i$ is a union of at least two intervals with disjoint closures, we define the modulus of P as

$$\mathrm{mod}\ P := \inf_{i \neq j} \quad \mathrm{mod}\ (I_i, I_j)$$

where $\quad \mathrm{mod}\ (I_i, I_j) := \quad \mathrm{mod}\ \left(\mathbb{S}^2 - (\overline{I}_i \cup \overline{I}_j)\right)$.

For $\varepsilon > 0$, define $\mathcal{R}_\varepsilon \subset \mathrm{RF}\, M$ as the following subset:

$$\mathcal{R}_\varepsilon := \{[g] : C_g \cap \Lambda \text{ contains a compact set of modulus} \geq \varepsilon \text{ containing } g^{\pm}\}.$$

Lemma 8.7. *For $\varepsilon > 0$, the set $\mathcal{R}_\varepsilon$ is closed.*

Proof. Suppose that $g_n \in \mathcal{R}_\varepsilon$ converges to some $g \in \mathrm{RF}\, M$. We need to show $g \in \mathcal{R}_\varepsilon$. Let $P_n \subset C_{g_n} \cap \Lambda$ be a compact set of modulus $\geq \varepsilon$ containing $g_n^{\pm}$. Since the set of all closed subsets of $\mathbb{S}^2$ is a compact space in the Hausdorff topology on closed subsets, we may assume P_n converges to some P_∞, by passing to a subsequence. This means that $P_\infty = \limsup_n P_n = \liminf_n P_n$ [19].

Write $C_g - P_\infty = \bigcup_{i \in I} I_i$ as the disjoint union of connected components. As $g^{\pm} \in P_\infty$, $|I| \geq 2$. Let $i \neq j \in I$, and write $I_i = (a_i, b_i)$ and $I_j = (a_j, b_j)$. There exist $a_{i,n}, b_{i,n}, a_{j,n}, b_{j,n} \in P_n$ converging to a_i, b_i, a_j and b_j respectively. Set $I_{i,n}$ and $I_{j,n}$ to be the intervals $(a_{i,n}, b_{i,n})$ and $(a_{j,n}, b_{j,n})$ respectively. Since $I_{i,n} \to I_i$, and $I_{j,n} \to I_j$, $I_{i,n} \cup I_{j,n} \subset C_{g_n} - P_n$ for all large n. Since $a_{i,n}, b_{i,n} \in P_n$, $I_{i,n}$ is a connected component of $C_{g_n} - P_n$. Similarly, $I_{j,n}$ is a connected component of $C_{g_n} - P_n$. Since $\quad \mathrm{mod}\ (I_{i,n}, I_{j,n}) \geq \varepsilon$ for all n, it follows that I_i and I_j have disjoint closures and $\quad \mathrm{mod}\ (I_i, I_j) \geq \varepsilon$. This shows that P_∞ is a compact subset of $C_g \cap \Lambda$ of modulus at least ε containing $g^{\pm}$. Therefore $g \in \mathcal{R}_\varepsilon$. $\square$

There exists $\kappa = \kappa(\varepsilon) > 1$ such that for any $x \in \mathcal{R}_\varepsilon$, $\mathsf{T}(x) := \{t : xu_t \in \mathcal{R}_\varepsilon\}$ is κ-thick (see [37, Prop. 4.3]); hence $\mathcal{R}_\varepsilon$ satisfies the hypothesis of Theorem 8.6.

In general, $\mathcal{R}_\varepsilon$ may be empty! However for geometrically finite acylindrical manifolds, there exists $\varepsilon > 0$ such that

$$F^* \subset \mathcal{R}_\varepsilon H \tag{8.1}$$

([37], [4]); hence Theorem 8.4 follows. The inclusion (8.1) is proved using *bridge* arguments devised in [37], and the monotonicity of conformal moduli, based on the property that for a convex cocompact acylindrical manifold M, Λ is a Sierpinski carpet of positive modulus, that is,

$$\inf_{i \neq j} \mathrm{mod}(\mathbb{S}^2 - (\overline{B}_i \cup \overline{B}_j)) > 0$$

where B_i's are components of $\mathbb{S}^2 - \Lambda$ (see [37] for details).

When M has cusps, the closures of some components of $\mathbb{S}^2 - \Lambda$ may meet each other, and hence Λ is not even a Sierpinski carpet in general. Nevertheless, under the assumption that M is a geometrically finite acylindrical manifold, Λ is still a *quotient of a Sierpinski carpet of positive modulus*, in the sense that we can present

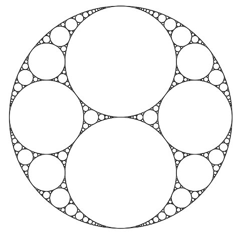

FIGURE 11. Apollonian gasket

$\mathbb{S}^2 - \Lambda$ as the disjoint union $\bigcup_\ell T_\ell$ where T_ℓ's are maximal *trees of components of* $\mathbb{S}^2 - \Lambda$ so that

$$\inf_{\ell \neq k} \ \mathrm{mod}(\mathbb{S}^2 - (\overline{T}_\ell \cup \overline{T}_k)) > 0.$$

Question: Let Γ be a Zariski dense geometrically finite subgroup of G with a connected limit set. Let $C \in \mathcal{C}^*$. If $C \cap \Lambda$ contains a Cantor set,

$$\text{is } \Gamma C \text{ either discrete or dense in } \mathcal{C}^*?$$

If $C \cap \Lambda$ contains a Cantor set of positive modulus, this question has been answered affirmatively in [4].

