Exercise IX

Math 305b/ ENAS 514b - Spring Semester 2017

Due Wednesday, 04/19/2015, 1:00 PM

1. Suppose that the sequence of functions $f_n(t) = \sum_{k=-n}^{n} a_k e^{ikt}$ converges in L^1-periodic to an integrable periodic function f. Show that if $|n| \geq |m|$, then $\widehat{f_n}(m) = \widehat{f}(m)$.

2. Let a be a complex number with $|a| < 1$; find the Fourier coefficients of the following functions:

   (a) $f(t) = (1 - ae^{it})^{-1}$

   (b) $g(t) = (1 - ae^{i2t})^{-1}$

   (c) $h(t) = \sum\limits_{n=1}^{\infty} a^n \sin(2^n t)$

3. Let f be a continuous periodic function; prove the following results:

   (a) f is an even function if and only if $\widehat{f}(-n) = \widehat{f}(n)$ for any $n \in \mathbb{Z}$.

   (b) f is an odd function if and only if $\widehat{f}(-n) = -\widehat{f}(n)$ for any $n \in \mathbb{Z}$.

   (c) $f = \bar{g}$ a.e., for some continuous periodic[1] g, if and only if $\widehat{f}(-n) = \overline{\widehat{g}(n)}$ for any $n \in \mathbb{Z}$.

4. Show that if $f_1, f_2, f_3, \ldots$ converges to f in L^1-periodic, then $\lim\limits_{k\to\infty} \widehat{f_k}(n) = \widehat{f}(n)$.

5. Use Parseval's identity to compute (a) $\sum_{n=1}^{\infty} n^{-2}$, and (b) $\sum_{n=1}^{\infty} n^{-4}$. *Hint*: consider the functions $f_1(t) = t$ and $f_2(t) = t^2$.

6. Show that if $h = f * g$ then $\widehat{h}(n) = \widehat{f}(n)\widehat{g}(n)$ for $n \in \mathbb{Z}$ when

   (a) f and g are continuous periodic functions

   (b) f is L^1-periodic and g is L^2-periodic

7. Show that if $g_1, g_2, \ldots$ form an approximate identity[2], then for any L^2-periodic function f we have $\lim\limits_{k\to\infty} \|g_k * f - f\|_2 = 0$,

8. Show that constants c_k can be chosen to that the sequence of functions $g_n(t) = c_k[1 + cos(t)]^k$ $(k = 0, 1, 2, 3, \ldots)$ is an approximate identity and compute the appropriate constants $\{c_k\}_{k=1}^{\infty}$.

[1]In this case f and g may take complex values, and complex conjugate is denoted by a *bar* over a given value.

[2]See definition in Beals 13H.