

HAUSDORFF DIMENSION OF DIRECTIONAL LIMIT SETS FOR SELF-JOININGS OF HYPERBOLIC MANIFOLDS

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ABSTRACT. The classical result of Patterson and Sullivan says that for a non-elementary convex cocompact subgroup $\Gamma < \mathrm{SO}^\circ(n, 1)$, $n \geq 2$, the Hausdorff dimension of the limit set of Γ is equal to the critical exponent of Γ . In this paper, we generalize this result for self-joinings of convex cocompact groups in two ways.

Let Δ be a finitely generated group and $\rho_i : \Delta \rightarrow \mathrm{SO}^\circ(n_i, 1)$ be a convex cocompact faithful representation of Δ for $1 \leq i \leq k$. Associated to $\rho = (\rho_1, \dots, \rho_k)$, we consider the following self-joining subgroup of $\prod_{i=1}^k \mathrm{SO}(n_i, 1)$:

$$\Gamma = \left(\prod_{i=1}^k \rho_i \right) (\Delta) = \{(\rho_1(g), \dots, \rho_k(g)) : g \in \Delta\}.$$

- (1) Denoting by $\Lambda \subset \prod_{i=1}^k \mathbb{S}^{n_i-1}$ the limit set of Γ , we first prove that

$$\dim_{\mathrm{H}} \Lambda = \max_{1 \leq i \leq k} \delta_{\rho_i}$$

where δ_{ρ_i} is the critical exponent of the subgroup $\rho_i(\Delta)$.

- (2) Denoting by $\Lambda_u \subset \Lambda$ the u -directional limit set for each $u = (u_1, \dots, u_k)$ in the interior of the limit cone of Γ , we obtain that for $k \leq 3$,

$$\frac{\psi_\Gamma(u)}{\max_i u_i} \leq \dim_{\mathrm{H}} \Lambda_u \leq \frac{\psi_\Gamma(u)}{\min_i u_i}$$

where $\psi_\Gamma : \mathbb{R}^k \rightarrow \mathbb{R} \cup \{-\infty\}$ is the growth indicator function of Γ .

CONTENTS

1.	Introduction	2
2.	Preliminaries	6
3.	Hausdorff dimension of Λ	8
4.	Fibered dynamical systems and $\ker \psi_u$ -coordinate map	10
5.	Hausdorff dimension of Λ_u and local behavior of ν_u	13
6.	Examples of symmetric growth indicator functions	18
	References	20

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1. INTRODUCTION

Let $k \geq 1$, and $G = \prod_{i=1}^k G_i$ where $G_i = \mathrm{SO}^\circ(n_i, 1)$ for $n_i \geq 2$. Consider the hyperbolic n_i -space $(X_i = \mathbb{H}^{n_i}, d_i)$ of constant curvature -1 . The Lie group G is the identity component of $\mathrm{Isom}(X)$, where (X, d) is the Riemannian product $X = \prod_{i=1}^k X_i$ with

$$d((x_i), (y_i)) = \sqrt{\sum_{i=1}^k d_i(x_i, y_i)^2}. \quad (1.1)$$

The Furstenberg boundary of G is then the Riemannian product $\mathcal{F} = \prod_{i=1}^k \mathbb{S}^{n_i-1}$ of the geometric boundaries $\partial X_i \simeq \mathbb{S}^{n_i-1}$. We consider a particular class of discrete subgroups of G , constructed as follows. Let Δ be a finitely generated group. For $1 \leq i \leq k$, let $\rho_i : \Delta \rightarrow G_i$ be a convex cocompact faithful representation of Δ . Let

$$\Gamma = \left(\prod_{i=1}^k \rho_i \right) (\Delta) = \{(\rho_1(\sigma), \dots, \rho_k(\sigma)) \in G : \sigma \in \Delta\}.$$

We will assume that each $\rho_i(\Delta)$ is Zariski dense in G_i and no two ρ_i 's are conjugate to each other; this implies that Γ is Zariski dense in G . The quotient $\Gamma \backslash X$ is a locally symmetric Riemannian manifold of rank k , which we call a self-joining of a hyperbolic manifold. Unless $k = 1$ and $\rho_1(\Delta)$ is a cocompact lattice of G_1 , $\Gamma \backslash X$ is of infinite volume.

For each i , we fix a basepoint $o_i \in X_i$. Denote by $\Lambda_{\rho_i} \subset \mathbb{S}^{n_i-1}$ the limit set of $\rho_i(\Delta)$, which is the set of all accumulation points of the orbit $\rho_i(\Delta)o_i$ in the compactification $X_i \cup \mathbb{S}^{n_i-1}$. We also denote by δ_{ρ_i} the critical exponent of $\rho_i(\Delta)$, which is the abscissa of convergence of the Poincaré series $\sum_{\sigma \in \Delta} e^{-s d_i(\rho_i(\sigma)o_i, o_i)}$ (that is, the infimum of the set of s for which the series converges). These two notions are independent of the choice of $o_i \in X_i$. A well-known theorem of Patterson [19] and Sullivan [27] says that δ_{ρ_i} is equal to the Hausdorff dimension of the limit set Λ_{ρ_i} :

$$\delta_{\rho_i} = \dim \Lambda_{\rho_i}.$$

The main aim of this paper is to investigate a higher rank analogue of this theorem. Let $o = (o_1, \dots, o_k) \in X$. The limit set of Γ is the set of all accumulation points of an orbit Γo in \mathcal{F} :

$$\Lambda = \left\{ (\xi_1, \dots, \xi_k) \in \mathcal{F} : \exists \text{ a sequence } \sigma_\ell \in \Delta \text{ s.t. } \forall 1 \leq i \leq k, \right. \\ \left. \xi_i = \lim_{\ell \rightarrow \infty} \rho_i(\sigma_\ell)(o_i) \right\}. \quad (1.2)$$

The Hausdorff dimension of a subset $S \subset \mathcal{F}$, which will be denoted by $\dim S$, is computed with respect to the Riemannian product metric of the spherical metrics on \mathbb{S}^{n_i-1} , $1 \leq i \leq k$.

Hausdorff dimension of the limit set. Our first result is the following (Theorem 3.1):

Theorem 1.1. *We have*

$$\dim \Lambda = \max_{1 \leq i \leq k} \delta_{\rho_i}. \quad (1.3)$$

Note that $\max_{1 \leq i \leq k} \delta_{\rho_i} = \delta_{\min}$ where δ_{\min} denotes the abscissa of the convergence of the series $\sum_{(\gamma_1, \dots, \gamma_k) \in \Gamma} e^{-s \min_i d_i(\gamma_i o_i, o_i)}$ (see the proof of Theorem 3.1). If δ denotes the critical exponent of Γ with respect to the Riemannian metric d on X , then $\sqrt{k}\delta \leq \delta_{\min}$ and moreover $\sqrt{k}\delta = \delta_{\min}$ if and only if ρ_i 's are all conjugate to each other [12]. It is therefore interesting to note that for $k \geq 2$, $\dim \Lambda$ is not in general equal to δ , in contrast to $k = 1$ case.

Limit cone and Growth indicator function. We also obtain estimates on the Hausdorff dimension of directional limit sets of Γ . To state the estimates, we need to recall the notion of the Cartan projection, the limit cone of Γ and the growth indicator function of Γ .

For $g = (g_1, \dots, g_k) \in G$, the Cartan projection of g is a vector-valued distance function:

$$\mu(g) = (d_1(g_1 o_1, o_1), \dots, d_k(g_k o_k, o_k)) \in \mathbb{R}_{\geq 0}^k;$$

note that the standard Euclidean norm $\|\mu(g)\|$ is equal to $d(go, o)$.

The limit cone \mathcal{L} of Γ is defined as the asymptotic cone of $\mu(\Gamma)$, i.e.,

$$\mathcal{L} = \left\{ \lim_{i \rightarrow \infty} t_i \mu(\gamma_i) \in \mathbb{R}_{\geq 0}^k : t_i \rightarrow 0, \gamma_i \in \Gamma \right\}.$$

This notion was introduced by Benoist, who also showed that \mathcal{L} is a convex cone with non-empty interior [2].