One particular case of interest is when Λ is the Apollonian gasket. The correspdoning geometrically finite hyperbolic 3-manifold is not acylindrical in this case, because its compact core is a handle body of genus 2, and hence it is not boundary incompressible; this can also be seen from the property that the Apollonian gasket contains a loop of three consecutively tangent disks.

Question: Can we classify all possible closures of U-orbits in a geometrically finite acylindrical group? In order to answer this question, we first need to classify all possible H-orbit closures in ∂F, which is unsettled yet.

9. Unipotent flows in higher dimensional hyperbolic manifolds

Let $\mathbb{H}^d$ denote the d-dimensional hyperbolic space for $d \geq 2$ with $\partial(\mathbb{H}^d) = \mathbb{S}^{d-1}$, and let $G := \mathrm{SO}^\circ(d,1)$, which is the isometry group $\mathrm{Isom}^+(\mathbb{H}^d)$. Any complete hyperbolic d-manifold is given as the quotient $M = \Gamma\backslash\mathbb{H}^d$ for a torsion-free discrete subgroup $\Gamma < G$ (also called a Kleinian group). The limit set of Γ and the convex core of M are defined just like the dimension 3 case. As we have seen in the dimension 3 case, the geometry and topology of hyperbolic manifolds becomes relevant in the study of unipotent flows in hyperbolic manifolds of infinite volume, unlike in the finite volume case. Those hyperbolic 3-manifolds in which we have a complete understanding of the topological behavior of unipotent flows are rigid acylindrical hyperbolic 3-manifolds.

9.1. **Convex cocompact hyperbolic manifolds with Fuchsian ends.** The higher dimensional analogues of rigid acylindrical hyperbolic 3-manifolds are as follows:

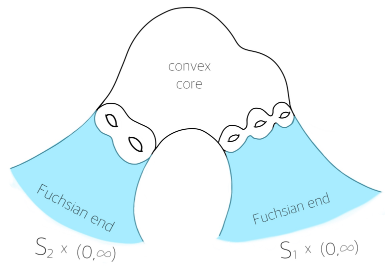

FIGURE 12. Convex cocompact manifolds with Fuchsian ends.

Definition 9.1. *A convex cocompact hyperbolic d-manifold M is said to have Fuchsian ends if the convex core of M has non-empty interior and has totally geodesic boundary.*

The term *Fuchsian ends* reflects the fact that each component of the boundary of core M is a $(d-1)$-dimensional closed hyperbolic manifold, and each component of the complement $M - \text{core}(M)$ is diffeomorphic to the product $S \times (0, \infty)$ for some closed hyperbolic $(d-1)$-manifold S. For $d = 2$, any convex cocompact hyperbolic surface has Fuchsian ends. For $d = 3$, these are precisely rigid acylindrical hyperbolic 3-manifolds.

Convex cocompact hyperbolic manifolds with non-empty Fuchsian ends are constructed from closed hyperbolic manifolds as follows. Begin with a closed hyperbolic d-manifold $N_0 = \Gamma_0 \backslash \mathbb{H}^d$ with a fixed collection of finitely many, mutually disjoint, properly embedded totally geodesic hypersurfaces. Cut N_0 along those hypersurfaces and perform the metric completion to obtain a compact hyperbolic manifold W with totally geodesic boundary hypersurfaces. Then $\Gamma := \pi_1(W)$ injects to $\Gamma_0 = \pi_1(N_0)$, and $M := \Gamma \backslash \mathbb{H}^d$ is a convex cocompact hyperbolic manifold with Fuchsian ends.

Unlike $d = 3$ case, Kerckhoff and Storm showed that if $d \geq 4$, a convex cocompact hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^d$ with Fuchsian ends does not allow any non-trivial local deformation, in the sense that the representation of Γ into G is infinitesimally rigid [21].

9.2. **Orbit closure of unipotent flows are relatively homogeneous.** We let $A = \{a_t\}$ be the one parameter subgroup of semisimple elements of G which give the frame flow, and let $N \simeq \mathbb{R}^{d-1}$ denote the contracting horospherical subgroup. We have a compact A-invariant subset $\mathrm{RF}\, M = \{x \in \Gamma \backslash G : xA \text{ is bounded}\}$.

