Exercise II

Math 305b/ ENAS 514b - Spring Semester 2017

Due Monday, 02/06/2015, 1:00 PM

1. Compute the outer measure $m^*(A)$ of the set

$$A = \{x \in \mathbb{R} : \text{the decimal expansion of } x \text{ does not contain the digit } 4\}.$$

2. Let $A \subset \mathbb{R}$. Prove that if for every $\varepsilon > 0$ there exists a measurable $X \subseteq \mathbb{R}$ such that

   (a) $A \subseteq X$, and

   (b) $m^*(X \setminus A) < \varepsilon$,

   then A is measurable.

3. Given a subset $E \subseteq \mathbb{R}$, let $E_n = E \cap [-n, n]$ for every $n \in \mathbb{N} \cup \{0\}$. Show that $m^*(E) = \lim_{n\to\infty} m^*(E_n)$.

4. Suppose $(A_n)_{n=1}^{\infty}$ is a sequence of subsets of $\mathbb{R}$. Define the following two sets:

$$\liminf_{n\to\infty} A_n = \{x : x \text{ belongs to } A_n \text{ for all but finitely many values of } n\}$$
$$\limsup_{n\to\infty} A_n = \{x : x \text{ belongs to } A_n \text{ for infinitely many values of } n\}$$

   (a) Demonstrate that these two sets may be different by giving an example of a sequence where $\liminf_{n\to\infty} A_n \neq \limsup_{n\to\infty} A_n$.

   (b) Show that $\liminf_{n\to\infty} A_n = \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} A_n$ and find a simlar expression for $\limsup_{n\to\infty} A_n$ as well.

   (c) Show that if each A_n is measurable then so are $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$.

5. Prove that the set of all real numbers that have a decimal expansion with infinitely many 0's is measurable.

6. Let A_n be a sequence of measurable sets, such that $\sum_{n=1}^{\infty} m(A_n) < \infty$. Prove that $m(\limsup_{n\to\infty} A_n) = 0$. *Hint:* notice that if $\sum_{n=1}^{\infty} x_n$ converges then $\lim_{N\to\infty} \sum_{n=N}^{\infty} x_n = 0$.

7. Give an example of a sequence of measurable sets $(A_n)_{n=1}^{\infty}$ with $A_1 \supseteq A_2 \supseteq \ldots$ such that $m(A_n) = \infty$ for each $n \in \mathbb{N}$, but $\cap_{n=1}^{\infty} A_n = \emptyset$.

8. For $n \in \mathbb{N}$, let D_n be the set of all $x \in \mathbb{R}$ that satisfy the inequality

$$|x - \tfrac{p}{q}| \leq \tfrac{1}{q^n},\ p \in \mathbb{Z},\ q \in \mathbb{N},$$

   for infinitely many values of q (some perspective on this condition can be found in Beals, Theorem 2.10). Prove that D_n is measurable for every n. Furthermore, prove that (a) $D_1 = \mathbb{R}$, (b) D_3 contains a dense set of irrational numbers, and (c) $m(D_3) = 0$.