Following Quint [22], the growth indicator function $\psi_\Gamma : \mathbb{R}^k \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as follows: for an open cone \mathcal{C} in \mathbb{R}^k , let $\tau_{\mathcal{C}}$ denote the abscissa of convergence of $\sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-sd(\gamma o, o)}$. Now for any non-zero $u \in \mathbb{R}^k$, let

$$\psi_\Gamma(u) := \|u\| \inf_{u \in \mathcal{C}} \tau_{\mathcal{C}} \quad (1.4)$$

where the infimum is taken over all open cones \mathcal{C} containing u , and let $\psi_\Gamma(0) = 0$. It is immediate that $\psi_\Gamma = -\infty$ outside \mathcal{L} and Quint [22] showed that ψ_Γ is a concave upper semi-continuous function satisfying

$$\mathcal{L} = \{\psi_\Gamma \geq 0\} \quad \text{and} \quad \psi_\Gamma > 0 \quad \text{on} \quad \text{int } \mathcal{L}.$$

Hausdorff dimension of the directional limit sets. For a vector $u = (u_1, \dots, u_k) \in \mathbb{R}_{>0}^k$, a point $(\xi_1, \dots, \xi_k) \in \mathcal{F}$ is called a u -directional limit point of Γ if the geodesic ray

$$\{(\xi_1(tu_1), \dots, \xi_k(tu_k)) : t \geq 0\}$$

accumulates on $\Gamma \setminus X$, where $\{\xi_i(t) : t \geq 0\}$ denotes a unit speed geodesic in $X_i = \mathbb{H}^{n_i}$ toward $\xi_i \in \mathbb{S}^{n_i-1}$. We denote by

$$\Lambda_u \subset \Lambda$$

the set of all u -directional limit points of Γ ; note that Λ_u depends only on the direction of u and it follows easily from the definition of Λ_u that $\Lambda_u = \emptyset$

for $u \notin \mathcal{L}$. For $k = 1$, the directional limit set is precisely the conical limit set. In a higher rank setting, the notion of directional limit sets was first considered by Burger [5] in the product of two rank one groups and then in [6] in general.

We obtain the following estimates on the Hausdorff dimension of directional limit sets in terms of the growth indicator function.

Theorem 1.2. *Assume that $k \leq 3$. For any $u = (u_1, \dots, u_k) \in \text{int } \mathcal{L}$, we have*

$$\frac{\psi_\Gamma(u)}{\max_i u_i} \leq \dim \Lambda_u \leq \frac{\psi_\Gamma(u)}{\min_i u_i}.$$

In particular, if $(1, \dots, 1) \in \text{int } \mathcal{L}$, then

$$\dim \Lambda_{(1, \dots, 1)} = \psi_\Gamma(1, \dots, 1).$$

See Theorem 5.2 for the upper bound, which is proved for all $k \geq 1$, and Corollary 5.9 for the lower bound.

Symmetric growth indicator functions. By the concavity of ψ_Γ and the strict convexity of the norm ball $\{\|v\| \leq 1\}$, there exists a unique unit vector u_Γ , called the direction of maximal growth, such that $\psi_\Gamma(u_\Gamma) = \sup_{\|u\|=1} \psi_\Gamma(u)$. By [22, Coro. III.1.4],

$$\delta = \psi_\Gamma(u_\Gamma).$$

In general, it is hard to determine u_Γ . However when the growth indicator function ψ_Γ is symmetric, that is, it is invariant under all permutations in coordinates, the concavity of ψ_Γ implies that $u_\Gamma = \frac{1}{\sqrt{k}}(1, \dots, 1)$ and hence by Theorem 1.2, we obtain the following *identity*:

Corollary 1.3. *If $1 \leq k \leq 3$ and ψ_Γ is symmetric, then $u_\Gamma = \frac{1}{\sqrt{k}}(1, \dots, 1)$ and*

$$\dim \Lambda_{u_\Gamma} = \sqrt{k}\delta. \tag{1.5}$$

If the ρ_i are all conjugate, the growth indicator function ψ_Γ is symmetric for an obvious reason. In section 6, we construct many geometric examples where no two of the ρ_i are conjugate to each other and ψ_Γ is symmetric.

On the proofs. The class of groups Γ we consider are precisely those of Anosov subgroups of G with respect to a minimal parabolic subgroup in the sense of Guichard and Wienhard [13], who generalized Labourie's notion of Anosov representations on Hitchin representations [14].

One important feature of these Anosov subgroups is that their limit sets consist entirely of conical limit points (2.3). This feature allows us to cover the limit set by shadows using which we can compare the Hausdorff dimension of the limit set and δ_{\min} and hence prove Theorem 1.1. This argument is an easy adaptation of Sullivan's proof on the rank one case.

The upper bound in Theorem 1.2 is obtained by a similar idea and holds for all $k \geq 1$. The proof of the lower bound in Theorem 1.2 is based on the reparametrization theorem for Anosov subgroups ([25], [4], [7]), which

provides us with a trivial vector bundle with fiber \mathbb{R}^{k-1} over a compact space Z associated to the dynamics of one-dimensional diagonal flow in the direction of u (see section 4). If we denote by ν_u the Patterson-Sullivan measure for the direction u , then the measure of maximal entropy \mathfrak{m}_u on Z is locally equivalent to $\nu_u \otimes \nu_u \otimes d\text{Leb}$ and \mathfrak{m}_u -almost all points have their \mathbb{R}^{k-1} -coordinate map decaying sublinearly along the flow (Theorem 4.2 and Corollary 4.4). Using this, we get estimates on the local size of ν_u at almost all points (Theorem 5.6). For $k \leq 3$, we have $\nu_u(\Lambda_u) = 1$ by [6], which enables us to use the mass distribution principle to prove Theorem 1.2. We remark that this is the exact reason for the hypothesis $k \leq 3$ in Theorem 1.2.

Our approach works for any Anosov subgroup of a semisimple real algebraic group of rank at most 3, provided the Hausdorff dimension of the limit set is computed with respect to a well-chosen metric on the Furstenberg boundary. The reason we have chosen to write this paper mainly for the product of $\text{SO}^\circ(n_i, 1)$'s is because \mathcal{F} in this case is simply $\prod_{i=1}^k \mathbb{S}^{n_i-1}$ and hence is equipped with a *natural* metric, that is, the Riemannian product of spherical metrics on \mathbb{S}^{n_i-1} 's. Our theorems are all valid when $\text{SO}^\circ(n_i, 1)$ is replaced by a rank one simple Lie group $G_i = \text{Isom}^\circ X_i$ (here X_i is a rank one Riemannian symmetric space), provided we use a certain sub-Riemannian metric on the Furstenberg boundary \mathcal{F} invariant under a maximal compact subgroup of G as described in [8] (see Remark 5.10).

Remark 1.4. We mention that a certain upper bound on the dimension of the limit set for projective Anosov representations was obtained in [11] and an equality between the Hausdorff dimension of the limit set and the first simple root critical exponent for certain hyperconvex representations in $\text{SL}_n(\mathbb{R})$ was obtained in [21]. Both papers neither address the cases of products of rank one groups nor yield the identity as in (1.3), not to mention that the directional limit sets were not considered at all.

Organization. In section 2, we review basic notions and state known results about Anosov subgroups of $\prod_{i=1}^k \text{SO}^\circ(n_i, 1)$. In section 3, we prove Theorem 1.1. In section 4, we discuss the trivial vector bundle mentioned above, and prove a result that the vector-valued coordinate map associated to u decays with speed $o(t)$ under the time t -flow $\exp tu$ (Theorem 4.2). In section 5, we prove Theorem 1.2 by studying the local behavior of the measures ν_u in Theorem 5.6. In the last section 6, we discuss some geometric examples with symmetric growth indicator functions.

Question. As mentioned, our proof for the lower bound in Theorem 1.2 requires the restriction $1 \leq k \leq 3$, whereas the upper bound holds for any $k \geq 1$. It would be interesting to understand whether lower bound is still valid for a general $k \geq 1$ or not.

Acknowledgements. Our work has been largely inspired by a pioneering paper of Marc Burger [5] on a higher rank Patterson-Sullivan theory. In

particular, for $k = 2$, the upper bound of Theorem 1.2 was already noted in [5, Thm. 2]. We would like to dedicate this paper to him on the occasion of his sixtieth birthday with affection and admiration. We are grateful to Dick Canary for helpful remarks on an earlier version of this paper, and in particular for pointing out how to strengthen our original version of Theorem 1.1. We would like to thank Minju Lee for helpful discussions. We are also grateful to anonymous referees for many useful comments.

2. PRELIMINARIES

We briefly recall the setup from the introduction. For each $1 \leq i \leq k$, $G_i = \mathrm{SO}^\circ(n_i, 1)$ and $X_i = (\mathbb{H}^{n_i}, d_i)$ for $n_i \geq 2$. Let $G = \prod_{i=1}^k G_i$, $X = \prod_{i=1}^k X_i$ with $d = \sqrt{\sum d_i^2}$ and $\mathcal{F} = \prod_{i=1}^k \mathbb{S}^{n_i-1}$.

Let Δ be a finitely generated group and $\rho_i : \Delta \rightarrow G_i$ a convex cocompact faithful representation with Zariski dense image for each $1 \leq i \leq k$. In the whole paper, let Γ be the subgroup of G defined as

$$\Gamma = \{(\rho_1(\sigma), \dots, \rho_k(\sigma)) \in G : \sigma \in \Delta\}.$$

We will assume that Γ is Zariski dense, or equivalently, no two ρ_i 's are conjugate to each other.