The following presents a generalization of Theorems 7.3 and 7.6 to any dimension:

Theorem 9.2. [22] *Let $d \geq 2$ and M be a convex cocompact hyperbolic d-manifold with Fuchsian ends. Let U be any connected closed subgroup of G generated by unipotent elements. Suppose that U is normalized by A. Then the closure of any U-orbit is relatively homogeneous in $\mathrm{RF}\, M$, in the sense that for any $x \in \mathrm{RF}\, M$,*

$$\overline{xU} \cap \mathrm{RF}\, M = xL \cap \mathrm{RF}\, M$$

for a connected closed reductive subgroup $U < L < G$ such that xL is closed.

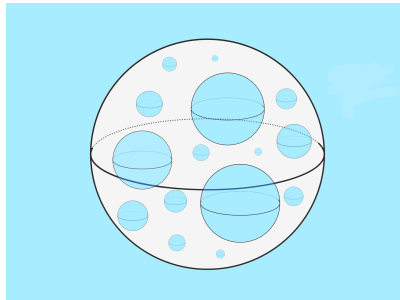

FIGURE 13. Limit set of a convex cocompact hyperbolic 4-manifold with Fuchsian boundary

When M has finite volume, this is a special case of Ratner's orbit closure theorem [45]. This particular case was also proved by Shah by topological methods [49].

Theorem 9.2 and its refinements made in [22] yield the analogous topological rigidity of geodesic planes and horocycles. A geodesic k-plane of M is the image of a totally geodesic immersion $f : \mathbb{H}^k \to M$.

Theorem 9.3. [22] *Let M be a convex cocompact hyperbolic d-manifold with Fuchsian ends. Then for any $2 \leq k \leq d-1$,*

(1) *the closure of any geodesic k-plane intersecting* $\operatorname{core} M$ *is a properly immersed geodesic m-plane for some $k \leq m \leq d$;*
(2) *a properly immersed geodesic k-plane is a convex cocompact (immersed) hyperbolic k-manifold with Fuchsian ends;*
(3) *there are at most countably many maximal properly immersed geodesic planes intersecting* $\operatorname{core} M$*;*
(4) *any infinite sequence of maximal properly geodesic planes intersecting* $\operatorname{core} M$ *becomes dense in M.*

A k-horosphere in $\mathbb{H}^d$ is a Euclidean sphere of dimension k which is tangent to a point in $\mathbb{S}^{d-1}$. A k-horosphere in M is simply the image of a k-horosphere in $\mathbb{H}^d$ under the covering map $\mathbb{H}^d \to M = \Gamma\backslash\mathbb{H}^d$.

Theorem 9.4. [22] *Let χ be a k-horosphere of M for $k \geq 1$. Then either*

(1) *χ is properly immersed; or*
(2) *$\overline{\chi}$ is a properly immersed m-dimensional submanifold, parallel to a convex cocompact geodesic m-plane of M with Fuchsian ends for some $m \geq k+1$.*

9.3. **Avoidance of singular set.** An important ingredient of the proof of Theorem 9.2 which appears newly for $d \geq 4$ is the avoidance of the singular set along the recurrence time of unipotent flows to $\mathrm{RF}\, M$.

Let $U = \{u_t\}$ be a one parameter unipotent subgroup of N. Extending the definition given by Dani-Margulis [13] to the infinite volume setting, we define the singular set $\mathscr{S}(U)$ as

$$\mathscr{S}(U) := \bigcup xL \cap \mathrm{RF}_+ M \tag{9.1}$$

where $\mathrm{RF}_+ M = \mathrm{RF}\, M \cdot N$, and the union is taken over all closed orbits xL of proper connected closed subgroups L of G containing U. Its complement in $\mathrm{RF}_+ M$ is denoted by $\mathscr{G}(U)$, and called the set of *generic* elements of U.

The structure of $\mathscr{S}(U)$ as the countable union of *singular tubes* is an important property which plays crucial roles in both measure theoretic and topological aspects of the study of unipotent flows. Let $\mathscr{H}$ denote the collection of all proper connected closed subgroups H of G containing a unipotent element such that $\Gamma\backslash\Gamma H$ is closed and $H\cap\Gamma$ is Zariski dense in H. For each $H \in \mathscr{H}$, we define the singular tube:

$$X(H,U) := \{g \in G : gUg^{-1} \subset H\}.$$

We have the following:

1. $\mathscr{H}$ is *countable*;
2. $X(H_1,U) \cap gX(H_2,U) = X(H_1 \cap gH_2g^{-1}, U)$ for any $g \in G$;
3. If $H_1, H_2 \in \mathscr{H}$ with $X(H_1 \cap H_2, U) \neq \emptyset$, there exists a closed subgroup $H_0 \subset H_1 \cap H_2$ such that $H_0 \in \mathscr{H}$.

