Differential equations and integral geometry

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1 Introduction

1. An example. Let $f(x)$ be a smooth function in $\mathbb{R}^m$ and

$$I : f(x) \longmapsto If(y;r) := \int_{|\omega|=1} f(y + \omega \cdot r) d\omega$$

be the operator of mean value over a radius r sphere centered at $y \in \mathbb{R}^m$. The integral transform I is clearly injective.

Let C be a compact hypersurface in $\mathbb{R}^m$ isotopic to a sphere.

Theorem 1.1 *Let $f(x)$ be a smooth function vanishing near C. Then one can recover f from its mean values along the spheres tangent to C, and the inversion is given by an explicit formula.*

In fact we will show that this theorem is true for any compact manifold C satisfying a mild condition. The only known before case was the family of all spheres tangent to a plane (horospheres in the hyperbolic geometry, see [GGV]).

The function $If(y;r)$ satisfies the Darboux differential equation

$$\Big(\frac{\partial^2}{\partial r^2} - \sum_{i=1}^{m} \frac{\partial^2}{\partial y_i^2} + \frac{m-1}{r}\frac{\partial}{\partial r}\Big) If(y;r) = 0$$

Let $Sol(\Delta_{\mathcal{D}}, C^{\infty}(\mathbb{R}^m \times R_+^*))$ be a space of smooth solutions of the Darboux equation $\Delta_{\mathcal{D}}$. We will construct an inverse operator J as a map

$$J : Sol(\Delta_{\mathcal{D}}, C^{\infty}(\mathbb{R}^m \times \mathbb{R}_+^*)) \longrightarrow C^{\infty}(\mathbb{R}^m)$$

Namely, let $\mathcal{A}^m(X)$ be the space of smooth differential m-forms on a manifold X. We will define a differential operator $\nu : C^\infty(\mathbb{R}^m \times \mathbb{R}^*_+) \to \mathcal{A}^m(\mathbb{R}^m \times \mathbb{R}^*_+)$ such that the m-form $\nu\varphi$ is closed if (and only if) $\Delta_{\mathcal{D}}\varphi = 0$. For a solution $\varphi(y, r)$ we define $(J\varphi)(x)$ integrating the (closed!) differential m-form $\nu\varphi(y, r)$ over a certain m-cycle. In particular restricting this form to the m-dimensional subvariety of all spheres tangent to C and integrating over it we get the theorem, see chapter 7 for details and generalizations.

2. General problem. Let X be a smooth manifold of dimension n and $\mathcal{M}$ a system of linear partial differential equations on X. Denote by $Sol(\mathcal{M}, C^\infty(X))$ the space of smooth solutions to $\mathcal{M}$.

Let $\mathcal{N}$ be a linear system of PDE on a manifold Y. Let $K(x, y)dx$ be a $(n, 0)$-form on $X \times Y$ with a compact support along X. Assume that it satisfies the system $\mathcal{N}$ along Y. Then the kernel $K(x, y)dx$ defines a linear map $I_K : C^\infty(X) \longrightarrow Sol(\mathcal{N}, C^\infty(X))$, $f(x) \longmapsto \int_X k(x, y)f(x)dx$. Its restriction to $Sol(\mathcal{M}, C^\infty(X))$ gives an operator $Sol(\mathcal{M}, C^\infty(X)) \longrightarrow Sol(\mathcal{N}, C^\infty(X))$. However if $\mathcal{M}$ is non trivial the functional dimension of $Sol(\mathcal{M}, C^\infty(X))$ is less then n, so many kernels represents the same operator.

In this paper I adress the following

Problem. What is the *natural description* for the linear maps

$$Sol(\mathcal{M}, C^\infty(X)) \longrightarrow Sol(\mathcal{N}, C^\infty(Y)) \tag{1}$$

When Y is a point we come to the question of *natural description* for linear functionals on the space $Sol(\mathcal{M}, C^\infty(X))$. On the other hand the composition of a linear map (1) with the evaluation at a point $y \in Y$ gives a linear functional on $Sol(\mathcal{M}, C^\infty(X))$. So these questions are closely related.

In chapters 5-6 we suggest a construction of operators between solution spaces of linear PDE called *natural linear maps*. Unlike the operators given by the Schwartz kernels $K(x, y)dx$, the natural linear maps are obtained by integration of closed forms over certain cycles in X. We apply these ideas to solve some old problems in integral geometry.

In chapter 5 a general construction of linear maps between solution spaces is given. The natural linear maps seems to be the most interesting particular case of that construction.

To discuss these questions we need the language of $\mathcal{D}$-modules.

3. Systems of linear PDE and $\mathcal{D}$-modules. Let $\mathcal{D}_X$ (or $\mathcal{D}$) be the sheaf of rings of differential operators on a smooth manifold X. Suppose we have a linear system $\mathcal{M}$ of p differential equations on q functions $f_1, ..., f_q$:

$$\mathcal{M} = \{\sum_{j=1}^{q} P_{ij} f_j = 0, i = 1, ..., p\}$$

Then we can assign to $\mathcal{M}$ a left coherent $\mathcal{D}$-module $\mathcal{M}$ with q generators $e_1, ..., e_q$ and p relations:

$$\mathcal{M} = \frac{\oplus \mathcal{D} \cdot e_j}{\Big(+\mathcal{D}(\sum P_{ij} f_j)\Big)} = Coker(\mathcal{D}^p \longrightarrow \mathcal{D}^q)$$

On the other hand a coherent $\mathcal{D}$-module $\mathcal{M} = Coker(\mathcal{D}^p \longrightarrow \mathcal{D}^q)$ provides a linear system of p differential equations on q functions.

A solution f to the system $\mathcal{M}$ in some space of functions $\mathcal{F}$ is nothing else then a morphism of $\mathcal{D}$-modules $\alpha_f : \mathcal{M} \longrightarrow \mathcal{F}$.

4. Natural functionals on solutions to $\mathcal{M}$. Below X usually will be an algebraic manifold over $\mathbb{R}$ of dimension n. Let $D'(X)$ be the space of distributions on X understood as the space of linear continuous maps on the space of smooth differential forms of top degree with compact support on X. Denote by $D^m(X)$ the space of m-currents on X, i.e linear continuous functionals on the space of smooth differential $(n - m)$-forms with compact support on X.

The de Rham complex $DR(\mathcal{M})^\bullet$ of a $\mathcal{D}$-module $\mathcal{M}$ is defined as follows:

$$\mathcal{M} \xrightarrow{d} \mathcal{M} \otimes_{\mathcal{O}} \Omega^1 \xrightarrow{d} ... \xrightarrow{d} \mathcal{M} \otimes_{\mathcal{O}} \Omega^{n-1} \xrightarrow{d} \mathcal{M} \otimes_{\mathcal{O}} \Omega^n \tag{2}$$

where $\mathcal{M}\otimes_{\mathcal{O}}\Omega^n$ is sitting in degree 0, d has degree +1 and given by

$$d(m\otimes\omega):=m\otimes d\omega+\sum\frac{\partial}{\partial x_i}m\otimes dx_i\wedge\omega$$

(it does not depend on coordinates x_i), and $\mathcal{O}$ is the structural sheaf of X.

Consider the complex

$$DR(\mathcal{M}\otimes_{\mathcal{O}}D'(X))^\bullet=DR(\mathcal{M})^\bullet\otimes_{\mathcal{O}}D'(X)$$

Notice that $DR(D'(X))^\bullet$ coincides with $D^\bullet(X)[n]$, the usual de Rham complex of currents on X shifted by n to the left. Therefore any $f\in Sol(\mathcal{M},C^\infty(X))$ defines a homomorphism of complexes

$$m\circ f:DR(\mathcal{M}\otimes_{\mathcal{O}}D'(X))^\bullet\longrightarrow D^\bullet(X)[n]$$

given by the composition

$$DR(\mathcal{M}\otimes_{\mathcal{O}}D'(X))^\bullet\xrightarrow{f}DR(C^\infty(X)\otimes_{\mathcal{O}}D'(X))^\bullet\xrightarrow{m}D^\bullet(X)[n]$$

Here m is induced by the homomorphism of $\mathcal{D}$-modules $C^\infty(X)\otimes_{\mathcal{O}}D'(X)\longrightarrow D'(X)$ provided by the multiplication. We get a pairing

$$H^m(DR(\mathcal{M}\otimes_{\mathcal{O}}D'(X))[-n])\otimes Sol(\mathcal{M},C^\infty(X))\longrightarrow H^m(X,\mathbb{R}) \tag{3}$$

$$(\kappa,f)\longrightarrow[(m\circ f)(\kappa)]$$

Evaluation on a homology class $[\gamma]\in H_m(X,\mathbb{R})$ leades to a functional

$$Sol(\mathcal{M},C^\infty(X))\longrightarrow\mathbb{R},\quad f\longrightarrow<[(m\circ f)(\kappa)],[\gamma]> \tag{4}$$

Such functionals are called the *natural functionals* on $Sol(\mathcal{M},C^\infty(X))$.

How the integer m depends on $\mathcal{M}$? Let $\Sigma_{\mathcal{M}}\subset T^*X$ be the characteristic variety for a $\mathcal{D}$-module $\mathcal{M}$. It is coisotropic, so $d_{\mathcal{M}}:=dim\Sigma_{\mathcal{M}}-dimX\geq 0$. The number $d_{\mathcal{M}}$ can often be viewed as the functional dimension of the solution space to $\mathcal{M}$. We will see in chapter 3 that

$$H^m(DR(\mathcal{M}\otimes_{\mathcal{O}}D'(X))[-n])=0\quad\text{for}\quad m>d_{\mathcal{M}}$$

In particular natural functionals for a non zero system of PDE are never given by integration over fundamental class of X.

For a $\mathcal{D}$-module $\mathcal{M}$ one may ask whether the described above natural functionals give all the dual to $Sol(\mathcal{M},C^\infty(X))$. Integral geometry (including the cohomological Penrose transform) provides a wide class of examples where the answer is positive.

Remark. If X is noncompact, the integration over a (may be noncompact) m-cycle γ_m defines a linear functional $f\longrightarrow\int_{\gamma_m}\kappa(f)$ on an appropriately chosen class of functions with certain decreasing conditions at infinity. In a sence a system of PDE "changes topology of the space", see examples in chapter 2 and s. 5.6 below. I will not pursue this point further and hope to return to it in future. (If $\mathcal{M}$ is holonomic the complex of solutions is a constructible complex of sheaves on X "changing" topology of X).