We remark that the class of these groups is precisely the class of Anosov subgroups of G with respect to a minimal parabolic subgroup in the sense of Guichard and Wienhard [13]. This follows from combining [13, Lem. 3.18, Coro. 4.16 and Thm. 5.15]: Γ is Anosov with respect to a minimal parabolic subgroup of G if and only if for each $1 \leq i \leq k$, $\rho_i(\Delta)$ is Anosov with respect to a minimal parabolic subgroup of G_i , and Anosov subgroups of G_i are precisely convex cocompact subgroups.

This enables us to use the general theory developed for Zariski dense Anosov subgroups. Fix a basepoint $o_i \in X_i$ for each i , and we write $o = (o_1, \dots, o_k) \in X$. Let

$$\mathfrak{a} = \mathbb{R}^k \quad \text{and} \quad \mathfrak{a}^+ = \{(u_1, \dots, u_k) \in \mathbb{R}^k : u_i \geq 0 \text{ for all } i\}.$$

We denote by $\|\cdot\|$ the standard Euclidean norm on \mathfrak{a} .

The limit set of Γ , which we denote by $\Lambda = \Lambda_\Gamma$, is defined as the set of all accumulation points of Γo in the Furstenberg boundary \mathcal{F} , as in (1.2). It is the unique Γ -minimal subset of \mathcal{F} ([2], [15, Lem. 2.13]).

For each $\xi = (\xi_1, \dots, \xi_k) \in \mathcal{F}$ and $(t_1, \dots, t_k) \in \mathfrak{a}$, we write

$$\xi(t_1, \dots, t_k) = (\xi_1(t_1), \dots, \xi_k(t_k)) \tag{2.1}$$

where $\{\xi_i(t) : t \geq 0\}$ denotes the unit speed geodesic from o_i to ξ_i in X_i . Set

$$\xi(\mathfrak{a}^+) := \{\xi(t_1, \dots, t_k) \in X : t_i \geq 0 \text{ for all } i\}. \tag{2.2}$$

Recall that $\xi \in \mathcal{F}$ is called a conical limit point if there exists a sequence $\gamma_j \in \Gamma$ such that

$$\sup_j d(\xi(\mathfrak{a}^+), \gamma_j o) < \infty.$$

If $\Lambda_{\mathbf{c}}$ denotes the set of all conical limit points, then it is a well-known property of an Anosov subgroup (cf. [15, Prop. 7.4]) that

$$\Lambda = \Lambda_{\mathbf{c}}. \quad (2.3)$$

The Cartan projection of $g = (g_i)_{i=1}^k \in G$ is given by

$$\mu(g) = (d_1(g_1 o, o), \dots, d_k(g_k o, o)) \in \mathfrak{a}^+.$$

In particular, $d(go, o) = \|\mu(g)\|$. We denote by $\mathcal{L} \subset \mathfrak{a}^+$ the limit cone of Γ , which is the asymptotic cone of $\mu(\Gamma)$. It is a convex cone with non-empty interior [2]. Let $\delta = \delta_{\Gamma}$ denote the critical exponent of Γ , which is the abscissa of convergence of the Poincaré series $\mathcal{P}_{\Gamma}(s) = \sum_{\gamma \in \Gamma} e^{-s\|\mu(\gamma)\|}$. It follows from the non-elementary assumption on $\rho_i(\Delta)$ that $\delta > 0$.

Let $\psi_{\Gamma} : \mathfrak{a} \rightarrow \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of Γ defined as in the introduction (see (1.4)). By the concavity of ψ_{Γ} and the strict convexity of the unit norm ball $\{\|u\| \leq 1\}$, there exists a unique unit vector $u_{\Gamma} \in \mathcal{L}$ such that

$$\delta = \sup_{\|u\|=1} \psi_{\Gamma}(u) = \psi_{\Gamma}(u_{\Gamma}). \quad (2.4)$$

As all $\rho_i : \Delta \rightarrow G_i$ are faithful convex cocompact, it follows that there exist constants $C, C' > 0$ such that for all $\sigma \in \Delta$ and $1 \leq i, j \leq k$, we have

$$d_i(\rho_i(\sigma)o_i, o_i) \geq C d_j(\rho_j(\sigma)o_j, o_j) - C'$$

(cf. [13, Thm. 5.15]). Therefore:

Theorem 2.1. *We have $\mathcal{L} \subset \text{int } \mathfrak{a}^+ \cup \{0\}$.*

The following theorem follows from the fact that Γ is a Zariski dense Anosov subgroup of $G = \prod_{i=1}^k \text{SO}^{\circ}(n_i, 1)$ with respect to a minimal parabolic subgroup [24, Lem. 4.8] and [20, Prop. 4.6 and 4.11].

Theorem 2.2. *We have $u_{\Gamma} \in \text{int } \mathcal{L}$.*

For $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k) \in X$, and $\xi = (\xi_1, \dots, \xi_k) \in \mathcal{F}$, the \mathfrak{a} -valued Busemann function is given as

$$\beta_{\xi}(x, y) = (\beta_{\xi_1}(x_1, y_1), \dots, \beta_{\xi_k}(x_k, y_k)) \in \mathfrak{a}$$

where $\beta_{\xi_i}(x_i, y_i) = \lim_{t \rightarrow +\infty} d_i(\xi_i(t), x_i) - d_i(\xi_i(t), y_i)$ is the Busemann function on $\mathbb{S}^{n_i-1} \times X_i \times X_i$.

Definition 2.3. For a linear form $\psi \in \mathfrak{a}^*$, a Borel probability measure ν on Λ is called a (Γ, ψ) -Patterson-Sullivan measure if the following holds: for any $\xi \in \Lambda$ and $\gamma \in \Gamma$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{-\psi(\beta_{\xi}(\gamma o, o))}$$

where $\gamma_*\nu(W) = \nu(\gamma^{-1}W)$ for any Borel subset $W \subset \Lambda$.

A linear form $\psi \in \mathfrak{a}^*$ is called *tangent to ψ_{Γ} at $u \in \mathfrak{a}$* if $\psi \geq \psi_{\Gamma}$ and $\psi(u) = \psi_{\Gamma}(u)$.

Theorem 2.4 ([10, Thm. 7.7 and Cor. 7.8], [15, Cor. 7.12]). *Let $u \in \text{int } \mathcal{L}$.*

- (1) *There exists a unique $\psi_u \in \mathfrak{a}^*$ which is tangent to ψ_Γ at u .*
- (2) *There exists a unique (Γ, ψ_u) -Patterson-Sullivan measure, say, ν_u .*
- (3) *The abscissa of convergence of the series*

$$\mathcal{P}_u(s) := \sum_{\gamma \in \Gamma} e^{-s\psi_u(\mu(\gamma))}$$

is equal to 1 and $\mathcal{P}_u(1) = \infty$.

We remark that the existence of (Γ, ψ_u) -Patterson-Sullivan measure was proved by Quint [23].

Construction of ν_u . Fix $u \in \text{int } \mathcal{L}$. By Theorem 2.1, $\mathcal{L} \subset \text{int } \mathfrak{a}^+ \cup \{0\}$; this implies that all the accumulation points of Γo lie in \mathcal{F} and hence in Λ . Therefore $\Gamma o \cup \Lambda$ is a compact space. For $s > 1$, by Theorem 2.4(3), $\mathcal{P}_u(s)$ is well-defined and hence we may consider the probability measure on $\Gamma o \cup \Lambda$ given by

$$\nu_{u,s} := \frac{1}{\mathcal{P}_u(s)} \sum_{\gamma \in \Gamma} e^{-s\psi_u(\mu(\gamma))} D_{\gamma o} \quad (2.5)$$

where $D_{\gamma o}$ denotes the Dirac measure on γo . Note that the space of probability measures on $\Gamma o \cup \Lambda$ is a weak* compact space. Therefore, by passing to a subsequence, it weakly converges to a probability measure, say $\tilde{\nu}_u$, on $\Gamma o \cup \Lambda$. Since $\mathcal{P}_u(1) = \infty$ by Theorem 2.4(3), $\tilde{\nu}_u$ is supported on Λ . It is standard to check that $\tilde{\nu}_u$ is a (Γ, ψ_u) -Patterson-Sullivan measure. Now the uniqueness of (Γ, ψ_u) -Patterson-Sullivan measure (Theorem 2.4(2)) implies that $\tilde{\nu}_u = \nu_u$, as given in Theorem 2.4(2). Therefore, as $s \rightarrow^+ 1$, $\nu_{u,s}$ weakly converges to ν_u .