In parictular $\mathscr{S}(U)$ can be expressed as the union of countable singular tubes:

$$\mathscr{S}(U) = \bigcup_{H\in\mathscr{H}} \Gamma\backslash\Gamma X(H,U) \cap \mathrm{RF}_+ M.$$

Remark 9.5. If $\Gamma < G = \mathrm{PSL}_2(\mathbb{C})$ is a uniform lattice, and U is the one-parameter subgroup as in (4.1), then $H \in \mathscr{H}$ if and only if $H = g^{-1}\,\mathrm{PSL}_2(\mathbb{R})g$ for $g \in G$ such that Γ intersects $g^{-1}\,\mathrm{PSL}_2(\mathbb{R})g$ as a uniform lattice. It follows that if $H_1, H_2 \in \mathscr{H}$ and $X(H_1,U)\cap X(H_2,U) \neq \emptyset$, then $H_1 = H_2$.

We note that $\mathscr{H}$ and hence $\mathscr{S}(U)$ may be empty in general; see Remark 7.5(1).

When the singular set $\mathscr{S}(U)$ is non-empty, it is very far from being closed in $\mathrm{RF}_+ M$; in fact, it is dense, which is an a posteriori fact. Hence presenting a compact subset of $\mathscr{S}(U)$ requires some care, and we will be using the following family of compact subsets $\mathscr{S}(U)$ in order to discuss the recurrence of U-flows relative to the singular set $\mathscr{S}(U)$. We define $\mathcal{E} = \mathcal{E}_U$ to be the collection of all subsets of $\mathscr{S}(U)$ which are of the form

$$\bigcup \Gamma\backslash\Gamma H_i D_i \cap \mathrm{RF}\, M$$

where $H_i \in \mathscr{H}$ is a finite collection, and D_i is a compact subset of $X(H_i,U)$.

The following theorem was obtained by Dani and Margulis [13] and independently by Shah [51] using the linearization methods, which translates the study of unipotent flows on $\Gamma\backslash G$ to the study of vector-valued polynomial maps via linear representations.

Theorem 9.6 (Avoidance theorem for lattice case)**.** [13] *Let $\Gamma < G$ be a uniform lattice, and let $U < G$ be a one-parameter unipotent subgroup. Then for any $\varepsilon > 0$, there exists a sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}$ such that $\mathscr{S}(U) = \bigcup_{n\geq 1} E_n$ which satisfies the following: Let x_j be a sequence converging to $x \in \mathscr{G}(U)$. For each $n \geq 1$, there exist a neighborhood $\mathcal{O}_n$ of E_n and $j_n \geq 1$ such that for all $j \geq j_n$ and for all $T > 0$,*

$$\ell\{t \in [0,T] : x_j u_t \in \bigcup_{i\leq n} \mathcal{O}_i\} \leq \varepsilon T \tag{9.2}$$

where ℓ denotes the Lebesgue measure.

If we set
$$\mathsf{T}_n := \{t \in \mathbb{R} : x_{j_n} u_t \notin \bigcup_{i \le n} \mathcal{O}_i\},$$
then for any sequence $\lambda_n \to \infty$, $\limsup \lambda_n^{-1} \mathsf{T}_n$ accumulates at 0 and ∞; and hence the sequence T_n has accumulating renormalizations.

When xL is a closed orbit of a connected closed subgroup of L containing U, the relative singular subset $\mathscr{S}(U, xL)$ *of* $xL \cap \mathrm{RF}_+ M$ is defined similarly by replacing $\mathscr{H}$ by its subcollection of proper connected closed subgroups of L, and $\mathscr{G}(U, xL)$ is defined as its complement inside $xL \cap \mathrm{RF}_+ M$. And Theorem 9.6 applies in the same way to $\mathscr{G}(U, xL)$ with the ambient space $\Gamma\backslash G$ replaced by xL.

In order to explain some ideas of the proof of Theorem 9.6, we will discuss the following (somewhat deceptively) simple case when $G = \mathrm{PSL}_2(\mathbb{C})$ and Γ is a uniform lattice. Let $U = \{u_t\}$ be as in (4.1).

Proposition 9.7. *Let $E \in \mathcal{E}_U$. If $x \in \mathscr{G}(U)$, then xU spends most of its time outside a neighborhood of E, more precisely, for any $\varepsilon > 0$, we can find a neighborhood $E \subset \mathcal{O}$ such that for all $T > 0$,*
$$\ell\{t \in [0, T] : xu_t \in \mathcal{O}\} \le \varepsilon T. \tag{9.3}$$

Proof. Since xU is dense in $\Gamma\backslash G$ a posteriori, xu_t will go into any neighborhood of E for an infinite sequence of t's, but that the proportion of such t is very small is the content of Proposition 9.7. In view of Remark 9.5, we may assume that E is of the form $\Gamma\backslash\Gamma\,\mathrm{N}(H)D$ where $H = \mathrm{PSL}_2(\mathbb{R})$, and $D \subset V$ is a compact subset; note $X(H, U) = \mathrm{N}(H)V$, and $\mathrm{N}(H)$ is generated by H and $\mathrm{diag}(i, -i)$.