5. An elementary description of natural functionals. Assume that a $\mathcal{D}$-module $\mathcal{M}$ has q generators. Then an m-chain κ in the de Rham complex $DR(\mathcal{M}\otimes_{\mathcal{O}}D'(X))[-n]$ may be written as

$$\kappa=\sum_{j=1}^{q}P_j\otimes\omega_j,\qquad P_j\in\mathcal{D}_X,\quad\omega_j\in D^m(X)$$

So we may think about it as of a differential operator

$$\bar{\kappa}:C^\infty(X)^q\longrightarrow D^m(X);\qquad\bar{\kappa}(f_1,...,f_q)\longmapsto\sum_{i=1}^{q}P_j(f_j)\cdot\omega_j$$

Suppose that κ is a cycle in the De Rham complex of $\mathcal{M}$. Then the m-current $\bar{\kappa}(f_1, ..., f_q)$ is *closed on solutions of the system* $\mathcal{M}$, i.e. $d\bar{\kappa}(f_1, ..., f_q) = 0$ whenever the functions $(f_1, ..., f_q)$ satisfy the system $\mathcal{M}$. In this case we will say that the differential operator $\bar{\kappa}$ is $\mathcal{M}$-closed.

Remark. This definition makes sence for any system of partial differential equations, not necessarily *linear*. It leads to a notion of conservation laws for a system of *nonlinear* PDE.

6. Natural linear maps between solution spaces: a naive version. Let $\mathcal{M}$ and $\mathcal{N}$ be systems of linear PDE on manifolds X and Y. A natural linear map

$$I : Sol(\mathcal{M}, C^\infty(X)) \longrightarrow Sol(\mathcal{N}, D'(Y))$$

is defined as follows. Let $\kappa_y : C^\infty(X) \longrightarrow D^m(X)$ be an $\mathcal{M}$-closed differential operator whose coefficients are distributions on Y satisfying the system $\mathcal{N}$. Then

$$I : f \longmapsto \int_{\gamma_m} \kappa_y f \in Sol(\mathcal{N}, D'(Y))$$

where γ_m is an m-cycle in X and by definition $\int_{\gamma_m} \kappa_y f :=< [\gamma_m], \kappa_y f >$. The key idea of this paper is the following:

If there is a (continuous) linear functional on solutions to a system of linear partial differential equations $\mathcal{M}$ or an operator between solution spaces to $\mathcal{M}$ and $\mathcal{N}$, then one should look for its natural realization.

7. Relation with integral geometry. Let B be a manifold of dimension n and a linear operator

$$I_K : C_0^\infty(B) \longrightarrow C^\infty(\Gamma) \qquad f(x) \longmapsto \int_B K(x, \xi)dx \tag{5}$$

enjoys the following properties:

It is injective, transforms functions $f(x)$ to solutions of a linear system of PDE $\mathcal{N}$ on Γ, and $I_K(C_0^\infty(B))$ is dense in $Sol(\mathcal{N}, C^\infty(\Gamma))$.

Usually $K(x, \xi)$ satisfies a holonomic system of differential equations.

Such a situation is typical in integral geometry and appears as follows. Let $\{B_\xi\}$ be a family of submanifolds of B parametrized by a manifold Γ. Suppose on $\{B_\xi\}$ densities μ_ξ (depending smoothly on ξ) are given. Then there is an integral operator

$$I : C_0^\infty(B) \longrightarrow C^\infty(\Gamma), \qquad f(x) \longmapsto \int_{B_\xi} f(x)\mu_\xi \tag{6}$$

So here $K(x, \xi) = \mu(x, \xi) \cdot \delta(A)db$, where db is a volume form on B, and

$$A := \{(x, \xi)| x \in B_\xi\} \subset B \times \Gamma$$

is the incidence subvariety. The integral transform I often satisfies the list of properties above. This was discovered by F.John [J] for the family of all lines in $\mathbb{R}^3$, and developed much further by Gelfand, Graev, Shapiro [GGrS]. Here are some examples.

Example 1. Consider the integral transformation

$$I : f(x_1, ..., x_{n+1}) \longrightarrow \int f(t_1, ..., t_n, a\sum_{i=1}^n t_i^2 + \sum_{i=1}^n b_i t_i + c)d^n t \tag{7}$$

related to the $(n+2)$-parametrical family of paraboloids in $\mathbb{R}^{n+1}$. Let $S(\mathbb{R}^{n+1})$ be the Schwartz space of functions in $\mathbb{R}^{n+1}$.

Lemma 1.2 *If $f \in S(\mathbb{R}^{n+1})$ then $(\frac{\partial^2}{\partial a \partial c} - \sum_{i=1}^n \frac{\partial^2}{\partial b_i^2})If = 0$.*

The integral transformation I is injective on $S(\mathbb{R}^{n+1})$.

Proof. Applying $\frac{\partial^2}{\partial a \partial c}$ to the right-hand side of (7) we get

$$\int \sum_{i=1}^{n} t_i^2 f''_{t_{n+1}}(t_1, ..., t_n, a\sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{n} b_i t_i + c) d^n t$$

Applying $\sum_{i=1}^{n} \frac{\partial^2}{\partial b_i^2}$ we get the same result. Let $a = 0$. Then I is the Radon transform and so the lemma follows from its standard properties.

Example 2. Consider the integral transformation

$$I_k : f(x_1, ..., x_n) \longrightarrow \int f(t_1, ..., t_k, \sum_{j=1}^{k} a_1^j t_j + a_1^0, ..., \sum_{j=1}^{k} a_{n-k}^j t_j + a_{n-k}^0) dt_1 ... dt_k$$

related to the family of k-planes in $\mathbb{R}^n$: $x_{k+i} = \sum_{j=1}^{k} a_i^j x_j + a_i^0$.

Theorem 1.3 *([GGrS]) a) If $f(x) \in S(\mathbb{R}^n)$ then*

$$\Big(\frac{\partial^2}{\partial a_{i_1}^{j_1} \partial a_{i_2}^{j_2}} - \frac{\partial^2}{\partial a_{i_1}^{j_2} \partial a_{i_2}^{j_1}}\Big) If = 0$$

where $k+1 \leq i_1, i_2 \leq n, \quad 0 \leq j_1, j_2 \leq k$.

b) I_k is injective on $S(\mathbb{R}^n)$ and provides an integral formula for solutions the system of PDE above.

Let us return to the integral transform I_K (see (5)). Its properties implies that its inverse would provide a continuous linear map

$$J_K : Sol(\mathcal{N}, C^\infty(\Gamma)) \longrightarrow C^\infty(B)$$

Definition 1.4 *An integral transform I_K admits a universal inversion formula if the inverce operator J_K is given by a natural linear map.*

To clarify the meaning of this definition consider the composition J_b of the operator J with the δ-functional at a point $b \in B$. Its natural realization is given by an $\mathcal{N}$-closed differential operator $\nu_b : C^\infty(\Gamma) \longrightarrow D^n(\Gamma)$ and a certain n-dimensional cycle γ_b in Γ such that

$$\int_{\gamma_b} \nu_b(I_K f) = c_{[\gamma_b]} \cdot f(b)$$

Here $c_{[\gamma_b]}$ is a constant depending linearly on the homology class of γ_b and $n = dim B$. We define the left hand side as $< \nu_b(I_K f), [\gamma_b] >$. To compute the integral we may use any cycle γ_b transversal to the wave front of the distribution $\kappa_b(I_K f)$. Then the restriction of this distribution to γ_b is defined and we can integrate it over the fundamental class of γ_b. So we can find the value $f(b)$ if we know only the values of $I_K(f)$ at an infinitesimal neighborhood of any such a cycle. This explanes the name "*universal* inversion formula".

8. Local and nonlocal inversion formulas in integral geometry. Let us discuss in more details the general Radon transform (6).

Definition 1.5 *A local universal inversion formula for the Radon transform (6) is given by a differential operator $\kappa_b : C^\infty(\Gamma) \longrightarrow \mathcal{A}^k(\Gamma_b)$ such that $\kappa_b(If)$ is closed (on Γ_b) and*

$$\int_{\gamma_k} \kappa_b(If) = c_{[\gamma_k]} \cdot f(b)$$

where $c_{[\gamma_k]}$ is a constant (depending linearly on the homology class $[\gamma_k]$).

In particular the value of any smooth function f on B at any point b can be recovered from its integrals over the submanifolds B_ξ passing through an *infinitesimal* neighborhood of b.

Let Γ_b be the variety parametrizing all the subvarieties B_ξ passing through a given point b. Set $k := dim B_\xi$. Notice that $dim\Gamma - dimB = dim\Gamma_b - dimB_\xi$. So if $dim\Gamma > dimB$ the degree of the form $\kappa_b(If)$ is less then $dim\Gamma_b$.

A first example of local universal inversion formula was discovered in 1967 by I.M.Gelfand, M.I.Graev and Z.Ya.Shapiro for integral transformation $I_k^{\mathbb{C}}$ related to the family of all k-planes in $\mathbb{C}^n$ ([GGS]). Here we treat complex planes as real submanifolds and integrate smooth functions along them. Later more generic examples were studied, including local universal inversion formulas for the families of complex curves, see [GGiG], [BG], [Gi].

However in integral geometry there are many examples where there are no *local* inversion formulas. This is quite typical in "real" integral geometry (i.e. we integrate over family of real submanifolds). For instance in examples 2 (resp 3) the inversion formula is nonlocal if the dimension of hyperboloids (resp. planes) is odd. It is always non local for integral transformations related to any family of real curves.

A very interesting approach to integral geometry on k-planes in $\mathbb{R}^n$ was suggested by I.M.Gelfand and S.G.Gindikin [GGi], (see also [GGR]). However it was based on the Fourier transform in $\mathbb{R}^n$ and so can not be generalized to families of "curved" submanifolds, like in examples 1-2. What is even more important, the differential k-form κ was replaced by a k-density, so a possibility to use the Stokes formula was missed. It seems that this approach to integral geometry was not really understood yet.