Hausdorff dimension. For $S \subset \mathcal{F}$ and $s > 0$, the s -dimensional Hausdorff measure of S is defined by $H^s(S) = \lim_{\varepsilon \rightarrow 0} \inf \{ \sum_j r_j^s : S \subset \bigcup_{j \in J} B(x_j, r_j) : 0 < r_j \leq \varepsilon \}$ where the infimum is taken over all countable covers of S by balls of radius at most ε . The Hausdorff dimension of S is defined as

$$\dim S := \inf \{ s \geq 0 : H^s(S) = 0 \},$$

or equivalently the supremum s such that $H^s(S) = +\infty$. We refer to [3] for general facts on Hausdorff dimension.

3. HAUSDORFF DIMENSION OF Λ

In this section, we prove Theorem 3.1, which implies Theorem 1.1.

Let δ_{\min} denote the abscissa of convergence of the series

$$\sum_{\gamma=(\gamma_1, \dots, \gamma_k) \in \Gamma} e^{-s \min_i d_i(o_i, \gamma_i o_i)}.$$

The notation δ_{ρ_i} means the critical exponent of $\rho_i(\Delta)$.

Theorem 3.1. *For any $k \geq 1$, we have*

$$\dim \Lambda = \delta_{\min} = \max_{1 \leq i \leq k} \dim \Lambda_{\rho_i} = \max_{1 \leq i \leq k} \delta_{\rho_i}. \quad (3.1)$$

We need to introduce some notations for the proof of this theorem. These notations will also be used in the next sections as well. Let $K_i \simeq \text{SO}(n_i)$ be the maximal compact subgroup of G_i given as the stabilizer of $o_i \in X_i$. Fixing a unit tangent vector \mathbf{v}_i at o_i , let $M_i := \text{Stab}(\mathbf{v}_i)$. We then have the following identification: $G_i/K_i = X_i$ and $G_i/M_i = \mathbb{T}^1 X_i$. Let $A_i = \{a_t^{(i)} : t \in \mathbb{R}\} < G_i$ denote the one-parameter subgroup of semisimple elements whose right translation action on G_i/M_i corresponds to the geodesic flow on $\mathbb{T}^1 X_i$. Set $K = \prod_{i=1}^k K_i < G$, $M = \prod_{i=1}^k M_i < G$, and $A = \prod_{i=1}^k A_i$. We also set $A^+ = \prod_{i=1}^k A_i^+$ where $A_i^+ = \{a_t^{(i)} : t \geq 0\}$. Then $X = G/K$.

For each i and $g_i \in G_i$, we denote by $g_i^+ \in \mathbb{S}^{n_i-1}$ and $g_i^- \in \mathbb{S}^{n_i-1}$ respectively the forward and backward endpoints of the geodesic determined by the tangent vector $[g_i] \in \mathbb{T}^1 \mathbb{H}^{n_i}$. For $g = (g_1, \dots, g_k) \in G$, we set

$$g^\pm = (g_1^\pm, \dots, g_k^\pm) \in \mathcal{F}. \quad (3.2)$$

The shadows are important tools in our proof:

Definition 3.2 (Shadows). For $R > 0$ and $x \in X$, the shadow $O_R(o, x)$ is defined as

$$O_R(o, x) = \{\eta \in \mathcal{F} : \exists g \in K, a \in A^+ \text{ s.t } g^+ = \eta \text{ and } d(gao, x) \leq R\}. \quad (3.3)$$

Proof of Theorem 3.1. For $\xi \in \mathcal{F}$ and $r > 0$, let $B(\xi, r)$ denote the ball in $\mathcal{F} = \prod_{i=1}^k \mathbb{S}^{n_i-1}$ centered at ξ of radius r . For $g = (g_i)$, we write

$$\min \mu(g) = \min d_i(o_i, g_i o_i).$$

For each $N \in \mathbb{N}$, let $\Lambda_N := \Lambda \cap \limsup_{\gamma \in \Gamma} O_N(o, \gamma o)$, that is,

$$\Lambda_N = \{\xi \in \Lambda : \exists \gamma_\ell \rightarrow \infty \text{ in } \Gamma \text{ such that } \xi \in O_N(o, \gamma_\ell o) \text{ for all } \ell \geq 1\}.$$

There exists a constant $c_N > 0$ such that for any $\gamma \in \Gamma$, the shadow $O_N(o, \gamma o)$ is contained in a ball $B(\xi_\gamma, c_N e^{-\min \mu(\gamma)})$ for some $\xi_\gamma \in \mathcal{F}$; in particular, the diameter of $O_N(o, \gamma o)$ is at most $2c_N e^{-\min \mu(\gamma)}$.

It follows from Theorem 2.1 that if a sequence $\gamma_i \in \Gamma$ tends to infinity, then $\min \mu(\gamma_i) \rightarrow \infty$. Hence, for any fixed $t > 0$, we have

$$\Lambda_N \subset \bigcup_{\gamma \in \Gamma, \min \mu(\gamma) > t} O_N(o, \gamma o). \quad (3.4)$$

Let $s > \delta_{\min}$ be any number. Since $\sum_{\gamma \in \Gamma} e^{-s \min \mu(\gamma)} < \infty$,

$$\lim_{t \rightarrow \infty} \sum_{\gamma \in \Gamma, \min \mu(\gamma) > t} e^{-s \min \mu(\gamma)} = 0.$$

Since $\{O_N(o, \gamma o) : \gamma \in \Gamma, \min \mu(\gamma) > t\}$ is a cover of Λ_N with subsets whose diameter is at most $2c_N e^{-\min \mu(\gamma)}$, this implies that the s -dimensional Hausdorff measure of Λ_N is equal to zero. Since $s > \delta_{\min}$ is arbitrary, we get

$$\dim \Lambda_N \leq \delta_{\min}.$$

Since Λ is equal to the conical limit set Λ_c by (2.3), we have $\Lambda = \cup_{N \in \mathbb{N}} \Lambda_N$. Consequently,

$$\dim \Lambda \leq \sup_{N \in \mathbb{N}} \dim \Lambda_N \leq \delta_{\min}. \quad (3.5)$$

Since, for any $s > 0$,

$$\sum_{\gamma \in \Gamma} e^{-s \min \mu(\gamma)} \leq \sum_{i=1}^k \sum_{\sigma \in \Delta} e^{-s d_i(\rho_i(\sigma) o, o)},$$

the series $\sum_{\gamma \in \Gamma} e^{-s \min \mu(\gamma)}$ converges when $s > \max_i \delta_{\rho_i}$. It implies that

$$\delta_{\min} \leq \max_i \delta_{\rho_i}. \quad (3.6)$$

Since the projection map $\Lambda \rightarrow \Lambda_{\rho_i}$ is Lipschitz, we have

$$\max_i \dim \Lambda_{\rho_i} \leq \dim \Lambda. \quad (3.7)$$

By combining (3.5), (3.6), (3.7), and the fact that $\dim \Lambda_{\rho_i} = \delta_{\rho_i}$ by Sullivan [27], we have

$$\max_i \dim \Lambda_{\rho_i} \leq \dim \Lambda \leq \delta_{\min} \leq \max_i \delta_{\rho_i} = \max_i \dim \Lambda_{\rho_i}.$$

This finishes the proof.

4. FIBERED DYNAMICAL SYSTEMS AND $\ker \psi_u$ -COORDINATE MAP

Recall that for $u \in \text{int } \mathcal{L}$, ψ_u denotes the unique linear form tangent to ψ_Γ at u , as was given in Theorem 2.4). In this section, for each direction u in the interior of the limit cone, we discuss the $\ker \psi_u$ -trivial bundle over a compact space Z , associated to the dynamics of one-dimensional diagonal flow in the direction u , and show that the $\ker \psi_u$ -coordinate map $\hat{K}_u(z, t)$ of this bundle decays sublinearly as time $t \rightarrow \infty$ for almost all z with respect to the measure m_u of maximal entropy (see Theorem 4.2 and Corollary 4.4). The results in this section will be used as main ingredients of our estimates on the Hausdorff dimension of directional limit sets in section 5.

We continue to use notation $M_i, K_i, A_i, M, K, A, g^\pm$ from the last section. Now for each $1 \leq i \leq k$, the map

$$[g_i] \rightarrow (g_i^+, g_i^-, \beta_{g_i^+}(o_i, g_i o_i))$$

gives an $\text{SO}^\circ(n_i, 1)$ -equivariant homeomorphism between $\text{SO}^\circ(n_i, 1)/M_i$ and $\{(\xi_i, \eta_i) \in \mathbb{S}^{n_i-1} \times \mathbb{S}^{n_i-1} : \xi_i \neq \eta_i\} \times \mathbb{R}$, where the left $\text{SO}^\circ(n_i, 1)$ -action on the latter space is given by

$$g_i \cdot (\xi_i, \eta_i, s) = (g_i \xi_i, g_i \eta_i, s + \beta_{\xi_i}(g_i^{-1} o_i, o_i)).$$

This homeomorphism is called the Hopf parametrization of $\text{SO}^\circ(n_i, 1)/M_i$ under which the right A_i -action on $\text{SO}^\circ(n_i, 1)/M_i$ corresponds to the translation flow on \mathbb{R} .