As remarked before, we prove this proposition using the linear representation and the polynomial-like behavior of unipotent action. As $\mathrm{N}(H)$ is the group of real points of a connected reductive algebraic subgroup, there exists an $\mathbb{R}$-regular representation $\rho : G \to \mathrm{GL}(W)$ with a distinguished point $p \in W$ such that $\mathrm{N}(H) = \mathrm{Stab}(p)$ and pG is Zariski closed. The set $pX(H, U) = pV$ is a real algebraic subvariety. [9]

Note that for $x = [g]$, the following are equivalent:

(1) $xu_t \in [e]\,\mathrm{N}(H)\mathcal{O}$;
(2) $p\gamma g u_t \in p\mathcal{O}$ for some $\gamma \in \Gamma$.

Therefore, we now try to find a neighborhood $p\mathcal{O}$ of pD so that the set
$$\{t \in [0, T] : xu_t \in [e]\,\mathrm{N}(H)\mathcal{O}\} \subset \bigcup_{q \in p\Gamma} \{t \in [0, T] : qgu_t \in p\mathcal{O}\}$$
is an ε-proportion of T. Each set $\{t \in [0, T] : qgu_t \in p\mathcal{O}\}$ can be controlled by the following lemma, which is proved using the property that the map $t \mapsto \|qgu_t\|^2$ is a polynomial of degree uniformly bounded for all $q \in p\Gamma$, and polynomial maps of bounded degree have uniformly *slow* divergence.

Lemma 9.8. [13, Prop. 4.2] *Let $\mathcal{A} \subset W$ be an algebraic variety. Then for any compact subset $C \subset \mathcal{A}$ and any $\varepsilon > 0$, there exists a compact subset $C' \subset \mathcal{A}$*

[9] We can explicitly take ρ and p as follows. Consider the Adjoint representation of G on its Lie algebra $\mathfrak{g}$. We then let ρ be the induced representation on the wedge product space $\wedge^3 \mathfrak{g}$ and set $p = w_1 \wedge w_2 \wedge w_3$ where w_1, w_2, w_3 is a basis of $\mathfrak{h}$.

such that the following holds: for any neighborhood Φ' of C' in W, there exists a neighbhorhood Φ of C of W such that for any $q \in W - \Phi'$ and any $T > 0$,

$$\ell\{t \in [0,T] : qu_t \in \Phi\} \le \varepsilon \cdot \ell\{t \in [0,T] : qu_t \in \Phi'\}.$$

Applying this lemma to $\mathcal{A} = pV$, and $C = pD$, we get a compact subset $C' = pD'$ for $D' \subset V$. Since $x \notin [e]\,\mathrm{N}(H)D'$, we can find a neighborhood $\mathcal{O}'$ so that $x \notin [e]\,\mathrm{N}(H)\mathcal{O}'$. Fix a neighborhood Φ' of C', so that $\Phi' \cap pG \subset p\mathcal{O}'$. We then get a neighborhood Φ of C such that if $\mathcal{O}$ is a neighborhood of D such that $p\mathcal{O} \subset \Phi$, then

$$\ell(J_q \cap [0,T]) \le \varepsilon \cdot \ell(I_q \cap [0,T]) \tag{9.4}$$

where $J_q := \{t \in \mathbb{R} : qgu_t \in p\mathcal{O}\}$ and $I_q := \{t \in \mathbb{R} : qgu_t \in p\mathcal{O}'\}$.

We now claim that in the case at hand, we can find a neighborhood $\mathcal{O}'$ of D' so that all I_q's are mutually disjoint:

$$\text{if } q_1 \ne q_2 \text{ in } p\Gamma, \text{ then } I_{q_1} \cap I_{q_2} = \emptyset. \tag{9.5}$$

Using (9.4), this would finish the proof, since

$$\ell\{t \in [0,T] : xu_t \in [e]\,\mathrm{N}(H)\mathcal{O}\} \le \sum_{q \in p\Gamma} \ell(J_q \cap [0,T]) \le \varepsilon \cdot \sum_{q \in p\Gamma} \ell(I_q \cap [0,T]) \le \varepsilon T.$$

To prove (9.5), we now observe the special feature of this example, namely, *no* singular tube $\Gamma\backslash\Gamma X(H,U)$ has *self-intersection*, meaning that

$$X(H,U) \cap \gamma X(H,U) = \emptyset \text{ if } \gamma \in \Gamma - \mathrm{N}(H). \tag{9.6}$$