As a result the nature of the form κ_b and inversion of the general integral transformations, especially if they do not admit a local inversion formula, were the key unsolved problems in integral geometry.

The main idea of this paper is that

Inversion formulas in integral geometry are given by natural linear maps between solution spaces of systems of partial differential equations.

Let me explain how the local universal inversion formulas fit in this concept. The form $\kappa_b(If)$ is a differential k-form on Γ_b. Since $n - k = dim\Gamma - dimB$, a k-form on Γ_b defines an n-current on Γ. *The n-current corresponding to $\kappa_b(If)$ leads to a natural liear map* given by integration of $\kappa_b(If)$ over an n-cycle $K \subset \Gamma$. We will demonstrate this for the Radon transform over spheres in $\mathbb{R}^m$.

In general our approach leads to a universal inversion formula where the functional J_b is represented by a differential n-form on Γ. The fact that this n-form does not concentrated on a subvariety Γ_b (or a certain bigger subvariety of Γ) means that we get a nonlocal universal inversion formula. So we treat simultaneously both local and nonlocal inversion formulas.

The form κ_b appeared in [GGS] as a construction "ad hoc" and looks like a very special phenomena. In our approach the universal inversion formula is a very general property of the corresponding system of linear PDE. Its locality, however, is a rather rare phenomena, which generalizes the Huygens principle or, more generally, the notion of lacunas for hyperbolic differential equations.

In particular in these examples our natural functionals describe the whole dual to the space of solutions of a linear system of PDE.

9. Some general remarks on analytic theory of overdetermined systems. The classical theory of PDE usually study systems of p linear partial differential equations on p unknown functions on X i.e. the characteristic variety of the system has codimension 1. (Of cource there are some exceptions with extremely reach analytic theory, like the Cauchy-Riemann system). It seems that one of the reasons is this. A system $P_1 f = P_2 f = 0$ of two *general* differentaial equations on one unknown function has no solutions because the corresponding $\mathcal{D}$-module is equal to zero (even if P_i are differential operators of order one). This shows that overdetermined systems (i.e. the ones where the codimension of the characteristic subvariety is greater then 1) can not describe a physical process in a way similar to systems of p equations on p unknown functions (like Laplas, Schrodinger, etc. equation): a small perturbation of the experimental data leads to a system without solutions. Therefore one should not expect an analytic theory of general overdetermined systems, i.e. a theory stable under a perturbation of a system

The theory of $\mathcal{D}$-modules is a tool providing nontrivial linear systems of PDE. We think that an interesting overdetermined system of PDE should be a part of a reacher data. For example for a system

$\mathcal{N}$ on a variety Γ appearing in integral geometry (see s.6) we should also remember the kernel $K(x,\xi)$ on $B\times\Gamma$. So perturbing such a system we should deform the whole data, not only the system $\mathcal{N}$ on Γ.

We may wonder about the goals of analytic theory for some *special* overdetermined systems. It seems that the problem of natural description of linear maps between solution spaces looks quite promising.

10. The structure of the paper. Chapter 2 contains examples of functionals and natural functionals on solution spaces of systems of PDE. In chapter 3 we recall some general information about $\mathcal{D}$-modules, including the duality on the derived category of $\mathcal{D}$-modules, needed for applications to integral geometry. In chapter 4 our key tool appears: the Green class of a $\mathcal{D}$-module. It generalizes the classical Green formula for a single differential operator. Chapter 5 contains a definition and properties of general linear maps between solution spaces of (complexes of) $\mathcal{D}$-modules. Then we define natural linear maps as a quite special case of general linear maps. The definitions uses the language of derived categories. This is necessary for many reasons including:

1) Even nice systemes like $\mathcal{M}=\{x_1\cdot f=...=x_k\cdot f=0\}$ may have no smooth solutions, so one should consider the spaces of "higher" solutions. (In the example above only $Ext^k_{\mathcal{D}}(\mathcal{M},C^\infty(\mathbb{R}^n))\neq 0$; it is isomorphic to $C^\infty(\mathbb{R}^{n-k})$).

2) The duality may send a $\mathcal{D}$-module to a complex of $\mathcal{D}$-modules.

In applications the dual complex for a $\mathcal{D}$-module $\mathcal{M}$ is often concentrated in just one degree. Such $\mathcal{M}$'s will be called *excellent* $\mathcal{D}$-modules. In chapter 6 we define natural linear maps between solution spaces for excellent $\mathcal{D}$-modules. This allows to eliminate derived categories and makes the story more elementary. I made this chapter independent of chapter 5, so those who interested only in applications to "nice" systems of PDE could go directly to chapter 6.

In chapter 7 we demonstrate how the general method works for the family of all spheres in $\mathbb{R}^m$ (see section 1.1 above). Our approach leads to *universal* inversion formulas which are *nonlocal* when m is odd and local when m is even. The corresponding problem of integral geometry was unsolved even for the family of circles in the plane.

In fact we study in chapter 7 integral operators I_λ more general then the Radon transform over the family of spheres. They are intertwiners for the group $O(m+1,1)$ acting from the space of sections of a line bundle over S^m to the space of sections of a line bundle over the manifold X_{m+1} of oriented hyperplane sections of S^m. (The hyperplane sections of S^m can be identified with spheres in $\mathbb{R}^m$ by a stereographic projection). The image of I_λ is described by differential equations. So the inverse operators gives examples of intertwiners which are well defined only on a subrepresentation.

The next problem after the definition of natural linear maps would be development of "calculas of natural linear maps". In particular there are the following questions:

a) How to compose natural linear maps.

b) How to compute their composition. For instance when the composition of two natural linear maps is equal to a given natural linear map.

A universal inversion formula for the integral transform I_K is a natural linear map $J_K: Sol(\mathcal{N},C^\infty(\Gamma))\longrightarrow C^\infty(B)$ such that the composition $J_K\circ I_K$ equals to the identity map, so this is a very special case of the problem b).

The development of this program should include a version of the theory of Fourier Integral Operators as a special case when $\mathcal{M}=\mathcal{D}_X$, $\mathcal{N}=\mathcal{D}_Y$.

In chapter 8 we study an algebraic version of the problem of composition of natural linear maps. It turns out that one can organize neatly the algebraic structures responsible for that introducing a *bicategory of $\mathcal{D}$-modules.*

The objects of this bicategory are pairs $(X,\mathcal{M})$, where $\mathcal{M}$ is a complex of $\mathcal{D}$-modules on a variety X. A 1-morphism $(X,\mathcal{M})\to(Y,\mathcal{N})$ is the algebraic part of the data needed to construct a linear map $RHom_{\mathcal{D}}(\mathcal{M},C^\infty(X))\longrightarrow RHom_{\mathcal{D}}(\mathcal{N},D'(Y))$. Composition of 1-morphisms mirrors the composition of linear maps. A 2-morphism between two 1-morphism reflects coincidence of the the corresponding maps on functions.

In the end of chapter 8 we consider the simplest examples of composition of 1-morphisms and 2-morphisms relevant to integral geometry.

The main results of this paper were annonced in [G1]. Another approach to integral geometry via $\mathcal{D}$-modules was independently developed by A. D'Agnolo and P. Schapira [A], [AS1-2]. Inversion formulas for real quadrics were also considered by S.G. Gindikin [Gi2].

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2 Examples

1. Analitycal functionals [GS]. Let $X = \mathbb{C}$ and $\mathcal{M}$ be the Cauchy - Riemann equation $\frac{\partial}{\partial \bar{z}} f(z, \bar{z}) = 0$. Let $g(z)$ be a holomorphic function. Then

$$f(z, \bar{z}) \longrightarrow g(z) f(z, \bar{z}) dz$$

is an $\mathcal{M}$-closed operator of order 0. The corresponding linear functional should be

$$f(z) \longrightarrow \int_{\gamma_1} g(z) f(z) dz \tag{8}$$

To make sense out of this consider the space Z_1 of holomorphic functions $f(z)$ such that $|z|^q \cdot |f(z)| \leq C_q \cdot e^{a \cdot Imz}$ for any $q > 0$ (the constants a and C_q may depend on f). Let $\bar{\mathbb{C}} = \{\mathbb{C} \cup S^1\}$ be a compactification of the complex plane by a circle such that each line compactifies by endpoints at + and - infinity and two lines have the same endpoints if and only if they are parallel. Let x_- and x_+ are the endpoints of the x - axis ($z = x + iy$). Let γ_1 be a cycle representing the nontrivial homology class in $H_1(\bar{\mathbb{C}}, x_- \cup x_+)$. Then the right side of (8) is convergent for $g(z) \in Z_1$ and defines a continuous linear functional on Z_1.

One can also take $g(z)$ to be a meromorphic function and integrate along compact 1-cycles in $\mathbb{C} \backslash \{\text{poles of } \ g(z)\}$. For example if $g(z) = \frac{1}{z - z_0}$ we get the Cauchy formula

$$f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0}$$

It can be interpreted as the *natural realization* for the δ-functional $f(z) \rightarrow f(z_0)$.

Now let $\mathcal{M}$ be the Cauchy-Riemann system in $\mathbb{C}^n$. Let $g(z)$ be a holomorphic function. Then

$$f(z, \bar{z}) \longrightarrow g(z) f(z, \bar{z}) dz_1 \wedge ... \wedge dz_n$$

is an $\mathcal{M}$-closed operator of order 0. The corresponding natural functional is

$$f(z) \longrightarrow \int_{\gamma_n} g(z) f(z) dz_1 \wedge ... \wedge dz_n$$

where $f(z)$ belongs to the space Z_n of holomorphic functions satisfying some grouth condition ([GS]). So any $g(z) \in Z_n$ defines an element of $DSol(\mathcal{M})_n$

However for $n > 1$ there are another $\mathcal{M}$ - closed operators. Namely, let us look at the classical Bochner - Martinelly formula

$$f(z_0) = \int_{s_{2n-1}} f(z) \frac{\omega^*(\bar{z}) \wedge \omega(z)}{|z - z_0|^{2n}} \tag{9}$$

where $\omega(z) = dz_1 \wedge ... \wedge dz_n$ and $\omega^*(z) = \sum_{i=1}^n (-1)^i z_i dz_1 \wedge ... \hat{dz_i} ... \wedge dz_n$ and $[s_{2n-1}]$ is a generator in $H_{2n-1}(\mathbb{C}^n \backslash z_0)$ The zero order operator

$$f(z, \bar{z}) \longrightarrow f(z) \frac{\omega^*(\bar{z}) \wedge \omega(z)}{|z - z_0|^{2n}}$$

represents a non-zero element of $DSol(\mathcal{M})_{2n-1}$ for $X = \mathbb{C}^n \setminus z_0$

In fact all "integral formulas" in complex analysis (like the Cauchy formula in a polydisc, the Weil formula ...) are examples of elements in $DSol(\mathcal{M})_m$ where $n \le m \le 2n-1$ given by zero order operators.