For $\xi \in \mathcal{F} = \prod_{i=1}^k \mathbb{S}^{n_i-1}$, we write ξ_i for its i -th component. We set $\mathcal{F}^{(2)} = \{(\xi, \eta) \in \mathcal{F} \times \mathcal{F} : \xi_i \neq \eta_i \text{ for all } i\}$. Then the Hopf parametrization of

$\mathrm{SO}^\circ(n_i, 1)/M_i$ extends to the Hopf parametrization of G/M componentwise, and gives the G -equivariant homeomorphism $G/M \simeq \mathcal{F}^{(2)} \times \mathfrak{a}$ given by

$$[g] \rightarrow (g^+, g^-, \beta_{g^+}(o, go)) \quad \text{where } g^\pm = (g_i^\pm).$$

Set $\Lambda^{(2)} = \mathcal{F}^{(2)} \cap (\Lambda \times \Lambda)$. Then $\Omega := \Gamma \backslash (\Lambda^{(2)} \times \mathfrak{a})$ is identified with the closed subspace $\{[g] \in \Gamma \backslash G/M : g^\pm \in \Lambda\}$ of $\Gamma \backslash G/M$ via the Hopf parameterization.

Trivial $\ker \psi_u$ -vector bundle. We fix a unit vector $u \in \mathrm{int} \mathcal{L}$ in the rest of this section. Consider the Γ -action on the space $\Lambda^{(2)} \times \mathbb{R}$ by

$$\gamma \cdot (\xi, \eta, s) = (\gamma\xi, \gamma\eta, s + \psi_u(\beta_\xi(\gamma^{-1}o, o))).$$

The reparametrization theorems for Anosov groups ([4, Prop. 4.1], [7, Thm. 4.15]) imply that Γ acts properly discontinuously and cocompactly on $\Lambda^{(2)} \times \mathbb{R}$. Hence $Z := \Gamma \backslash (\Lambda^{(2)} \times \mathbb{R})$ is a compact space. Now the Γ -equivariant projection $\Lambda^{(2)} \times \mathfrak{a} \rightarrow \Lambda^{(2)} \times \mathbb{R}$ given by $(\xi, \eta, v) \mapsto (\xi, \eta, \psi_u(v))$ induces an affine bundle with fiber $\ker \psi_u$:

$$\pi : \Omega = \Gamma \backslash (\Lambda^{(2)} \times \mathfrak{a}) \rightarrow Z = \Gamma \backslash (\Lambda^{(2)} \times \mathbb{R}).$$

It is well-known that such a bundle is indeed a trivial vector bundle, and hence we can choose a continuous global section

$$\mathfrak{s} : Z \rightarrow \Omega$$

so that $\pi \circ \mathfrak{s} = \mathrm{id}_Z$. Denote by $\{\tau_t : t \in \mathbb{R}\}$ the flow on Z given by translations by t on \mathbb{R} . For $v = (v_1, \dots, v_k) \in \mathfrak{a}$, we write

$$a_v = (a_{v_1}^{(1)}, \dots, a_{v_k}^{(k)}) \in A.$$

Definition 4.1 ($\ker \psi_u$ -coordinate map). We define a continuous $\ker \psi_u$ -valued map

$$\hat{K}_u : Z \times \mathbb{R} \rightarrow \ker \psi_u$$

as follows: for $z \in Z$ and $t \in \mathbb{R}$,

$$\mathfrak{s}(z)a_{tu} = \mathfrak{s}(z\tau_t)a_{\hat{K}_u(z,t)}. \quad (4.1)$$

Let \mathfrak{m}_u denote the ψ_u -Bowen-Margulis-Sullivan measure on Z ; that is, \mathfrak{m}_u is the unique τ_t -invariant probability measure on Z which is locally equivalent to $\nu_u \otimes \nu_u \otimes ds$. It follows from [4] that \mathfrak{m}_u is the measure of maximal entropy and in particular ergodic for the τ_t -flow.

Theorem 4.2. For \mathfrak{m}_u -a.e. $z \in Z$, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \hat{K}_u(z, t) = 0. \quad (4.2)$$

Proof. Combining the reparametrization theorem [4, Prop. 4.1] and [26, Prop. 3.5], we deduce that there exists a Hölder continuous function $F : Z \rightarrow \ker \psi_u$ with $\int_Z F d\mathfrak{m}_u = 0$ such that for all $z \in Z$ and $t \in \mathbb{R}$,

$$\hat{K}_u(z, t) = \int_0^t F(z\tau_s) ds + E(z) - E(z\tau_t) \quad (4.3)$$

for some bounded measurable function $E : Z \rightarrow \ker \psi_u$. The Birkhoff ergodic theorem for the τ_s flow on (Z, \mathfrak{m}_u) implies that for \mathfrak{m}_u -almost all $z \in Z$, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t F(z\tau_s) ds = \int_Z F d\mathfrak{m}_u = 0;$$

hence

$$\lim_{t \rightarrow \infty} \frac{1}{t} \hat{K}_u(z, t) = 0$$

since E is bounded. \square

Fix a compact subset $D \subset G/M$ such that $\mathfrak{s}(Z) = \Gamma \backslash \Gamma D$, and for each $z \in Z$, write $\mathfrak{s}(z) = \Gamma \tilde{\mathfrak{s}}(z)$ for some $\tilde{\mathfrak{s}}(z) \in D$. Hence

$$\Lambda^{(2)} \times \mathfrak{a} = \Gamma D a_{\ker \psi_u}. \quad (4.4)$$

We will sometimes consider D as a right M -invariant subset of G by abuse of notation.

Lemma 4.3. *For any $g \in G$ with $g^\pm \in \Lambda$, there exist $z_g \in Z$ and $w_g \in \ker \psi_u$ such that for all $t \in \mathbb{R}$, there exists $\gamma_{g,t} \in \Gamma$ satisfying*

$$\gamma_{g,t} g a_{tu} = \tilde{\mathfrak{s}}(z_g \tau_t) a_{\hat{K}_u(z_g, t) + w_g}. \quad (4.5)$$

Proof. By (4.4), there exist $\gamma \in \Gamma$, $z \in Z$ and $w \in \ker \psi_u$ such that $\gamma g = \tilde{\mathfrak{s}}(z) a_w$, and hence

$$\gamma g a_{tu} = \tilde{\mathfrak{s}}(z) a_{tu+w}.$$

On the other hand, by (4.1), there exists $\gamma_{z,t} \in \Gamma$ such that

$$\gamma_{z,t} \tilde{\mathfrak{s}}(z) a_{tu} = \tilde{\mathfrak{s}}(z \tau_t) a_{\hat{K}_u(z, t)}.$$

Therefore,

$$\gamma_{z,t} \gamma g a_{tu} = \tilde{\mathfrak{s}}(z \tau_t) a_{\hat{K}_u(z, t) + w}.$$

It remains to set $\gamma_{g,t} = \gamma_{z,t} \gamma$. \square

For each $g \in G$ with $g^\pm \in \Lambda$, we choose $z_g \in Z$ and $w_g \in \ker \psi_u$ as given by Lemma 4.3. We also set $K_u^\dagger(g, t) := \hat{K}_u(z_g, t) + w_g \in \ker \psi_u$, so that for all $t \in \mathbb{R}$,

$$\gamma_{g,t} g a_{tu} \in D a_{K_u^\dagger(g, t)}. \quad (4.6)$$

Corollary 4.4. *For ν_u -a.e. $\xi \in \Lambda$, there exists $\Lambda(\xi) \subset \Lambda$ with $\nu_u(\Lambda(\xi)) = 1$ such that for any $g \in G$ with $g^+ = \xi$ and $g^- \in \Lambda(\xi)$, we have*

$$\lim_{t \rightarrow \infty} \frac{1}{t} K_u^\dagger(g, t) = 0. \quad (4.7)$$

Proof. Since \mathfrak{m}_u is equivalent to $\nu_u \otimes \nu_u \otimes ds$, Theorem 4.2 implies that for ν_u -a.e. $\xi \in \Lambda$, there exists a Γ -invariant measurable subset $\Lambda(\xi) \subset \Lambda$ with $\nu_u(\Lambda(\xi)) = 1$ such that $\lim_{t \rightarrow \infty} \frac{1}{t} \hat{K}_u([\xi, \eta, s], t) = 0$ for any $\eta \in \Lambda(\xi)$ and $s \in \mathbb{R}$. For each ξ satisfying this, let $g \in G$ with $g^+ = \xi$ and $g^- \in \Lambda(\xi)$. It suffices to show that g satisfies (4.7). Let $z = [(\xi_1, \eta_1, s)] \in Z$ be such that $\gamma_{g,0} g = \tilde{\mathfrak{s}}(z) a_{K^\dagger(g,0)}$ as given by Lemma 4.3 and (4.6). It follows that

