Exercise VIII

Math 305b/ ENAS 514b - Spring Semester 2017

Due Wednesday, 04/12/2015, 1:00 PM

1. Prove that if $f_1, \dots, f_N \in L^2(\mathbb{R})$ are mutually orthogonal (i.e., $\langle f_n, f_k \rangle = 0$ when $n \neq k$) functions, then $\left\| \sum_{n=1}^{N} f_n \right\|_2^2 = \sum_{n=1}^{N} \|f_n\|_2^2$.

2. Let $f, f_1, f_2, \dots \in L^2(\mathbb{R})$ such that $\lim_{n\to\infty} \|f_n - f\|_2 = 0$ (*this is called convergence in* L^2 *and denoted by* $f_n \xrightarrow{L^2} f$). Prove that $\lim_{n\to\infty} \langle f_n, g \rangle = \langle f, g \rangle$ for any $g \in L^2(\mathbb{R})$.

3. Prove that if $F(x) = \int_0^x f(t)dt$ for $f \in L^2(\mathbb{R})$ then there exists a constant C such that $|F(x) - F(y)| \leq C|x - y|^{1/2}$ for all $x, y \in \mathbb{R}$.

4. Prove the following extension of the Cauchy-Schwarz inequality: Let $p, q \in \mathbb{N}$ such that $1/p + 1/q = 1$ and let[1] $f \in L^p, g \in L^q$, then $\left|\int fg\right| \leq \|f\|_p \|g\|_q$. (*This result is called Hölder's inequality.*)

5. Compute the Fourier coefficients of the following periodic functions defined via their values on the interval $(-\pi, \pi]$: (a) $f(t) = t$, (b) $g(t) = |t|$, (c) $h(t) = t^2$, and (d) $\chi = \mathbb{1}_{[0,\pi]} - \mathbb{1}_{(-\pi,0)}$

6. Prove that if g is an *integrable periodic function*(i.e., $g(t + 2\pi) = g(t)$ a.e., and $g \cdot \mathbb{1}_{(-\pi,\pi]}$ is integrable) then $\int_\tau^{\tau+2\pi} g(t)dt = \int_{-\pi}^{\pi} g(t)dt$. *Hint:* first prove this result for a continuous g and then for an arbitrary g.

7. Let f be an integrable periodic function, and let $f_a(t) = f(t - a)$ be it's translation by $a \in \mathbb{R}$. Prove that $\widehat{f_a}(n) = e^{-ina}\widehat{f}(n)$ for every $n \in \mathbb{Z}$.

8. Let f be a continuous periodic function with a continuous derivative $f' = \frac{df}{dt}$. Show that $\widehat{f'}(n) = in\widehat{f}$ for every $n \in \mathbb{Z}$.

[1] The normed space L^p is defined as the space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ with $\|f\|_p^p = \int |f|^p < \infty$