2. The Green function of a differential operator and a natural realization of the δ-functional. Let $P = \sum a_I(x)\partial_x^I$ be a differential operator and $P^t = \sum \partial_x^I a_I(x)$ the transposed one (I is the multyindex). The classical Green formula is

$$(Pu \cdot v - u \cdot P^t v) dx_1 \wedge ... \wedge dx_n = d\omega_{n-1}(u; P; v) \tag{10}$$

where $\omega_{n-1}(u; P; v)$ is an $(n-1)$-form depending linearly on u and v. For example if $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ then

$$\omega_{n-1}(\Delta; u, v) = \sum_{i=1}^n (u'_{x_i} v - u v'_{x_i}) dx_1 \wedge ... \hat{d}x_i ... \wedge dx_n$$

A Green function $g(x, y)$ for P is a generalized function on $X \times X$ satisfying

$$P_x^t g(x, y) = P_y g(x, y) = \delta(x - y)$$

Let us put in (10) $u := g(x, y)$ and suppose $Pv = 0$. Then

$$v(x) = \int_{s_{n-1}} \omega_{n-1}(g(x, y); P; v)$$

where s_{n-1} is a small $(n-1)$-sphere around x. Therefore $\omega_{n-1}(g(x, y); P; v)$ provides a natural realization of the δ-functional $f \to f(x)$.

There are differential operators that do not have a Green function, for example the H. Lewy operator $1/2(\partial_{x_1} + i\partial_{x_2}) - (x_1 + ix_2)\partial_{x_3}$ or the operator $\partial_x - ix\partial_y$.

3. A Green form for an arbitrary system $\mathcal{M}$.

Definition 2.1 *A Green form for a system $\mathcal{M}$ is an element $g_y \in Sol(\mathcal{M})_{n-1}$ such that*

$$dg_y \overset{\mathcal{M}}{=} \delta(x - y) dx_1 \wedge ... \wedge dx_n$$

Here $\overset{\mathcal{M}}{=}$ means $\mathcal{M}$-equivalence, i.e.

$$dg_y(f) = f(y) dx_1 \wedge ... \wedge dx_n \qquad f \in Sol(\mathcal{M}, C^\infty(X))$$

If g_y is a Green form for $\mathcal{M}$ then for any $f \in Sol(\mathcal{M}, C^\infty(X))$ one has

$$f(y) = \int_{s_{n-1}} g_y(f)$$

Here s_{n-1} is a small sphere around y. This follows from the Stokes formula.

Example 2.2 *$g_y : f \to \omega_{n-1}(P; g(x, y), f)$ (see s 2.2 above) is the Green form for a differential equation $Pf = 0$.*

Example 2.3 *The Bochner - Martinelly form (9) provides a Green form*

$$f \longrightarrow f \frac{\omega^*(\bar{z}) \wedge \omega(z)}{|z - z_0|^{2n}} \tag{11}$$

for the Cauchy-Riemann system in $\mathbb{C}^n$.

4. A universal solution of a boundary value problem. Let $\mathcal{M}$ be a system of PDE on X and $m := d_{\mathcal{M}}$. Let $Id : Sol(\mathcal{M}, C^\infty(X)) \longrightarrow Sol(\mathcal{M}, C^\infty(X))$. be the identity map. Its natural realization should be given by an $\mathcal{M}$-closed operator $G_x : C^\infty(X) \longrightarrow D^m(X)$ depending on a parameter $x \in X$ whose coefficients considered as functions on x are also solutions to $\mathcal{M}$. For a given solution $\phi \in Sol(\mathcal{M}, C^\infty(X))$ one has $G_x(\phi)$ is an m-form on X such that for any closed m-dimensional manifold Y one has

$$\int_Y G_x(\phi) = c_{[Y]}\phi(x) \tag{12}$$

where $c_{[Y]}$ is a constant depending linearly on the homology class of Y.

According to the definition to compute $G_x(\phi)$ at a point $y \in Y$ one has to know the restriction and a finite number of transversal derivatives of ϕ at y. So formula (12) is a universal solution to the Cauchy problem for $\mathcal{M}$ on Y. The fact that $d_{\mathcal{M}}$ can be often viewed as a "functional dimension" of the space of solutions to $\mathcal{M}$) looks very natural from this point of view. G_x will be referred to as the boundary value problem Green form.

Remark 2.5. There are two different realizations for the identity map given in s. 2.3 and s. 2.4. I would like to emphasize the following differences between them. The realization given in s. 2.3 is not a *natural* one because the form is not $\mathcal{M}$-closed. However one may interpret it as a natural realization for a modification of $\mathcal{M}$ at x. Further, in general the cycles for g_y and G_y are of different dimension and in fact of different nature. Namely, for a Green form g_y the cycle always exist and represents a class in $H_{n-1}(X\backslash x)$, while for the boundary value problem Green form G_y the cycle in (12) represents a homology class of X of dimension m and its existence is a nontrivial problem.

3 Basic facts about $\mathcal{D}$-modules

For conviniense of the reader I will recall some material about $\mathcal{D}$-modules (see [Be], [Bo]).

1. The bimodule $\mathcal{D}_X^\Omega$. I will assume that X is an algebraic manifold, $\mathcal{D} = \mathcal{D}_X$ is the sheaf of regular differential operators on X and Ω_X the $\mathcal{O}_X$ - sheaf of regular differential forms of highest degree on X. Ω_X has a right $\mathcal{D}_X$-module structure given by

$$\omega \cdot f = f\omega, \quad \omega \cdot \xi := -L_\xi \omega$$

where $f \in \mathcal{O}_X$ and ξ is a vector field. Here $L_\xi\omega := di_\xi\omega$. Set

$$\mathcal{D}_X^\Omega := \mathcal{D}_X \otimes_{\mathcal{O}} \Omega_X^{-1} = Hom_{\mathcal{O}_X}(\Omega_X, \mathcal{D}_X^r) \tag{13}$$

where $\mathcal{D}_X^r$ is $\mathcal{D}_X$ viwed as a right $\mathcal{D}$-module via right multiplication. Then (13) carries 2 commuting left $\mathcal{D}_X$-modules structures. The first is provided by the left multiplication on $\mathcal{D}_X$, and the second, $''\circ''$, is given by

$$\xi \circ (\lambda)(\omega) = \lambda(\omega \cdot \xi) - \lambda(\omega) \cdot \xi \tag{14}$$

where ξ is a vector field and $\lambda \in Hom_{\mathcal{D}_X}(\Omega_X, \mathcal{D}_X^r)$.

The two natural commuting left $\mathcal{D}_X$-module structures on $\mathcal{D}_X^\Omega$ let us to consider $\mathcal{D}_X^\Omega$ as a $\mathcal{D}$-module on $X \times X$. It is canonically isomorphic to the $\mathcal{D}_{X\times X}$-modules δ_Δ of δ-functions on the diagonal $\Delta \subset X \times X$. There exists a canonical involution on $\mathcal{D}_X^\Omega$ interchanging the two left $\mathcal{D}_X$-module structures. For δ_Δ it is induced by the switch of the factors of $X \times X$.

2. The duality functor. Let $D^b_{coh}(\mathcal{D}_X)$ be the derived category of bounded complexes of $\mathcal{D}_X$-modules whose cohomology groups are coherent $\mathcal{D}_X$-modules. Let us define a duality $\star : D^b_{coh}(\mathcal{D}_X)^0 \longrightarrow D^b_{coh}(\mathcal{D}_X)$ by

$$\star\mathcal{M} := RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega)[dimX]$$

The second $\mathcal{D}_X$-structure on $\mathcal{D}_X^\Omega$ provides a left $\mathcal{D}_X$-module structure on the sheaves $Ext^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega)$.

To compute $\star\mathcal{M}$ we should find a bounded complex $\mathcal{P} = \{\longrightarrow \mathcal{P}^{-1} \longrightarrow \mathcal{P}^0 \longrightarrow \mathcal{P}^1 \longrightarrow ...\}$ of locally projective coherent $\mathcal{D}$-modules quasiisomorphic to $\mathcal{M}$ and set $\star\mathcal{M} = \star\mathcal{P}$ where $(\star\mathcal{P})^i = \star(\mathcal{P}^{-dimX-i}) := Hom_{\mathcal{D}_X}(\mathcal{P}^{-dimX-i}, \mathcal{D}_X^\Omega)$. It is easy to see that $\star\star\mathcal{P}$ is isomorphic to $\mathcal{P}$. Therefore $\star\star = Id$.

The object $\star\mathcal{M}$ represents the functor

$$\mathcal{N} \longrightarrow RHom_{\mathcal{D}_{X\times X}}(\mathcal{N}\boxtimes\mathcal{M}, \delta_\Delta)[dimX]$$

i.e. one has

$$RHom_{\mathcal{D}_X}(\mathcal{N}, \star\mathcal{M}) = RHom_{\mathcal{D}_{X\times X}}(\mathcal{N}\boxtimes\mathcal{M}, \delta_\Delta)[dimX] \tag{15}$$

Indeed, there is canonical morphism

$$RHom_{\mathcal{D}_{X\times X}}(\mathcal{N}\boxtimes\mathcal{M}, \mathcal{D}_X^\Omega) \longrightarrow RHom_{\mathcal{D}_X}(\mathcal{N}, RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega))$$

It is obviously an isomorphism when $\mathcal{N} = \mathcal{M} = \mathcal{D}_X$, and so using locally free resolutions we see that it is an isomorphism in general.

Let $SS\mathcal{M}$ be the singular support of a $\mathcal{D}$-module $\mathcal{M}$. The following important result was proved by Roos.