$g^+ = \xi \in \Gamma\xi_1$ and $g^- \in \Gamma\eta_1$, and hence $z_g = [(\xi, \eta_1, s)]$ for $\eta_1 \in \Lambda(\xi)$. Therefore, $\hat{K}_u(z_g, t)/t \rightarrow 0$ as $t \rightarrow \infty$. Since $K_u^\dagger(g, t) - \hat{K}_u(z_g, t) = w_g$ which is independent of t , we get $K_u^\dagger(g, t)/t \rightarrow 0$ as $t \rightarrow \infty$. \square

Let m_u^{BMS} denote the Bowen-Margulis-Sullivan measure on $\Omega \subset \Gamma \backslash G$ given by $m_u^{\text{BMS}} = m_u \otimes \text{Leb}|_{\ker \psi_u} \otimes dm$ where dm denotes the Haar measure on M ; this is an A -invariant ergodic (infinite) Radon measure, as shown in [16]. We also remark that by [6], m_u^{BMS} is $\{a_{tu} : t \in \mathbb{R}\}$ -ergodic if and only if $k \leq 3$. In terms of this measure, Corollary 4.4 can be formulated as the following which may be regarded as an analogue of Sullivan's result [27, Coro. 19], which predates his logarithm law.

Theorem 4.5. *For m_u^{BMS} -a.e. $x \in \Gamma \backslash G$, we have*

$$\lim_{t \rightarrow \infty} \frac{d(xa_{tu}o, o)}{t} = 0.$$

Since D is compact, this theorem follows from Corollary 4.4 in view of (4.6).

5. HAUSDORFF DIMENSION OF Λ_u AND LOCAL BEHAVIOR OF ν_u

For each $u = (u_1, \dots, u_k) \in \mathfrak{a}^+$, the u -directional limit set $\Lambda_u \subset \Lambda$ is defined as

$$\Lambda_u := \{\xi \in \mathcal{F} : \liminf_{t \rightarrow +\infty} d(\xi(tu_1, \dots, tu_k), \Gamma o) < \infty\}$$

where $\xi(\cdot)$ is defined as in (2.1).

In this section, we obtain estimates on $\dim \Lambda_u$ for $u \in \text{int } \mathcal{L}$. We will obtain an upper bound for $\dim \Lambda_u$ for any $k \geq 1$ but our lower bound is obtained only when $k \leq 3$; the main reasons are that

- (1) the lower bound is deduced from local estimates on ν_u (Theorem 5.6) using the mass distribution principle and
- (2) the directional limit set Λ_u has positive ν_u -measure if and only if $k \leq 3$ (Theorem 5.7).

In the whole section, we fix a unit vector

$$u = (u_1, \dots, u_k) \in \text{int } \mathcal{L}.$$

However, note that the statements below still hold for an arbitrary vector in $\text{int } \mathcal{L}$ since all quantities are homogeneous. We also set

$$M_u = \max_{1 \leq i \leq k} u_i, m_u = \min_{1 \leq i \leq k} u_i, \text{ and } \delta_u := \psi_u(u) = \psi_\Gamma(u) > 0. \quad (5.1)$$

Upper bound for dimension. For any $N \in \mathbb{N}$, set

$$\Gamma_N(u) := \{\gamma \in \Gamma : \|\mu(\gamma) - t_\gamma u\| \leq N \text{ for some } t_\gamma > 0\}$$

and

$$\Lambda_N^*(u) := \limsup_{t \rightarrow \infty} \bigcup_{\gamma \in \Gamma_N(u), \|\mu(\gamma)\| \geq t} O_N(o, \gamma o),$$

where $O_N(o, \gamma o)$ is a shadow defined as in (3.3).

We will use the following simple observation:

Lemma 5.1. *We have*

$$\Lambda_u \subset \bigcup_{N \in \mathbb{N}} \Lambda_N^*(u).$$

Proof. Let $\xi \in \Lambda_u$. Choose any $g \in K$ such that $g^+ = \xi$. Then $\xi(tu) = ga_{tu}o$, $t \geq 0$, is a geodesic ray toward ξ . By the definition of Λ_u , there exist $N > 0$ and sequences $t_\ell \rightarrow \infty$, $\gamma_\ell \in \Gamma$ such that $d(\xi(t_\ell u), \gamma_\ell o) \leq N$ for all $\ell \geq 1$. Note that the absolute value of each component of $\mu(\gamma_\ell) - t_\ell u$ is bounded by N , and hence $\|\mu(\gamma_\ell) - t_\ell u\| \leq kN$. Replacing N with kN , if necessary, we may assume that $\gamma_\ell \in \Gamma_N(u)$ for all $\ell \geq 1$. By the definition of shadows in (3.3), it follows that $\xi \in O_N(o, \gamma_\ell o)$ for all $\ell \geq 1$. As $\gamma_\ell \in \Gamma_N(u)$, we have $\|\mu(\gamma_\ell)\| \geq \|u\|t_\ell - N$ and hence $\mu(\gamma_\ell) \rightarrow \infty$ as $\ell \rightarrow \infty$. Therefore $\xi \in \Lambda_N^*(u)$; this completes the proof. \square

Theorem 5.2. *For any $k \geq 1$, we have*

$$\dim \Lambda_u \leq \frac{\delta_u}{m_u}. \quad (5.2)$$

Proof. Fix $N \in \mathbb{N}$. For each $\gamma \in \Gamma_N(u)$, we fix $t_\gamma > 0$ such that $\|\mu(\gamma) - t_\gamma u\| \leq N$, which exists by the definition of $\Gamma_N(u)$. Then there exists $d_N > 0$ such that for any $\gamma \in \Gamma_N(u)$, the shadow $O_N(o, \gamma o)$ is contained in a ball $B(\xi_\gamma, d_N e^{-t_\gamma m_u})$ for some $\xi_\gamma \in \mathcal{F}$. Since $\|\mu(\gamma) - t_\gamma u\| \leq N$, by applying ψ_u , we get

$$|\psi_u(\mu(\gamma))\delta_u^{-1} - t_\gamma| \leq N\delta_u^{-1}\|\psi_u\|_{\text{op}}$$

where $\|\psi_u\|_{\text{op}}$ denotes the operator norm of ψ_u .

Therefore, for some constant $d'_N \geq 1$, we have that for all $\gamma \in \Gamma_N(u)$, we have

$$O_N(o, \gamma o) \subset B(\xi_\gamma, d_N e^{-t_\gamma m_u}) \subset B(\xi_\gamma, d'_N e^{-m_u \delta_u^{-1} \psi_u(\mu(\gamma))});$$

in particular, the diameter of $O_N(o, \gamma o)$ is at most $2d'_N e^{-m_u \delta_u^{-1} \psi_u(\mu(\gamma))}$. Moreover, for any $t > 1$, $\{O_N(o, \gamma o) : \gamma \in \Gamma_N(u), \|\mu(\gamma)\| \geq t\}$ is a cover of $\Lambda_N^*(u)$.

Let $s > \delta_u/m_u$ be any number. By Theorem 2.4(3), we have

$$\lim_{t \rightarrow \infty} \sum_{\gamma \in \Gamma, \|\mu(\gamma)\| \geq t} e^{-s m_u \delta_u^{-1} \psi_u(\mu(\gamma))} = 0.$$

It implies that the s -dimensional Hausdorff measure of $\Lambda_N^*(u)$ is zero. Since $s > \delta_u/m_u$ is arbitrary, it follows that

$$\dim \Lambda_N^*(u) \leq \frac{\delta_u}{m_u}.$$

Since

$$\Lambda_u \subset \bigcup_{N \in \mathbb{N}} \Lambda_N^*(u)$$

by Lemma 5.1, this implies the desired bound: $\dim \Lambda_u \leq \frac{\delta_u}{m_u}$. \square

Remark 5.3. We can replace $B(\xi_\gamma, d_N e^{-t_\gamma m_u})$ with $e^{t_\gamma(M_u - m_u)} \sum_{i=1}^k (n_i - 1)$ balls of radius $d_N e^{-t_\gamma M_u}$. We then have the upper bound

$$\dim \Lambda_u \leq \frac{\delta_u + (M_u - m_u) \sum_{i=1}^k (n_i - 1)}{M_u}$$

which is smaller than the upper bound in (5.2) when $m_u \sum_{i=1}^k (n_i - 1) < \delta_u$.

