Exercise V - mandatory

Math 320a/520a - Fall Semester 2017

Due Tuesday, 10/10/2017, 2:30 PM

1. Show that the definitions of the Lebesgue integral on simple functions are consistent. Specifically, if a simple function is written as $s(x) = \sum_{i=1}^{m} a_i \chi_{E_i}$ or as $s(x) = \sum_{j=1}^{n} b_j \chi_{F_j}$, where $E_1, \ldots, E_m$ and $F_1, \ldots, F_n$ are in the σ-algebra, then $\sum_{i=1}^{m} a_i \mu(\chi_{E_i}) = \sum_{i=j}^{n} b_j \mu(\chi_{F_i}) = \int s^+ d\mu - \int s^- d\mu$, where the integrals on the right hand side are defined using the supremum definition (i.e., $\int f d\mu = \sup\{\int s_0 \, d\mu : s_0$ is simple and $0 \leq s_0 \leq f\}$).

2. Lusin's Theorem states that if $f : [a, b] \to \mathbb{R}$ (for some real $a < b$) is a Borel measurable function, then for every $\varepsilon > 0$ there exists a compact set $E \subseteq [a, b]$ with Lebesgue measure $m(E) > b - a - \varepsilon$ such that the restriction $f|_E : E \to \mathbb{R}$ for f to E is a continuous function (on E). Prove that this statement holds for:

   (a) Characteristic functions;
   (b) Simple functions;
   (c) Bounded nonnegative functions;
   (d) All functions from $[a, b]$ to $\mathbb{R}$.

   Can this theorem be generalized by replacing $[a, b]$ with a Borel set A with finite Lebesgue measure and replacing $m(E) > b - a - \varepsilon$ with $m(E) > m(A) - \varepsilon$? Explain your answer.

3. Show that if $f : X \to [0, \infty)$ is μ-integrable then for every $\varepsilon > 0$ there exists a measurable set $E \subseteq X$ with finite measure $\mu(E) < \infty$ such that $\int_E f d\mu \geq \int f d\mu - \varepsilon$.

4. Show that if $f : X \to \mathbb{R}$ is measurable and nonnegative, and μ is σ-finite then there exist simple functions $s_1, s_2, \ldots$ increasing to f at each point such that $\mu(\{x : s_n(x) \neq 0\}) < \infty$ for each n.

5. Prove the following properties of the Lebesgue integral:

   (a) If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are integrable, and $f(x) \leq g(x)$ for all x, then $\int f d\mu \leq \int g d\mu$.
   (b) If $f : X \to \mathbb{C}$ is integrable then $\int c f d\mu = c \int f d\mu$ for all complex c
   (c) If $\mu(A) = 0$ then $\int_A f d\mu = 0$ for all integrable f.

6. Show that if μ and ν are two **finite** measures on the Borel σ-algebra on $[0, 1]$ such that $\int f d\mu = \int f d\nu$ for all real-valued continuous functions on $[0, 1]$, then $\mu = \nu$.

7. Show that if $f_1, f_2, \ldots : X \to [0, \infty)$ are integrable, $f_n \to f$ and $\int f_n d\mu \to \int f d\mu < \infty$, then for every $A \in \mathcal{A}$, $\int_A f_n d\mu \to \int_A f d\mu$.

8. Consider two integrable functions $f, g : X \to \mathbb{R}$. Suppose there are integrable $f_1, f_2, \ldots$ such that $f_n \to f$ a.e., and integrable $g_1 \geq |f_1|, g_2 \geq |f_2|, \ldots$ such that $g_n \to g$ a.e. and $\int g_n d\mu \to \int g d\mu$. Prove that $\int f_n d\mu \to \int f d\mu$.