Theorem 3.1 *a) $\mathcal{M}$ has a finite resolution by locally projective $\mathcal{D}_X$-modules.*
b) $codimSS(Ext^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega) \geq i$
c) If $codimSS(\mathcal{M}) = k$, then $Ext^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega) = 0$ for $i < k$.

Notice that $H^i(\star\mathcal{M}) = Ext^{dimX+i}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X^\Omega)$ The Roose theorem implies that $\star\mathcal{M}$ is concentrated in degrees $[-dimX, -codimSS\mathcal{M}]$.

Lemma 3.2 $DR(\mathcal{M}) = \Omega_X \overset{L}{\otimes}_{\mathcal{D}_X} \mathcal{M}$.

Proof. Using the Koszul complex we see that $DR(\mathcal{D}_X)$ is a locally free resolution for the right $\mathcal{D}_X$-module Ω_X. One has $DR(\mathcal{M}) = DR(\mathcal{D}) \otimes_{\mathcal{O}} \mathcal{M}$. Let $D^b(Sh_X)$ be the bounded derived category of sheaves on X.

Theorem 3.3 *Let $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$ and $\mathcal{N} \in D^b(\mathcal{D}_X)$. Then there is an isomorphism in $D^b(Sh_X)$ functorial with respect to $\mathcal{M}$ and $\mathcal{N}$*

$$DR(\star\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N})[-dimX] = RHom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \tag{16}$$

This nature of this isomorphism and the fact that $\mathcal{N}$ may not be coherent plays a crucial role, so we will scetch its proof following [Bo], ch. 6. Let us replace $\mathcal{M}$ and $\mathcal{N}$ by bounded locally projective resolutions $\mathcal{P}^\bullet_{\mathcal{M}}$ and $\mathcal{P}^\bullet_{\mathcal{N}}$. One can suppose $\mathcal{P}^\bullet_{\mathcal{M}}$ to be locally free from certain low degree on. Therefore according to lemma 3.2 to prove the theorem it is sufficient to construct for a given coherent $\mathcal{D}_X$-modules $\mathcal{M}$ and $\mathcal{N}$ a natural morphism (functorial with respect to $\mathcal{M}$ and $\mathcal{N}$)

$$\alpha : \Omega_X \otimes_{\mathcal{D}_X} \Big(Hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}^r_X \otimes \Omega_X^{-1}) \otimes_{\mathcal{O}_X} \mathcal{N}\Big) \longrightarrow Hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) \tag{17}$$

which will be an isomorphism if $\mathcal{M} = \mathcal{D}_X$.

The functors $\Omega_X \otimes_{\mathcal{D}_X}$ and $Hom_{\mathcal{D}_X}$ in the left-hand side of (17) are defined using different commuting left $\mathcal{D}_X$-structures on $\mathcal{D}_X \otimes \Omega_X^{-1}$. So we can interchange them and get the canonical morphism from the left-hand side of (17) to

$$Hom_{\mathcal{D}_X}\Big(\mathcal{M}, \Omega_X \otimes_{\mathcal{D}_X} \Big(\mathcal{D}_X \otimes \Omega_X^{-1}) \otimes_{\mathcal{O}_X} \mathcal{N}\Big)\Big) \tag{18}$$

There is canonical isomorphism of $\mathcal{D}_X$-modules

$$\Omega_X \otimes_{\mathcal{D}_X} \Big(\mathcal{D}_X \otimes \Omega_X^{-1}) \otimes_{\mathcal{O}_X} \mathcal{N}\Big) = \mathcal{N} \tag{19}$$

Indeed, the left structure on $(\mathcal{D}_X \otimes \Omega_X^{-1})$ we used to define $\Omega_X \otimes_{\mathcal{D}_X} (\mathcal{D}_X \otimes \Omega_X^{-1})$ is provided by the left multiplication in $\mathcal{D}_X$, therefore $\Omega_X \otimes_{\mathcal{D}_X} (\mathcal{D}_X \otimes \Omega_X^{-1}) = \mathcal{O}$. So (18) is canonically isomorphic to $\mathcal{N}$ as $\mathcal{O}$-module. This isomorphism commutes with the action of vector fields. So (19) is canonically isomorphic to $Hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N})$. Theorem 3.3 is proved.

Placing to (16) $\mathcal{N} = C^\infty(X)$ and using $\star\star\mathcal{M} = \mathcal{M}$ we get

Corollary 3.4
$$DR(\mathcal{M} \otimes_{\mathcal{O}} C^{\infty}(X))[-dimX] = RHom_{\mathcal{D}_X}(\star\mathcal{M}, C^{\infty}(X)) \tag{20}$$

In particular
$$Hom_{\mathcal{D}_X}(\star\mathcal{D}_X, D'(X)) \longrightarrow DR(\mathcal{D}_X^{\Omega} \otimes_{\mathcal{O}} D'(X)) = D^n(X)$$

Corollary 3.5 *For any $\mathcal{A}, \mathcal{B} \in D^b_{coh}(X)$, $\mathcal{C} \in D^b(X)$ one has a canonical functorial isomorphism in $D^b(Sh_X)$:*
$$RHom_{\mathcal{D}_X}(\mathcal{A}, \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{C}) = RHom_{\mathcal{D}_X}(\mathcal{A} \otimes_{\mathcal{O}_X} \star\mathcal{B}, \mathcal{C})$$

Proof. By the theorem above both parts are isomorphic to
$$RHom_{\mathcal{D}_X}(\mathcal{O}_X, \star\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{C})$$

In particular for $\mathcal{M} \in D^b_{coh}(X)$ there is a canonical morphism of $\mathcal{D}_X$-modules
$$i_{\mathcal{M}} : \mathcal{O}_X \longrightarrow \star\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M} \tag{21}$$

More precisely, there exists a canonical section over X of the sheaf $H^0(RHom_{\mathcal{D}_X}(\mathcal{O}_X, \star\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M})$, or, what is the same, a canonical morphism $\mathbb{C}_X \to RHom_{\mathcal{D}_X}(\mathcal{O}_X, \star\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M})$ in $D^b(Sh_X)$.

3. Functors between the derived categories of $\mathcal{D}$-modules . Let $Y \to X$ be a morphism of varieties and $d_{Y,X} := dimY - dimX$. Let p^+ be the naive inverse image functor on $\mathcal{D}$-modules. Then $p^! := Lp^+[d_{Y,X}]$. If $p : Y \to X$ is smooth then p_* can be computed via relative De Rham complex $p_*\mathcal{M} = Rp_\bullet(DR_{Y|X}(\mathcal{M}))$, where $DR_{Y|X}(\mathcal{M}) := \Omega^\bullet_{Y|X} \otimes \mathcal{M}[d_{Y,X}]$.

Lemma 3.6 *Suppose p is smooth. Then there is a canonical isomorphism of functors on $D^b_{coh}(\mathcal{D})$*
$$p^* := \star p^! \star = p^![-2d_{Y,X}]$$

Proof. See proof of proposition 9.13 in [Bo].

Let $p_! := \star p_* \star$.

Theorem 3.7 *Suppose p is proper. Then $p_! = p_*$ on $D^b_{coh}(\mathcal{D})$ and the functor $p_!$ (resp. p^*) is left adjoint to $p^!$ (resp p_*), i.e.*
$$RHom_{\mathcal{D}}(p^*\mathcal{M}, \mathcal{N}) = RHom_{\mathcal{D}}(\mathcal{M}, p_*\mathcal{N})$$
$$RHom_{\mathcal{D}}(p_!\mathcal{M}, \mathcal{N}) = RHom_{\mathcal{D}}(\mathcal{M}, p^!\mathcal{N})$$

Proof. See proof of theorem 9.12 in [Bo].

Lemma 3.8 *Let $p : X \longrightarrow *$ be projection to the point and $\Delta : X \hookrightarrow X \times X$ be the diagonal. Then $Rp_\bullet RHom_{\mathcal{D}}(\star\mathcal{M}, \mathcal{N}) = p_*\Delta^!(\star\mathcal{M} \boxtimes \mathcal{N})$*

4 The Green class of $\mathcal{M}$

"... we can say that there is only one formula (which we shall call "fundamental formula") in the whole theory of partial differential equations, no matter to which type they belong."

J.Hadamard, Lectures on the Cauchy problem.

0. The definition. For any $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$ the identity map in $Hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{M})$ provides a canonical element
$$G_{\mathcal{M}} \in H^{dimX}\Big(DR(\star\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M})\Big) \tag{22}$$

I will call it *the Green class of* $\mathcal{M}$.

The right hand side of (22) is a sheaf on X, and $G_{\mathcal{M}}$ is a canonical section of this sheaf. A more concrete way to think about it is this. Choose a locally projective resolution $\mathcal{M}^{\bullet}$ for $\mathcal{M}$. Take a Cech covering $\{\mathcal{U}_i\}$ of X (in the classical or Zariski topology). Then there exists a section in the Cech complex $C\Big(\mathcal{U}_{\bullet}, DR(\star\mathcal{M}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{M}^{\bullet})\Big)$ with coefficients in the complex of sheaves $DR(\star\mathcal{M}^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{M}^{\bullet})$ which represents the Green class.

1. The Green class and the classical Green formula. Let P be a differential operator. Set

$$\mathcal{P} = \frac{\mathcal{D}_X}{\mathcal{D}_X \cdot P} \quad \text{and} \quad \tilde{\star}\mathcal{P} = \frac{\mathcal{D}_X^{\Omega}}{P \cdot \mathcal{D}_X^{\Omega}}$$

Here $\mathcal{D}_X^{\Omega}$ is considered as a left $\mathcal{D}$-module with respect to the second structure. Notice that $Hom_{\mathcal{D}}(\mathcal{D}_X^{\Omega}, C^{\infty}(X)) = \mathcal{A}^n(X)$. Let $v \in \mathcal{A}^n(X)$. According to the Green formula there exists an $(n-1)$-form $\omega_{n-1}(\varphi; P; v)$ on X such that (P^* is the adjoint operator on $\mathcal{A}^n(X)$)

$$P\varphi \cdot v - \varphi \cdot P^* v = d\omega_{n-1}(\varphi; P; v) \tag{23}$$

Of course neither $(n-1)$-form $\omega_{n-1}(\varphi; P; v)$ nor its cohomology class $[\omega_{n-1}(\varphi; P; v)]$ are defined canonically by (23). However there is a way to define a cohomology class in $H^{n-1}(X, \mathbb{R})$ starting form the Green formula. Namely, locally there exists an algebraic bidifferential operator

$$\omega_P : C^{\infty}(X) \otimes_{\mathcal{O}} \mathcal{A}^n(X) \to \mathcal{A}^{n-1}(X) \quad \text{such that} \quad d\omega_P = P \otimes 1 - 1 \otimes P^*$$

so $\omega_{n-1}(\varphi; P; v) := \omega_P(\varphi \otimes v)$. For two different algebraic bidifferential operators ω_P and ω'_P there exists an algebraic bidifferential operator

$$\omega''_P : C^{\infty}(X) \otimes_{\mathcal{O}} \mathcal{A}^n(X) \to \mathcal{A}^{n-2}(X) \quad \text{such that} \quad d\omega''_P = \omega_P - \omega'_P$$

and so on. So choosing a covering and taking a partition of unity corresponding to it we get a well defined cohomology class $[\omega_{n-1}(\varphi; P; v)]$. Below we explane how to get it without computaions in local coordinates, using the $\mathcal{D}$-modules instead. (On the other hand the approach we scetched leads to an equivariant cohomology class of the group of diffeomorphisms of X).