The local size of ν_u . We define the following subset of Λ :

$$\Lambda_u^* = \left\{ \xi \in \Lambda : \begin{array}{l} \exists \Lambda(\xi) \subset \Lambda \text{ with } \nu_u(\Lambda(\xi)) = 1 \text{ such that} \\ \lim_{t \rightarrow \infty} \frac{1}{t} K_u^\dagger(g, t) = 0 \\ \text{for any } g \in G \text{ with } g^+ = \xi \text{ and } g^- \in \Lambda(\xi) \end{array} \right\}. \quad (5.3)$$

Note that Λ_u^* is not necessarily a subset of Λ_u .

By Corollary 4.4, we have

$$\nu_u(\Lambda_u^*) = 1.$$

We will be using the following two lemmas.

Lemma 5.4. *There exists a compact subset $\mathcal{S} \subset G$ such that for any $\xi \in \Lambda$ and for any measurable subset $\Lambda' \subset \Lambda$ with $\nu_u(\Lambda') = 1$, there exists $g \in \mathcal{S}$ such that $g^+ = \xi$ and $g^- \in \Lambda'$.*

Proof. This lemma is proved in [15, Lem. 10.6] for $\Lambda' = \Lambda$. It suffices to replace \mathcal{S} by the one-neighborhood of \mathcal{S} , say, \mathcal{S}_0 . Let $\xi \in \Lambda$, and $g \in \mathcal{S}$ be such that $g^+ = \xi$ and $g^- \in \Lambda$. Then we can find a neighborhood \mathcal{O} of g^- such that for any $\eta \in \mathcal{O}$, there exists $h \in \mathcal{S}_0$ such that $h^+ = \xi$ and $h^- = \eta$. Since $\nu_u(\Lambda') = 1$, we have Λ' is dense in Λ , and hence $\Lambda' \cap \mathcal{O} \neq \emptyset$. This implies the claim. \square

Note that the proof of this lemma can be extended to general Anosov subgroups as it only uses [15, Lem. 10.6].

The following shadow lemma is obtained for any Γ -conformal measure of any discrete Zariski dense subgroup $\Gamma < G$:

Lemma 5.5 (Shadow lemma). [15, Lem. 7.8] *There exists $R_0 > 0$ such that for all $R > R_0$, there exists $c = c(\psi_u, R) \geq 1$ such that for any $\gamma \in \Gamma$,*

$$c^{-1} \cdot e^{-\psi_u(\mu(\gamma))} \leq \nu_u(O_R(o, \gamma o)) \leq c \cdot e^{-\psi_u(\mu(\gamma))}. \quad (5.4)$$

For $r > 0$, let $B(\xi_i, r)$ denote the ball in \mathbb{S}^{n_i-1} centered at ξ_i of radius r . The following theorem is one of two key ingredients of our proof for the lower bound of $\dim \Lambda_u$ (Corollary 5.9):

Theorem 5.6. *Let $k \geq 1$. There exists $C_1, C_2 > 0$ such that for any $\xi = (\xi_1, \dots, \xi_k) \in \Lambda_u^*$, and for any sufficiently small $\varepsilon > 0$, there exists $t_0 = t_{\varepsilon, \xi} > 0$ such that for all $t \geq t_0$,*

$$C_1 \cdot e^{-\delta_u(1+\varepsilon)t} \leq \nu_u \left(\prod_{i=1}^k B(\xi_i, e^{-u_i t}) \right) \leq C_2 \cdot e^{-\delta_u(1-\varepsilon)t}. \quad (5.5)$$

Proof. Choose $g \in \mathcal{S}$ such that $g^+ = \xi$ and $g^- \in \Lambda(\xi)$ where $\Lambda(\xi)$ is given in (5.3) and \mathcal{S} is a compact subset of G given in Lemma 5.4. Let $\varepsilon > 0$. By the definition (5.3) of Λ_u^* , there exists $t_0 = t_{\varepsilon, g} > 0$ such that for each $1 \leq i \leq k$, the absolute value of the i -th component of $K^\dagger(g, t) \in \mathfrak{a} = \mathbb{R}^k$ is

$$\text{at most } \frac{\varepsilon u_i t}{4} \quad \text{for all } t > t_0.$$

Recall the definition of $\gamma_{g,t}$ from (4.6): $\gamma_{g,t} g a_{tu} = d_t a_{K^\dagger(g,t)}$ where $d_t \in D$. Therefore $\gamma_{g,t}^{-1} = g a_{tu - K^\dagger(g,t)} d_t^{-1}$. Let q be the diameter of $D^{-1}o$.

Note that there exists $c_0 > 0$ such that for all $t > t_0$,

$$O_1(o, \gamma_{g,t}^{-1}o) \subset O_{q+1}(o, g a_{tu - K^\dagger(g,t)}o) \subset \prod_{i=1}^k B(\xi_i, c_0 e^{-u_i(1-\varepsilon/4)t});$$

the first inclusion is immediate from the definition of the shadows. Hence we deduce from Lemma 5.5 that for all $t > \max(t_0, 2 \log c_0 / (u_i \varepsilon))$,

$$\beta \cdot e^{-\delta_u t} \leq \nu_u \left(\prod_{i=1}^k B(\xi_i, c_0 e^{-u_i(1-\varepsilon/4)t}) \right) \leq \nu_u \left(\prod_{i=1}^k B(\xi_i, e^{-u_i(1-\varepsilon/2)t}) \right)$$

for some constant $\beta = \beta(\psi_u, D) > 0$. By reparametrizing $(1 - \varepsilon/2)t = s$, this implies the lower bound in (5.5).

On the other hand, for all $t \geq t_0$,

$$\begin{aligned} \prod_{i=1}^k B(\xi_i, e^{-u_i(1+\varepsilon/4)t}) &\subset O_p(o, g a_{(1+\varepsilon/4)tu}o) \\ &\subset O_p(o, g a_{tu - K^\dagger(g,t)}o) \subset O_{p+q}(o, \gamma_{g,t}^{-1}o) \end{aligned}$$

where p depends only on \mathcal{S} . Hence, by (5.4),

$$\nu_u \left(\prod_{i=1}^k B(\xi_i, e^{-u_i(1+\varepsilon/4)t}) \right) \leq c e^{-\psi_u(\mu(\gamma_{g,t}^{-1}))}$$

where $c = c(\psi_u, p + q)$. Recalling that $\gamma_{g,t}^{-1} = g a_{tu - K^\dagger(g,t)} d_t^{-1}$, we have

$$\|\mu(\gamma_{g,t}^{-1}) - (tu - K^\dagger(g,t))\| \leq \|\mu(g)\| + \|\mu(d_t)\| \leq \beta'$$

where $\beta' = 2 \max\{\|\mu(h)\| : h \in \mathcal{S} \cup D\}$.

Since $K^\dagger(g, t) \in \ker \psi_u$, we have for all $t > t_0$, $|\psi_u(\mu(\gamma_{g,t}^{-1})) - t\delta_u| \leq \|\psi_u\|_{\text{op}}\beta'$. Therefore we have

$$\nu_u \left(\prod_{i=1}^k B(\xi_i, e^{-u_i(1+\varepsilon/4)t}) \right) \leq C_2 e^{-\delta_u t}$$

where $C_2 > 0$ depends only on \mathcal{S} , D and ψ_u . In other words, for all $t > 2t_0$,

$$\nu_u \left(\prod_{i=1}^k B(\xi_i, e^{-u_i t}) \right) \leq C_2 e^{-(1-\varepsilon)\delta_u t}.$$

This proves the upper bound in (5.5). \square

Lower bound for dimension. The second key ingredient of the proof of Corollary 5.9 is the following recent result:

Theorem 5.7. [6, Thm. 1.6] *We have*

$$\nu_u(\Lambda_u) = \begin{cases} 1 & \text{if } k \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

This together with Corollary 4.4 implies:

Corollary 5.8. *If $k \leq 3$, then $\nu_u(\Lambda_u^* \cap \Lambda_u) = 1$.*

We are now ready to prove the following lower bound on $\dim \Lambda_u$:

Corollary 5.9. *For $k \leq 3$, we have*

$$\dim \Lambda_u \geq \dim(\Lambda_u^* \cap \Lambda_u) \geq \frac{\delta_u}{M_u}$$

where δ_u and M_u are given in (5.1).