If non-empty, by Remark 9.5, we must have $H \cap \gamma H \gamma^{-1} = H$, implying that $\gamma \in \mathrm{N}(H)$. Now if $t \in I_{p\gamma_1} \cap I_{p\gamma_2}$, then $gu_t \in \gamma_1^{-1}HV \cap \gamma_2^{-1}HV$ and hence $\gamma_1\gamma_2^{-1} \in \mathrm{N}(H)$. So $p\gamma_1 = p\gamma_2$, proving (9.5). □

In the higher dimensional case, we cannot avoid self-intersections of $\Gamma X(H,U)$; so I_q's are not pairwise disjoint, which means a more careful study of the nature of the self-intersection is required. Thanks to the countability of $\mathscr{H}$, an inductive argument on the dimension of $H \in \mathscr{H}$ is used to take care of the issue, using the fact that the intersections among $\gamma X(H,U)$, $\gamma \in \Gamma$ are essentially of the form $X(H_0,U)$ for a proper connected closed subgroup H_0 of H contained in $\mathscr{H}$ (see [13] for details).

In order to illustrate the role of Theorem 9.6 in the study of orbit closures, we prove the following sample case: let $G = \mathrm{SO}^\circ(4,1)$, $H = \mathrm{SO}^\circ(2,1)$ and $L = \mathrm{SO}^\circ(3,1)$; the subgroups H and L are chosen so that $A < H < L$ and $H \cap N$ is a one-parameter unipotent subgroup. The centralizer $\mathrm{C}(H)$ of H is $\mathrm{SO}(2)$. We set $H' = H\,\mathrm{C}(H)$.

Proposition 9.9. *Let $\Gamma < G$ be a uniform lattice. Let $X = \overline{xH'}$ for some $x \in \Gamma\backslash G$. If X contains a closed orbit zL properly, then $X = \Gamma\backslash G$.*

A geometric consequence of this proposition is as follows: let M be a closed hyperbolic 4-manifold, and let $P \subset M$ be a geodesic 2-plane. If $\overline{P}$ contains a properly immersed geodesic 3-plane P', then the closure $\overline{P}$ is either P' or M.

Proof. Let $U_1 = H \cap N$ and $U_2 = H \cap N^+$ where N^+ is the expanding horospherical subgroup of G. Then the subgroups U_1 and U_2 generate H, and the intersection of the normalizers of U_1 and U_2 is equal to $A\,\mathrm{C}(H)$. Since zL is compact, each U_ℓ acts ergodically on zL by Moore's ergodicity theorem. Therefore we may choose z so that zU_ℓ is dense in zL for each $\ell = 1, 2$.

It suffices to show X contains either N or N^+-orbit. Since zL is a proper subset of X, there exists $g_n \to e$ in $G - L\,\mathrm{C}(H)$ such that $x_n = zg_n \in X$. As L is reductive, the Lie algebra of G decomposes into $\mathrm{Ad}(\mathfrak{l})$-invariant subspaces $\mathfrak{l} \oplus \mathfrak{l}^\perp$ with $\mathfrak{l}$ the Lie algebra of L. Hence we write $g_n = \ell_n r_n$ with $\ell_n \in L$ and $r_n \in \exp \mathfrak{l}^\perp - \mathrm{C}(H)$. As $g_n \notin \mathrm{C}(H)$, there exists $1 \le \ell \le 2$ such that no r_n belongs to the normalizer of U_ℓ, by passing to a subsequence. We set $U = U_\ell$. Without loss of generality we assume $U = H \cap N$; otherwise replace N by N^+ in the argument below.

Note that $\overline{zU} = zL$, in particular, z is a generic point: $z \in \mathcal{G}(U, zL) = zL - \mathscr{S}(U, zL)$. We replace the sequence $z\ell_n$ by $z\ell_{j_n}$ with j_n given by Theorem 9.6.

Define

$$\mathsf{T}_n := \{t \in \mathbb{R} : z\ell_n u_t \notin \bigcup_{i \le n} \mathcal{O}_i\}. \tag{9.7}$$

By Theorem 9.6 applied to $zL = z\,\mathrm{SO}^\circ(3,1)$, T_n has accumulating renormalizations.

Now by a similar argument as in the proof of Lemma 5.5(3), we can show that

$$\limsup\{u_t r_n u_{-t} : t \in \mathsf{T}_n\}$$

accumulates at 0 and ∞ in V where V is the one-dimensional unipotent subgroup $(L \cap N)V = N$. In particular, there exists $v \in V$ of arbitrarily large size such that $v = \lim u_{-t_n} r_n u_{t_n}$ for some $t_n \in \mathsf{T}_n$.

