Exercise VII

Math 305b/ ENAS 514b - Spring Semester 2017

Due Wednesday, 04/05/2017, 1:00 PM

1. Compute the limit $\lim_{n\to\infty} \int_0^{\pi/2} \sqrt{n \sin(x/n)} dx$.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded, measurable function that is equal to zero outside some interval $I \neq \emptyset$, and let $\omega_n(x) = \sup\{|f(y) - f(z)| : y, z \in I \cap (x - 1/n, x + 1/n)\}$. Show that for any $a \in \mathbb{R}$, the set $\{\omega_n > a\}$ is an open set.
3. Is every nonnegative function in L^1 bounded? Prove or give a counter example.
4. Let $f \in L^1$, and let $f_a(x) = f(x - a)$ be its translation by $a \in \mathbb{R}$ (notice that $f_a \in L^1$ as well). Prove that $\lim_{a\to 0} \|f_a - f\|_1 = 0$.
5. Prove that if f is integrable, then for every $\varepsilon > 0$ there exists a set A with $m(A) < \varepsilon$ such that f is continuous on[1] the complement of A. *This result is called Lusin's Theorem.*
6. Prove that the version of *Lusin's Theorem* in the previous question still applies when the integrability condition is relaxed to be "If $f < \infty$ a.e., then ..."
7. A sequence of functions $\{f_n\}$ is said to *converge in measure* to f when $\lim_{n\to\infty} m(\{|f_n - f| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$. Prove that if f_n converges to f in L^1, then f_n also converges in measure to f. Is the converse true? (prove or give a counter example)
8. Show that $f_n(x) = (sin(nx))^n \cdot \mathbb{1}_{[0,\pi]}$, $n \in \mathbb{N}$, converges to zero in L^1.

[1] Notice that this does not mean f is continuous at every point in A^c, but rather that it is equal to a continuous function when both are restricted to A^c.