Lemma 4.1 *$\star\mathcal{P}[-1]$ is isomorphic to $\tilde{\star}\mathcal{P}$, so the Green class $G_{\star\mathcal{P}}$ is an element of $H^{n-1}DR(\tilde{\star}\mathcal{P} \otimes_{\mathcal{O}} \mathcal{P})$.*

Proof. Let $\mathcal{D}_X \stackrel{P}{\longrightarrow} \mathcal{D}_X$ be the obvious free resolution for $\mathcal{P}$. It is concentrated in degrees [-1,0]. So

$$RHom_{\mathcal{D}_X}(\mathcal{P}, \mathcal{D}_X^{\Omega}) = (\mathcal{D}_X^{\Omega} \stackrel{P^*}{\longrightarrow} \mathcal{D}_X^{\Omega})[n]$$

(the complex is concentrated in degrees [-n,-(n-1)]), where $P^* : Q \to P \circ Q$. Recall that there is canonical involution on $\mathcal{D}_X^{\Omega}$ interchanging two left $\mathcal{D}_X$-structures. If we choose a volume form ω this involution sends $P \otimes \omega^{-1}$ just to $P^t \otimes \omega^{-1}$ where P^t is the transposed to P defined using ω. The lemma follows immediately from these remarks.

Let

$$\mathcal{R}_{\mathcal{P}} := \mathcal{D}_X \stackrel{P}{\longrightarrow} \mathcal{D}_X, \quad \mathcal{R}_{\mathcal{P}^*} := \mathcal{D}_X^{\Omega} \stackrel{P^*}{\longrightarrow} \mathcal{D}_X^{\Omega}$$

be the resolutions for $\mathcal{P}$ and $\tilde{\star}\mathcal{P}$. Their tensor product over $\mathcal{O}$

$$\mathcal{R}_{\mathcal{P}} \otimes_{\mathcal{O}} \mathcal{R}_{\mathcal{P}^*} = \begin{array}{ccc} \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} & \stackrel{P\otimes 1}{\longrightarrow} & \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \\ 1 \otimes P^* \uparrow & & \uparrow -P^* \otimes 1 \\ \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} & \stackrel{P\otimes 1}{\longrightarrow} & \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \end{array}$$

sits in degrees $[-(n-2), -n]$. Let

$$... \xrightarrow{d} \mathcal{R}_{\mathcal{P}} \otimes_{\mathcal{O}} \mathcal{R}_{\mathcal{P}^*} \otimes_{\mathcal{O}} \Omega^{n-1} \xrightarrow{d} \mathcal{R}_{\mathcal{P}} \otimes_{\mathcal{O}} \mathcal{R}_{\mathcal{P}^*} \otimes_{\mathcal{O}} \Omega^{n}$$

be the de Rham complex $DR(\mathcal{R}_{\mathcal{P}} \otimes_{\mathcal{O}} \mathcal{R}_{\mathcal{P}^*})$. Its degree $-(n-1)$ part is

$$\Big(\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \otimes_{\mathcal{O}} \Omega^n \quad \oplus \quad \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \otimes_{\mathcal{O}} \Omega^n\Big) \quad \oplus \quad \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \otimes_{\mathcal{O}} \Omega^{n-1} \tag{24}$$

Since $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \otimes_{\mathcal{O}} \Omega^n = \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$, there is a canonical element $(1 \otimes 1, 1 \otimes 1)$ in the left summand of (24).

Choose a covering $\{\mathcal{U}_i\}$ of X. Consider the Cech complex

$$C\Big(\{\mathcal{U}_\bullet\}, DR(\mathcal{R}_{\mathcal{P}} \otimes_{\mathcal{O}} \mathcal{R}_{\mathcal{P}^*})\Big)$$

An $-(n-1)$-cocycle $\tilde{G}_{\mathcal{M}}$ in this complex such that

$$\text{the component in} \quad C\Big(\mathcal{U}_i, (\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \otimes_{\mathcal{O}} \Omega^n \quad \oplus \quad \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \otimes_{\mathcal{O}} \Omega^n)\Big) \quad \text{is} \quad (1 \otimes 1, 1 \otimes 1)$$

represents the Green class. Its existence follows from general theory.

Let $\omega^{(i)}_{n-1}$ be the componenet of $\tilde{G}_{\mathcal{M}}$ in $C(\mathcal{U}_i, \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \otimes_{\mathcal{O}} \Omega^{n-1})$, and $\omega^{(i,j)}_{n-2}$ the component in $C(\mathcal{U}_{i,j}, \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}^{\Omega} \otimes_{\mathcal{O}} \Omega^{n-2})$. Then $d\omega^{(i,j)}_{n-2} = \omega^{(i)}_{n-1} - \omega^{(j)}_{n-1}$. To relate this cocycle with the discussion in s. 4.1 notice that $\omega^{(i)}_{n-1}$ can be vied as an "algebraic bidifferential operator".

2. The Green formula and the Bar construction. Let E^1 and E^2 be vector bundles over an n-dimensional manifold X and $E^1 \xrightarrow{P} E^2$ be a differential operator. Set $V_i := E^{i^*} \otimes \mathcal{A}^n$. There are canonical pairings

$$\Gamma_0(X, E^i) \otimes \Gamma(X, V_i) \longrightarrow \mathbb{R} \qquad (\varphi, g \otimes \omega) \longrightarrow \int_X (\varphi, g)\omega$$

So one has the adjoint operator $V_1 \xleftarrow{P^*} V_2$. It is a differential operator of the same order as P uniquely defined by the property $(\varphi_1, P^* v_2) = (P\varphi_1, v_2)$.

Now suppose we have a sequence (not necessarily a complex) of differential operators

$$E^0 \xrightarrow{P_1} E^1 \xrightarrow{P_2} ... \xrightarrow{P_k} E^k$$

Consider the sequence of adjoint differential operators

$$V_0 \xleftarrow{P_1^*} V_1 \xleftarrow{P_2^*} ... \xleftarrow{P_k^*} V_k$$

Theorem 4.2 *For any k there exists forms $\omega_{n-k}(\varphi_0; P_1, ..., P_k; v_k)$ such that $\omega_n(\varphi; 1; v) := \varphi \cdot v$ and*

$$d\omega_{n-k}(\varphi_0; P_1, ..., P_k; v_k) = \omega_{n-k+1}(P_1\varphi_0; P_2, ..., P_k; v_k) +$$

$$\sum_{i=1}^{k-1} (-1)^i \omega_{n-k+1}(\varphi_0; P_1, ..., P_i \circ P_{i+1}, ..., P_k; v_k) + (-1)^k \omega_{n-k+1}(\varphi_0; P_1, ..., P_{k-1}; P_k^* v_k)$$

3. How to compute the Green class. Let us call a $\mathcal{D}$-module $\mathcal{M}$ *excellent* if the object $\star\mathcal{M}$ is concentrated in just one degree, i.e. $H^i(\star\mathcal{M}) = 0$ for all i but one. By the Roos theorem this degree is $-d_{\mathcal{M}}$. In this case set $\tilde{\star}\mathcal{M} := H^{-d_{\mathcal{M}}}(\star\mathcal{M})$. Consider a locally free resolution of a $\mathcal{D}$-module $\mathcal{M}$:

$$\mathcal{P}^\bullet = \{\mathcal{P}^{-k} \longrightarrow ... \longrightarrow \mathcal{P}^{-2} \longrightarrow \mathcal{P}^{-1} \longrightarrow \mathcal{P}^0\}$$

Let

$$\star\mathcal{P}^\bullet = \{*(\mathcal{P}^0) \longrightarrow *(\mathcal{P}^1) \longrightarrow *(\mathcal{P}^2) \longrightarrow ... \longrightarrow *(\mathcal{P}^k)\}[d_X]$$

be the dual complex. Then $E^\bullet := Hom_{\mathcal{D}}(\mathcal{P}^\bullet, C^\infty(X))$ is a complex of differential operators between vector bundles:

$$E^0 \xrightarrow{P_1} E^1 \xrightarrow{P_2} ... \xrightarrow{P_k} E^k$$

The adjoint complex

$$V_\bullet := \quad \{V_k \xrightarrow{P_k^*} V_{k-1} \xrightarrow{P_{k-1}^*} ... \xrightarrow{P_1^*} V_0\}$$

is canonically isomorphic to $Hom_{\mathcal{D}}(\star\mathcal{P}^\bullet, C^\infty(X))[-d_X - k]$.