Note that $z\ell_n u_{t_n}$ is contained in the compact subset $zL - \bigcup_{i \le n} \mathcal{O}_i$. Since $\bigcup_i \mathcal{O}_i$ is a neighborhood of $\mathscr{S}(U, zL)$, $z\ell_n u_{t_n}$ converges to some

$$z_0 \in \mathscr{G}(U, zL). \tag{9.8}$$

Therefore

$$zg_n u_{t_n} = z\ell_n u_{t_n}(u_{-t_n} r_n u_{t_n}) \to z_0 v.$$

Since $z_0 \in \mathscr{G}(U, zL)$, by Proposition 7.8, we have

$$X \supset \overline{z_0 v U} = \overline{z_0 U} v = zLv.$$

As v can be taken arbitrarily large, we get a sequence $v_n \to \infty$ in V such that $X \supset zLv_n$. Using the A-invariance of X, we get $X \supset zL(Av_nA) \supset z(L \cap N)V_+$ for some one-parameter semigroup V_+ of V. Since $X \supset zv_n(L \cap N)v_n^{-1}V_+$, and $\limsup v_n^{-1}V_+ = V$, X contains an N orbit, finishing the proof. $\square$

Roughly speaking, if H is a connected closed subgroup of G generated by unipotent elements, the proof of the theorem that $\overline{xH}$ is homogeneous uses an inductive argument on the codimension of $H \cap N$ in N and involves repeating the following two steps:

1. (1) Find a closed orbit zL inside $\overline{xH}$ for some connected reductive subgroup $L < G$.
2. (2) If $\overline{xH} \ne zL$, then enlarge zL, i.e., find a closed orbit zL' inside $\overline{xH}$ with $\dim(L' \cap N) > \dim(L \cap N)$.

The proof of Proposition 9.9 is a special sample case of the step (2), demonstrating the importance of getting accumulating renormalizations for the sequence of return time avoiding the exhausting sequence of compact subsets of the singular set.

The following version of the avoidance theorem in [22] is a key ingredient in the proof of Theorem 9.2:

Theorem 9.10 (Avoidance theorem). *Let $M = \Gamma\backslash\mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Let $U < N$ be a one-parameter unipotent subgroup. There exists a sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}$ such that $\mathscr{S}(U) \cap \mathrm{RF}\, M = \bigcup_{n\geq 1} E_n$ which satisfies the following: Let $x_j \in \mathrm{RF}\, M$ be a sequence converging to $x \in \mathscr{G}(U)$. For each $n \geq 1$, there exist a neighborhood $\mathcal{O}_n$ of E_n and $j_n \geq 1$ such that for all $j \geq j_n$,*

$$\mathsf{T}^{\diamond}(x_j) := \{t \in \mathbb{R} : x_j u_t \in \mathrm{RF}\, M - \mathcal{O}_n\} \tag{9.9}$$

has accumulating renormalizations.

Note that in the lattice case, one can use the Lebesgue measure ℓ to understand the return time away from the neighborhoods $\mathcal{O}_n$ to prove Theorem 9.6, as was done in [13] (also see the proof of Proposition 9.7). In the case at hand, the relevant return time is a subset of $\{t \in \mathbb{R} : x_n u_t \in \mathrm{RF}\, M\}$ on which it is not clear if there exists any *friendly* measure. This makes the proof of Theorem 9.10 very delicate, as we have to examine each return time to $\mathrm{RF}\, M$ and handpick the time outside $\mathcal{O}_n$. First of all, we cannot reduce a general case to the case $E \subset \Gamma\backslash\Gamma X(H, U)$ for a single $H \in \mathscr{H}$. This means that not only do we need to understand the *self-intersections* of $\Gamma X(H, U)$, but we also have to control intersections among different $\Gamma X(H, U)$'s in $\mathscr{S}(U)$, $H \in \mathscr{H}$.

We cannot also use an inductive argument on the dimension of H. When $G = \mathrm{SO}(3, 1)$, there are no intersections among closed orbits in $\mathscr{S}(U)$ and the proof is much simpler in this case. In general, our arguments are based on the k-thick recurrence time to $\mathrm{RF}\, M$, a much more careful analysis on the graded intersections of among $\Gamma X(H, U)$'s, $H \in \mathscr{H}$, and a combinatorial inductive search argument. We prove that there exists $\kappa > 1$, depending only on Γ such that $\mathsf{T}^{\diamond}(x_n)$ is κ-thick in the sense that for any $r > 0$,

$$\mathsf{T}^{\diamond}(x_n) \cap \pm[r, \kappa r] \neq \emptyset.$$

We remark that unlike the lattice case, we are not able to prove that $\{t \in \mathbb{R} : x_n u_t \in \mathrm{RF}\, M - \bigcup_{j\leq n} \mathcal{O}_j\}$ has accumulating renormalizations. This causes an issue in carrying out a similar proof as in Proposition 9.9, as we cannot conclude the limit of $x_n u_{t_n}$ for $t_n \in \mathsf{T}^{\diamond}(x_n)$ belongs to a generic set as in (9.8).

Fortunately, we were able to devise an inductive argument (in the proof of Theorem 9.11 below) which involves an extra step of proving equidistribution of translates of maximal closed orbits and overcome this difficulty.

