Exercise VI

Math 305b/ ENAS 514b - Spring Semester 2017

Due Wednesday, 03/08/2015, 1:00 PM

1. Show that if f is an integrable function, then $\lim_{n\to\infty} m(\{|f| > n\}) = 0$.
2. State and prove Fatou's lemma.
3. Show an example where equality does not hold in Fatou's lemma.
4. Show that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if there are sequences of step functions $(g_n)_{n=1}^{\infty}$ and $(h_n)_{n=1}^{\infty}$ such that $g_n \leq f \leq h_n$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \int (h_n - g_n) = 0$.
5. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable nonnegative functions such that $0 \leq f_1 \leq f_2 \leq \cdots$, and let $f : \mathbb{R} \to [0, \infty]$ be their pointwise limit $f = \lim_{n\to\infty} f_n$. Prove that if $\sup\{\int f_n : n \in \mathbb{N}\} < \infty$ then $f < \infty$ a.e.
6. Show that if $f : \mathbb{R} \to \mathbb{R}$ is an integrable function and $f = g$ a.e. for some $g : \mathbb{R} \to \mathbb{R}$, then g is also integrable with $\int g = \int f$.
7. Let $f : \mathbb{R} \to \mathbb{R}$ be nonnegative and measurable, and let $E_1 \subseteq E_2 \subseteq \cdots$ be a sequence of measirable sets. Prove that $\int_{\cup E_n} f = \lim_{n\to\infty} \int_{E_n} f$.
8. Let $f(x) = \frac{sinx}{x}$ for $x \neq 0$ and $f(0) = 1$. Show that $\lim_{n\to\infty} \int_{[-n,n]} f(x)dx$ exists and is finite, but f is not an integrable function.