Suppose that a $\mathcal{D}$-module $\mathcal{M}$ is excellent and admits a locally free resolution of the minimal possible length $k = d_{\mathcal{M}}$. (This usually happen in integral geometry). Then

$$Sol(\mathcal{M}, C^\infty(X)) = KerP_1 \quad \text{and} \quad Sol(\tilde{*}\mathcal{M}, D'(X)) = KerP_k^*$$

Therefore for any $\varphi_0 \in KerP_1$ and $v_k \in KerP_k^*$ one has $d\omega_{n-d_{\mathcal{M}}}(\varphi_0; \mathcal{P}^\bullet; v_k) = 0$

Theorem 4.3 *The cohomology class of the form $\omega_{n-d_{\mathcal{M}}}(\varphi_0; \mathcal{P}^\bullet; v_k)$ coincides with the Green class $G_{\mathcal{M}}(\varphi_0; v_k)$.*

Proof. It is similar to the proof of lemma (4.1). Since $\mathcal{P}^{-i}$ is a locally free $\mathcal{D}$-module, there is a canonical element 1_i in

$$\mathcal{P}^{-i} \otimes_{\mathcal{O}} Hom_{\mathcal{D}}(\mathcal{P}^{-i}, \mathcal{D}^\Omega) \otimes \Omega \tag{25}$$

Namely, locally $\mathcal{P}^{-i} = \mathcal{D} \otimes_{\mathbb{C}} V$, so (25) is

$$V \otimes_{\mathbb{C}} V^* \otimes_{\mathbb{C}} \mathcal{D} \otimes_{\mathcal{O}} Hom_{\mathcal{D}}(\mathcal{D}, \mathcal{D}^\Omega) \otimes \Omega = End(V) \otimes_{\mathbb{C}} \mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$$

and we take $Id_V \otimes 1 \otimes 1$. A 0-cycle in $DR(\mathcal{P} \otimes_{\mathcal{O}} \star\mathcal{P})$ whose component in $\mathcal{P} \otimes_{\mathcal{O}} \star\mathcal{P} \otimes_{\mathcal{O}} \Omega$ is $\sum 1_i$ represents the Green class.

5 General linear maps and natural linear maps between solution spaces

Denote by $RHom^c(\cdot, \cdot)$ the $RHom$ with compact supports in the category of sheaves. For $\mathcal{M}, \mathcal{N} \in D^b(\mathcal{D}_X)$ set

$$RHom^c_{\mathcal{D}_X}(\mathcal{M}, \mathcal{N}) := R\Gamma_c(X; RHom_{\mathcal{D}_{\mathcal{X}}}(\mathcal{M}, \mathcal{N}))$$

We will define a canonical morphism

$$RHom^c_{\mathcal{D}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2, D'(X_1 \times X_2)) \quad \otimes \quad RHom_{\mathcal{D}}(\mathcal{M}_1, C^\infty(X_1)) \longrightarrow \tag{26}$$

$$RHom_{\mathcal{D}}(\mathcal{M}_2, D'(X_2))[-n]$$

Any linear map

$$RHom_{\mathcal{D}}(\mathcal{M}_1, C^\infty(X_1)) \longrightarrow RHom_{\mathcal{D}}(\mathcal{M}_2, D'(X_2))[-n] \tag{27}$$

continuous in an appropriate topology is given by a unique element in

$$RHom^c_{\mathcal{D}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2, D'(X_1 \times X_2)) \tag{28}$$

(this follows from lemma (5.1)), so the space (28) gives us the general linear maps (27). Our goal in this paper is to construct and study an interesting subspace in (28), the subspace of *natural* linear maps.

To make more clear the relation with natural functionals we will spell the construction of the map (26) using the Green class

$$G_{\mathcal{M}} \in H^{dimX}(DR(\star\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M})) \tag{29}$$

and using the canonical morphism in $D^b(\mathcal{D}_X)$

$$i_{\mathcal{M}} : \mathcal{O}_X \longrightarrow \star\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{M} \tag{30}$$

They are, of cource, equivalent. To clarify the main point we will start from the case when X_2 is a point.

1. The canonical pairing via the Green class. Let Δ_X be the orientation sheaf of X. Set $\tilde{D}'(X) := D'(X) \otimes_{\mathbb{Z}} \Delta_X$. We will define the canonical pairing

$$R^i Hom^c_{\mathcal{D}_X}(\star\mathcal{M}, \tilde{D}'(X)) \otimes R^{n-i} Hom_{\mathcal{D}_X}(\mathcal{M}, C^\infty(X)) \longrightarrow \mathbb{C}$$

If $\mathcal{A}_i, \mathcal{B}_j$ are sheaves on X, $a \in R^i Hom(\mathcal{A}_1, \mathcal{A}_2)$ and $b \in R^j Hom(\mathcal{B}_1, \mathcal{B}_2)$ then the tensor product over $\mathbb{C}$ provides an element

$$a \otimes_{\mathbb{C}} b \in R^{i+j} Hom(\mathcal{A}_1 \otimes_{\mathbb{C}} \mathcal{B}_1, \mathcal{A}_2 \otimes_{\mathbb{C}} \mathcal{B}_2)$$

If $\mathcal{A}_i, \mathcal{B}_j$ are sheaves of $\mathcal{O}$-modules on X we can make a tensor product over $\mathcal{O}$:

$$a \otimes_{\mathcal{O}} b \in R^{i+j} Hom(\mathcal{A}_1 \otimes_{\mathcal{O}} \mathcal{B}_1, \mathcal{A}_2 \otimes_{\mathcal{O}} \mathcal{B}_2)$$

Suppose X is a smooth variety over $\mathbb{R}$ of dimension n and

$$v \in R^i Hom^c_{\mathcal{D}_X}(\star\mathcal{M}, D'(X)), \qquad f \in R^{n-i} Hom_{\mathcal{D}_X}(\mathcal{M}, C^\infty(X))$$

Their tensor product over $\mathcal{O}_X$ is an element

$$v \otimes_{\mathcal{O}} f \in R^n Hom_{\mathcal{D}_X}(\star\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}, \tilde{D}'(X) \otimes_{\mathcal{O}} C^\infty(X)) \tag{31}$$

So the multiplication $m : \tilde{D}'(X) \otimes_{\mathcal{O}_X} C^\infty(X) \longrightarrow \tilde{D}'(X)$ leads to an element

$$m(v \otimes_{\mathcal{O}} f) \in R^{i+j} Hom_{\mathcal{D}_X}(\star\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}, \tilde{D}'(X))$$

Applying this element to the Green class (29) we get a cohomology class

$$m(v \otimes_{\mathcal{O}} f)(G_{\mathcal{M}}) \in H^n_c\Big(DR(\tilde{D}'(X)[n]\Big) = H^n_c(\tilde{X}, \mathbb{C}) = \mathbb{C}$$

where $\tilde{X} = X$ if X is orientable and it is a two fold covering given by the orientation class if it is not.

2. The canonical pairing via the morphism $i_{\mathcal{M}}$. Taking the Koszule resolution of the $\mathcal{D}$-module $\mathcal{O}_X$ we see that the complex of sheaves $RHom_{\mathcal{D}_X}(\mathcal{O}_X, D'(X))$ is canonically quasiisomorphic to the De Rham complex of currents on X:

$$D'(X) \xrightarrow{d} D'(X) \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} D'(X) \otimes_{\mathcal{O}_X} \Omega^{n-1}_X \xrightarrow{d} D'(X) \otimes_{\mathcal{O}_X} \Omega^n_X$$

(the last group sitting in degree n). If we take the $RHom$'s with compact support we get the De Rham complex of currents with compact support.

There is the trace map given by integration over the fundamental class of X:

$$\int_X : R^n Hom^c_{\mathcal{D}_X}(\mathcal{O}_X, \tilde{D}'(X)) \longrightarrow \mathbb{C}$$

The composition of the morphism $i_{\mathcal{M}}$ (30) with the element $m(v \otimes_{\mathcal{O}} f)$ (31) gives

$$m \circ i_{\mathcal{M}}(v \otimes_{\mathcal{O}} f) \in R^n Hom^c_{\mathcal{D}_X}(\mathcal{O}_X, \tilde{D}'(X))$$

Applying $\int_X$ we get a pairing

$$v \otimes f \longmapsto \int_X m \circ i_{\mathcal{M}}(v \otimes_{\mathcal{O}} f) \in \mathbb{C}$$

Recall that there is the Grothendieck duality theory for topological vector spaces [Gr]. In particular $C^\infty(X)$ has a natural topology of a Fréchet nuclear space, and $D'_0(X)$ has a natural topology of a dual to a Fréchet nuclear space, so they are topologically dual. An immediate consequence of this is the following simple duality lemma. (For a more general result see theorem 6.1 in [[KS]).

Lemma 5.1 *Suppose X is a smooth variety over $\mathbb{R}$ of dimension n. Then*

$$R^i Hom_{\mathcal{D}_X}(\mathcal{M}, C^\infty(X)) \tag{32}$$

has a topology of a Fréchet nuclear space and

$$R^{n-i} Hom^c_{\mathcal{D}_X}(\star\mathcal{M}, D'(X)) \tag{33}$$

has a topology of a dual to a Fréchet nuclear space. The spaces (32) and (33) are dual to each other.

Proof. Consider first the classical case of a single differential operator. Let P be a differential operator acting on smooth functions and P^* the adjoint acting on the distributions with compact supports:

$$C^\infty(X) \stackrel{P}{\longrightarrow} C^\infty(X)$$

$$D^n_0(X) \stackrel{P^*}{\longleftarrow} D^n_0(X)$$

The canonical pairing boils down to the obvious duality between $KerP$ and the closure of $CokerP^*$, and the closure of $CokerP$ and $KerP^*$. The general statement for any $\mathcal{M} \in D^b(\mathcal{D}_X)$ we get similarly taking a locally projective resolution.