9.4. **Induction.** For a connected closed subgroup $U < N$, we denote by $H(U)$ the smallest closed simple Lie subgroup of G which contains both U and A. If $U \simeq \mathbb{R}^k$, then $H(U) \simeq \mathrm{SO}^{\circ}(k + 1, 1)$. A connected closed subgroup of G generated by one-parameter unipotent subgroups is, up to conjugation, of the form $U < N$ or $H(U)$ for some $U < N$.

We set $F_{H(U)} := \mathrm{RF}_+ M \cdot H(U)$, which is a closed subset. We define the following collection of closed connected subgroups of G:

$$\mathcal{L}_U := \left\{ L = H(\widehat{U})C : \begin{array}{c} \text{for some } z \in \mathrm{RF}_+ M,\ zL \text{ is closed in } \Gamma\backslash G \\ \text{and } \mathrm{Stab}_L(z) \text{ is Zariski dense in } L \end{array} \right\}.$$

where $U < \widehat{U} < N$ and C is a closed subgroup of the centralizer of $H(\widehat{U})$. We also define:

$$\mathcal{Q}_U := \{vLv^{-1} : L \in \mathcal{L}_U \text{ and } v \in N\}.$$

Theorem 9.2 follows from the following:

Theorem 9.11. [22] *Let $M = \Gamma\backslash\mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends.*

(1) *For any $x \in \mathrm{RF}\, M$,*

$$\overline{xH(U)} = xL \cap F_{H(U)}$$

where xL is a closed orbit of some $L \in \mathcal{L}_U$.

(2) *Let $x_0\widehat{L}$ be a closed orbit for some $\widehat{L} \in \mathcal{L}_U$ and $x_0 \in \mathrm{RF}\, M$.*

  (a) *For any $x \in x_0\widehat{L} \cap \mathrm{RF}_+ M$,*

$$\overline{xU} = xL \cap \mathrm{RF}_+ M$$

  where xL is a closed orbit of some $L \in \mathcal{Q}_U$.

  (b) *For any $x \in x_0\widehat{L} \cap \mathrm{RF}\, M$,*

$$\overline{xAU} = xL \cap \mathrm{RF}_+ M$$

  where xL is a closed orbit of some $L \in \mathcal{L}_U$.

(3) *Let $x_0\widehat{L}$ be a closed orbit for some $\widehat{L} \in \mathcal{L}_U$ and $x_0 \in \mathrm{RF}\, M$. Let $x_iL_i \subset x_0\widehat{L}$ be a sequence of closed orbits intersecting $\mathrm{RF}\, M$ where $x_i \in \mathrm{RF}_+ M$, $L_i \in \mathcal{Q}_U$. Assume that no infinite subsequence of x_iL_i is contained in a subset of the form y_0L_0D where y_0L_0 is a closed orbit of $L_0 \in \mathcal{L}_U$ with $\dim L_0 < \dim \hat{L}$ and $D \subset \mathrm{N}(U)$ is a compact subset. Then*

$$\lim_{i\to\infty} (x_iL_i \cap \mathrm{RF}_+ M) = x_0\widehat{L} \cap \mathrm{RF}_+ M.$$

We prove (1) by induction on the co-dimension of U in N, and (2) and (3) by induction on the co-dimension of U in $\widehat{L} \cap N$. Let us say $(1)_m$ holds, if (1) is true for all U satisfying $\text{co-dim}_N(U) \le m$. We will say $(2)_m$ holds, if (2) is true for all U and $\widehat{L}$ satisfying $\text{co-dim}_{\widehat{L}\cap N}(U) \le m$ and similarly for $(3)_m$.

We then deduce $(1)_{m+1}$ from $(2)_m$ and $(3)_m$, and $(2)_{m+1}$ from $(1)_{m+1}$, $(2)_m$, and $(3)_m$ and finally $(3)_{m+1}$ from $(1)_{m+1}$, $(2)_{m+1}$ and $(3)_m$. In proving Theorem 9.2 for lattice case, we don't need $(3)_m$ in the induction proof. In the case at hand, $(3)_m$ is needed since we could not obtain a stronger version of Theorem 9.10 with $\mathcal{O}_n$ replaced by $\cup_{j\le n}\mathcal{O}_j$.

We remark that in the step of proving $(2)_{m+1}$ the following geometric feature of convex cocompact hyperbolic manifolds M of Fuchsian ends is used to insure that $\mathscr{S}(U, x_0\widehat{L}) \neq \emptyset$.

Proposition 9.12. *For any $2 \le k \le d$, any properly immersed geodesic k-plane of M is either compact or contains a compact geodesic $(k-1)$-plane.*

This proposition follows from the hereditary property that any properly immersed geodesic k-plane P of M is a convex cocompact hyperbolic k-manifold of Fuchsian ends; hence either P is compact (when P has empty ends) or the boundary of core P provides a co-dimension one compact geodesic plane.

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