Proof. Recall that $B(\xi, r)$ denotes the ball of radius $r > 0$ centered at $\xi = (\xi_1, \dots, \xi_k)$ in $\mathcal{F} = \prod_{i=1}^k \mathbb{S}^{n_i-1}$ with respect to the Riemannian metric. Since $u_i/M_u \leq 1$, we have that for all $t > 0$

$$B(\xi, e^{-t}) \subset \prod_{i=1}^k B(\xi_i, e^{-u_i t/M_u}).$$

Fix $\varepsilon > 0$. Therefore Theorem 5.6 implies that there exists $C > 0$, independent of $\varepsilon > 0$, such that for any $\xi \in \Lambda_u^*$ and for all sufficiently small $r = r_{\varepsilon, \xi} > 0$,

$$\nu_u(B(\xi, r)) \leq C \cdot r^{(1-\varepsilon)\delta_u/M_u}.$$

Since $\nu_u(\Lambda_u^* \cap \Lambda_u) = 1$ by Corollary 5.8, the Mass distribution property (more precisely, Rogers-Taylor theorem [3, Theorem 4.3.3]) now implies that

$$\dim(\Lambda_u^* \cap \Lambda_u) \geq (1 - \varepsilon)\delta_u/M_u.$$

Since $\varepsilon > 0$ is arbitrary, this proves the claim. \square

Remark 5.10. Our proofs of Theorems 1.1 and 1.2 work in the same way for the product $G = \prod_{i=1}^k G_i$ where $G_i = \text{Isom}^\circ(X_i)$ is a simple Lie group for a Riemannian symmetric space X_i of rank one. The Furstenberg boundary \mathcal{F} of G is the product $\prod_{i=1}^k \partial X_i$ of geometric boundaries of X_i , and the Hausdorff dimension of the limit set of $\Gamma < G$ is to be computed with respect to a certain sub-Riemannian metric on \mathcal{F} which is invariant under a maximal compact subgroup of G , as described in [8]. In these situations, shadows are comparable to metric balls by [8, Thm. 2.2] and $\dim \Lambda_{\rho_i} = \delta_{\rho_i}$ by [9, Thm 6.1]. Given these, the discussions in sections 2–5 remain valid.

6. EXAMPLES OF SYMMETRIC GROWTH INDICATOR FUNCTIONS

Given a self-joining subgroup $\Gamma < G$, there doesn't seem to be any general method to compute the maximal growth direction. In this section, we provide a class of geometric examples of Γ whose growth indicator functions are symmetric, and hence whose maximal growth direction u_Γ is parallel to $(1, \dots, 1)$.

Let Δ be a finitely generated group, and $\text{Out } \Delta$ denote its outer automorphism group, i.e., the group of automorphisms of Δ modulo the inner automorphisms. Note that, for a representation $\rho : \Delta \rightarrow \text{SO}^\circ(n, 1)$ and $\iota \in \text{Out } \Delta$, $\rho \circ \iota$ is well-defined up to conjugation in $\text{SO}^\circ(n, 1)$.

Lemma 6.1. *Let $k \geq 2$. Let $\rho_1 : \Delta \rightarrow \text{SO}^\circ(n, 1)$ be a non-elementary convex cocompact faithful representation and $\iota \in \text{Out } \Delta$ be of order k . Let $\rho_i = \rho_1 \circ \iota^{i-1}$ for $2 \leq i \leq k$ and let $\Gamma_\iota := (\prod_{i=1}^k \rho_i)(\Delta)$. Then*

$$\psi_{\Gamma_\iota} = \psi_{\Gamma_\iota} \circ \theta \quad \text{and} \quad u_{\Gamma_\iota} = \frac{1}{\sqrt{k}}(1, \dots, 1)$$

where θ denotes the cyclic permutation $(x_1, \dots, x_k) \mapsto (x_2, \dots, x_k, x_1)$.

Proof. For each $1 \leq n \leq k$, let $\Gamma^{(n)} = (\prod_{i=1}^k \rho_i \circ \iota^n)(\Delta)$. Since $\iota^k = 1$ in $\text{Out } \Delta$, $\Gamma^{(n)}$ can be regarded as a group obtained by permuting coordinates in a cyclic way. Hence,

$$\mathcal{L}_{\Gamma^{(n)}} = \theta(\mathcal{L}_{\Gamma^{(n-1)}}) \quad \text{and} \quad \psi_{\Gamma^{(n)}} = \psi_{\Gamma^{(n-1)}} \circ \theta^{-1}. \quad (6.1)$$

However, $\Gamma^{(n)} = \Gamma_\iota$ for all n ; since applying an automorphism to all coordinates does not change the group. Hence, (6.1) implies that $\mathcal{L}_{\Gamma_\iota}$ and ψ_{Γ_ι} are invariant under the cyclic permutation θ of coordinates. \square

Examples in $\mathbb{H}^2 \times \mathbb{H}^2$. Let us describe some examples to which Lemma 6.1 can be applied. We begin in dimension 2. For a closed orientable surface S of genus $g \geq 2$, one can obtain homeomorphisms $\iota : S \rightarrow S$ of order 2 in a number of ways. Figure 1 indicates how this can be done: Arrange the surface in \mathbb{R}^3 so that it is symmetric by a 180° rotation. There are several possibilities distinguished by the number of intersection points of the surface with the rotation axis, which yield fixed points of ι .

In order for the example $(\rho, \rho \circ \iota)$ not to be trivial, we need the representations not to be conjugate in $\text{SO}^\circ(2, 1)$. That is, ρ should not represent a

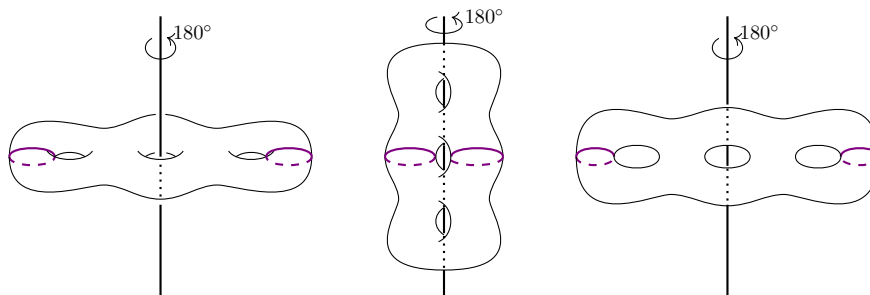


FIGURE 1. Examples of involutions $\iota \in \text{Out } \pi_1(S)$ where S is of genus 3. Indicated curves are mapped to each other by ι .

point of Teichmüller space $\mathcal{T}(S)$ which is fixed by ι . This is always possible when $g \geq 3$; to see this, note that there are disjoint, non-homotopic simple closed curves exchanged by ι in each case. They can be assigned different lengths by a hyperbolic structure, which would then not be fixed by ι . In genus 2, one just needs to avoid the hyperelliptic involution – the one with 6 fixed points – which fixes every point in $\mathcal{T}(S)$. All other rotations will do.

Examples in $\mathbb{H}^3 \times \mathbb{H}^3$. Examples involving 3-manifolds are also plentiful. Consider for example a “book of I -bundles” constructed as follows (see Anderson-Canary [1]). Let S_1, \dots, S_ℓ be ℓ copies of a surface of genus $g \geq 1$ with one boundary component and let Y be the 2-complex obtained by identifying all the boundary circles to one. A choice of cyclic order c on the ℓ surfaces determines a thickening of Y to a 3-manifold N_c : form $S_i \times [-1, 1]$ for each i , and identify the annulus $\partial S_i \times [0, 1]$ with $\partial S_j \times [-1, 0]$ whenever j follows i in the order c (the identification should take $[0, 1] \rightarrow [-1, 0]$ by an orientation-reversing homeomorphism, and should respect the original identification of the boundary circles). See Figure 2.

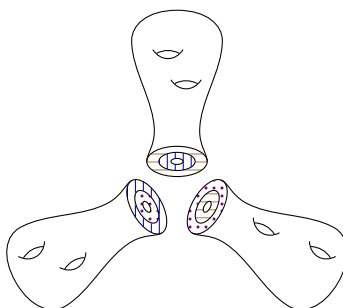


FIGURE 2. Book of I -bundles with three surfaces and patterns indicating the identification.

The result N_c is homotopy equivalent to Y , and has ℓ boundary components of genus $2g$. It admits many convex cocompact hyperbolic structures: it is easy to construct one “by hand” by attaching Fuchsian structures along

the common boundary using the Klein-Maskit combination theorem [18]. The Ahlfors-Bers theory parametrizes all convex cocompact representations as the Teichmüller space of ∂N_c (cf. [17]). A permutation of $(1, \dots, \ell)$ induces a homeomorphism of Y which extends to a homotopy equivalence of N_c which, if the permutation does not preserve or reverse the cyclic order, will not correspond to a homeomorphism. Selecting such a permutation of order 2, we have an automorphism that cannot be an isometry for any hyperbolic structure on N_c . (Even if it does correspond to a homeomorphism one can choose the hyperbolic structure on N_c using a point in $\mathcal{T}(\partial N_c)$ that is not symmetric with respect to the involution).

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