# Exercise III - mandatory

Math 320a/520a - Fall Semester 2017

Due Tuesday, 09/26/2017, 2:30 PM

1. Let $\mu$ be a measure defined on the Borel $\sigma-$algebra. Show that if $\mu\Big((a,b)\Big) \leq b-a$ for any $a < b \in \mathbb{R}$ then $\mu(\mathbb{Q}) = 0$.

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and consider

$$\bar{\mathcal{A}} = \left\{E \cup F : E \in \mathcal{A}, \text{ and } F \subseteq N \text{ for some } N \in \mathcal{A},\ \mu(N) = 0\right\}.$$

   (a) Verify that $\mathcal{A} \subseteq \bar{\mathcal{A}}$.

   (b) Show that $\bar{\mathcal{A}}$ a $\sigma-$algebra.

   (c) Find an extension $\bar{\mu}$ of the measure $\mu$ (i.e., $\bar{\mu}(A) = \mu(A)$ for any $A \in \mathcal{A}$) such that $(X, \bar{\mathcal{A}}, \bar{\mu})$ is a complete measure space.
   *Notice:* you need to show that $\bar{\mu}$ is indeed a measure on $(X, \bar{\mathcal{A}})$, and that together they satisfy the definition of a complete measure space.

3. For any set $X \subseteq [0,1]^2$, define $\mu_n(X) = \frac{1}{n^2}\#\left\{(i,j) \in \{0,\ldots,n-1\}^2 : \left(\frac{i}{n},\frac{j}{n}\right) \in X\right\}$ and $\mu(X) = \limsup_{n\to\infty}\mu_n(X)$. Construct a set $X$ with Lebesgue measure $m(X) = 0$ such that $\mu(X) > 0$.

4. Prove the following properties of the Cantor set $C$ that is defined in Example 4.11 (page 32) in Bass's textbook:

   (a) The Cantor set is compact and it contains no intervals.

   (b) For any $x \in C$, every open set containing $x$ also contains some $x \neq y \in C$.

   (c) The Cantor set is uncountable but its Lebesgue measure is zero.

5. Describe the simplest mathematrical formulation you can find for **all** finite measures on the measurable space $\Big(\mathbb{N}, \mathcal{P}(\mathbb{N})\Big)$.

6. Give an example of a set $X$, an outer measure $\mu^*$, and a set $A \subseteq X$ such that $A$ is not $\mu^*-$measurable.

7. For any two sets $A, B \subseteq \mathbb{R}$, we define $A + B \triangleq \{a + b : a \in A, b \in B\}$. Is there a finite constant $C$ such that the Lebesgue measure satisfies $m(A+B) \leq C\Big(m(A)+m(B)\Big)$ whenever $A, B, A+B \in \mathcal{B}_{\mathbb{R}}$?

8. Let $\mathcal{B}_{\mathbb{R}}$ be the Borel $\sigma$-algebra on $\mathbb{R}$, and let $m$ and $n$ be two measures on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that $m\Big((a,b)\Big) = n\Big((a,b)\Big) < \infty$ whenever $-\infty < a < b < \infty$. Prove that $m(A) = n(A)$ whenever $A \in \mathcal{B}_{\mathbb{R}}$.