Exercise I - optional*

Math 320a/520a - Fall Semester 2017

Due Tuesday, 09/12/2017, 2:30 PM

1. Let $N \subset [0,1)$ be a set with exactly one member from each equivalence class of the relation $x \sim y \Leftrightarrow |x-y| \in \mathbb{Q}$ over the interval $[0,1)$. Let N_r, $r \in \mathbb{Q} \cap [0,1)$ be defined as $N_r = \{x+r : x \in N \cap [0, 1-r)\} \cup \{x+r-1 : x \in N \cap [1-r, 1)\}$. Prove the following two statements that were used in class:

   (a) $\bigcup_{r \in \mathbb{Q} \cap [0,1)} N_r = [0,1)$.

   (b) $N_r \cap N_q = \emptyset$ for any $q \neq r$ in $\mathbb{Q} \cap [0,1)$.

2. Show that any countable union of countable sets is also countable.
   Remember: by 'countable' we mean either finite, or infinite with the same cardinality as the natural numbers.

3. A set $A \subseteq \mathbb{R}$ is open when every $x \in A$ has some $r > 0$ such that $(x-r, x+r) \subseteq A$. A set $A \subseteq \mathbb{R}$ is closed when its complement A^c is open. Let A^o and $\bar{A}$ denote the interior and the closure of a set A, as they are defined on page 3 of Bass's textbook[1]. Prove the following two statements:

   (a) A set $A \subseteq \mathbb{R}$ is open if and only if $A = A^o$.

   (b) A set $A \subseteq \mathbb{R}$ is closed if and only if $A = \bar{A}$.

   These statements show that the definitions of open and closed sets provided here are equivalent to the ones in the textbook.

4. Using the definitions in the previous question, prove that

   - *any* union of open sets is open, and
   - any *finite* intersection of open sets is open.

   Is this also true for any countable intersection of open sets? Explain (either prove or show a counterexample). *You can assume that only subsets of* $\mathbb{R}$ *are considered.*

---

*This exercise is not counted as part of the mandatory submissions, and its grade will be taken in consideration only if it improves the final grade

[1] Real Analysis for Graduate Students, Richard F. Bass, Version 3.1