3. A construction of the map (26). Let $RHom^{c_1}$ be the $RHom$ with compact supports along the factor X_1. Choose

$$K \in RHom^{c_1}_{\mathcal{D}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2, D'(X_1 \times X_2)) \quad f \in RHom_{\mathcal{D}}(\mathcal{M}_1, C^\infty(X_1))$$

Their product $K \otimes_{\mathcal{O}_{X_1}} f$ over X_1 belongs to

$$RHom^{c_1}_{\mathcal{D}_{X_1\times X_2}}\Big(\star\mathcal{M}_1 \otimes_{\mathcal{O}_{X_1}} \mathcal{M}_1 \boxtimes \mathcal{M}_2, \tilde{D}'(X_1) \otimes_{\mathcal{O}_{X_1}} C^\infty(X_1) \boxtimes D'(X_2)\Big)$$

Using the multiplication map

$$m_{X_1} : \tilde{D}'(X_1) \otimes_{\mathcal{O}_{X_1}} C^\infty(X_1) \boxtimes D'(X_2) \longrightarrow \tilde{D}'(X_1) \boxtimes D'(X_2)$$

we get a class

$$m_{X_1}(K \otimes_{\mathcal{O}_{X_1}} f) \in RHom^{c_1}_{\mathcal{D}_{X_1\times X_2}}\Big(\star\mathcal{M}_1 \otimes_{\mathcal{O}_{X_1}} \mathcal{M}_1 \boxtimes \mathcal{M}_2, \tilde{D}'(X_1) \boxtimes D'(X_2)\Big)$$

The canonical morphism $i_{\mathcal{M}_1} : \mathcal{O}_{X_1} \longrightarrow \star\mathcal{M}_1 \otimes_{\mathcal{O}_{X_1}} \mathcal{M}_1$ provides an element

$$(m_{X_1} \circ i_{\mathcal{M}_1})(K \otimes_{\mathcal{O}_{X_1}} f) \in RHom^{c_1}_{\mathcal{D}_{X_1\times X_2}}\Big(\mathcal{O}_{X_1} \boxtimes \mathcal{M}_2, \tilde{D}'(X_1) \boxtimes D'(X_2)\Big)$$

Applying $\int_{X_1} : R^n Hom^c_{\mathcal{D}_{X_1}}(\mathcal{O}_{X_1}, \tilde{D}'(X_1)) \longrightarrow \mathbb{C}$ we get

$$\bar{K}(f) \in RHom_{\mathcal{D}_{X_2}}(\mathcal{M}_2, D'(X_2))$$

4. Natural functionals. Recall that $R^{n-m}Hom^c(\mathbb{C}, \Delta_X) = H_m(X, \mathbb{C})$ and the Poincare duality is given by

$$R^i Hom(\mathbb{C}, \mathbb{C}) \otimes R^{n-i} Hom^c(\mathbb{C}, \Delta_X) \longrightarrow R^n Hom^c(\mathbb{C}, \Delta_X) \stackrel{\int_X}{\longrightarrow} \mathbb{C}$$

The first arrow is the composition of $Hom's$. Tensor product over $\mathbb{C}$ provides a canonical map

$$R^j Hom^c(\mathbb{C}, \Delta_X) \quad \otimes \quad R^i Hom_{\mathcal{D}_X}(\mathcal{M}, D'(X)) \longrightarrow R^{i+j} Hom^c(\mathcal{M}, \tilde{D}'(X))$$

Combining it with the canonical pairing we get a map

$$< \cdot, \cdot, \cdot >_{\mathcal{M}}:$$

$$H_{i+j}(X,\mathbb{C})\otimes R^iHom_{\mathcal{D}_X}(\star\mathcal{M},D'(X))\otimes R^jHom_{\mathcal{D}_X}(\mathcal{M},C^\infty(X))\longrightarrow\mathbb{C} \tag{34}$$

By definition the natural functionals on the space $R^jHom_{\mathcal{D}_X}(\mathcal{M},C^\infty(X))$ are the functionals $<\gamma,v,\cdot>_{\mathcal{M}}$ provided by a homology class $\gamma\in H_{i+j}(X,\mathbb{C})$ and $v\in R^iHom_{\mathcal{D}_X}(\star\mathcal{M},\tilde{D}'(X))$.

5. Natural linear maps. There is a map

$$R\Gamma_c(X_1,\Delta_{X_1})\quad\otimes\quad Hom_{\mathcal{D}}(\star\mathcal{M}_1\boxtimes\mathcal{M}_2,D'(X_1\times X_2))\quad\longrightarrow$$
$$RHom^c_{\mathcal{D}}(\star\mathcal{M}_1\boxtimes\mathcal{M}_2,D'(X_1\times X_2))$$

So we get a canonical morphism

$$R\Gamma_c(X_1,\Delta_{X_1})\otimes RHom_{\mathcal{D}}(\star\mathcal{M}_1\boxtimes\mathcal{M}_2,D'(X_1\times X_2))\otimes RHom_{\mathcal{D}}(\mathcal{M}_1,C^\infty(X_1))\longrightarrow$$
$$RHom_{\mathcal{D}}(\mathcal{M}_2,D'(X_2))[-n] \tag{35}$$

In particular it induces a map

$$H_{i+j}(X_1,\mathbb{Z})\otimes R^{i+k}Hom_{\mathcal{D}_{X_1\times X_2}}(\star\mathcal{M}_1\boxtimes\mathcal{M}_2,D'(X_1\times X_2))\otimes R^jHom_{\mathcal{D}}(\mathcal{M}_1,C^\infty(X_1))\longrightarrow$$
$$R^kHom_{\mathcal{D}_{X_2}}(\mathcal{M}_2,D'(X_2)) \tag{36}$$

By definition a natural linear map

$$\bar{K}_\gamma:R^jHom_{\mathcal{D}_{X_1}}(\mathcal{M}_1,C^\infty(X_1))\longrightarrow R^kHom_{\mathcal{D}_{X_2}}(\mathcal{M}_2,D'(X_2))$$

is given by a "kernel"

$$K=K(x_1,x_2)\in R^{i+k}Hom_{\mathcal{D}_{X_1\times X_2}}(\star\mathcal{M}_1\boxtimes\mathcal{M}_2,D'(X_1\times X_2))$$

and $\gamma\in H_{i+j}(X_1,\mathbb{Z})$.

6. Examples. Suppose $\mathcal{M}$ is an excellent $\mathcal{D}_X$-module, $m:=d_{\mathcal{M}}$. Recall that $\tilde{\star}\mathcal{M}:=H^{-m}(\star\mathcal{M})$ is the dual system to $\mathcal{M}$. Taking $k=m$, $i=0$, $l=m$ we get

$$Hom_{\mathcal{D}}(\mathcal{M},C^\infty(X))\otimes Hom_{\mathcal{D}}(\tilde{\star}\mathcal{M},D'(X))\otimes H_m(X,\mathbb{R})\longrightarrow\mathbb{R}$$

Let $\tilde{G}_{\mathcal{M}}(\cdot,\cdot)$ be a cocycle in $DR(\tilde{\star}\mathcal{M}\otimes_{\mathcal{O}}\mathcal{M})$ representing the Green class. Let $f(x)$ be a smooth solution of the system $\mathcal{M}$. Choose a distributional solution $v(x)$ of $\tilde{\star}\mathcal{M}$. Then we get a closed differential form $\tilde{G}_{\mathcal{M}}(v(x),f(x))$ of degree $d_{\mathcal{M}}$ on X. Choose a cycle γ of dimension $d_{\mathcal{M}}$ in X. Then

$$<\gamma,v,f>=\int_\gamma\tilde{G}_{\mathcal{M}}(v(x),f(x))$$

is a functional on smooth solutions of $\mathcal{M}$.

Example 0. Suppose $\mathcal{M}=\mathcal{D}_X$. Then $\star\mathcal{M}=\mathcal{D}^\Omega_X[n]$ and $\tilde{\star}\mathcal{M}=\mathcal{D}^\Omega_X$. Recall that $Hom^c_{\mathcal{D}}(\mathcal{D}^\Omega_X,D'(X))=D^n_0(X)$. We get the usual paring $C^\infty(X)\otimes D^n_0(X)\longrightarrow\mathbb{C}$.

The following examples show a wider class of functionals on solution spaces then the natural functionals we just defined above. The point is that sometimes we can integrate the differential form $\tilde{G}_{\mathcal{M}}(v(x),f(x))$ not only over cycles, but also over some chains (which do not represent a homology class in general sense) still getting a functional on smooth solutions of a system $\mathcal{M}$.

Example 1. Let $\mathcal{M}$ be a $\mathcal{D}$-module on $\mathbb{R}^n$ corresponding to the system $x_1\cdot f=x_2\cdot f=...=x_m\cdot f=0$, $0<m<n$. Then

$$RHom^i_{\mathcal{D}}(\mathcal{M},C^\infty(X))=0\quad\text{for}\quad i\neq m$$
$$RHom^m_{\mathcal{D}}(\mathcal{M},C^\infty(X))=\frac{C^\infty(X)}{(C^\infty(X)\cdot x_i)}$$

and

$$RHom^i_{\mathcal{D}}(\mathcal{M},D'(X))=0\quad\text{for}\quad i\neq 0$$

$$Hom_{\mathcal{D}}(\mathcal{M}, D'(X)) = \delta(x_1)\delta(x_2)...\delta(x_m) \cdot D'(X)$$

So there is a natural pairing

$$RHom^m_{\mathcal{D}}(\mathcal{M}, C^\infty(X)) \otimes Hom_{\mathcal{D}}(\mathcal{M}_1, D'(X)) \longrightarrow \mathbb{R}$$

It should correspond to the case $i = 0, j = m, k = m$. However $H_m(\mathbb{R}^n, \mathbb{R}) = 0$ in any topological sense.

Comparing the general and natural functionals. Let P be a differential operator on X. Recall that the general functionals on $KerP$ we get from the closure of $CokerP^*$, see the proof of proposition (5.1). The natural functionals we get in a different way. Take $f \in KerP$ and $v \in D'(X)$, $v \in KerP^*$. Notice that if, for example, P is an operator with constant coefficients, then the restriction of $KerP \cap D'_0(X) = 0$, so it is essential that v is not necessarily compactly supported. Then make the Green class $[\omega_{n-1}(v; P; f)]$ and integrate it over a homology class $[\gamma]$. A simplest example is given in the example 2 below.

An advantage of natural functionals on $KerP$ is that they correspond to "functions", i.e. elements of the subspace $KerP^*$, rather then to elements of the quotient $CokerP^*$.

Example 2. Let $\mathcal{L}_a$ be the system $(\sum_{i=1}^n x_i \partial_{x_i} - a)f(x) = 0$ on $\mathbb{R}^n \backslash 0$. Then $\tilde{\star}\mathcal{L}_a = \mathcal{L}_{-a-n}$. Consider the following differential $(n-1)$-form

$$\sigma_n(x, dx) := \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge ... \wedge \hat{dx_i} \wedge ... \wedge dx_n \tag{37}$$

Then

$$\tilde{G}_{\mathcal{L}_a}(v(x), f(x)) = v(x) f(x) \sigma_n(x, dx)$$

Let γ be an $(n-1)$-cycle generating $H_{n-1}(\mathbb{R}^n \backslash 0)$. Then

$$\int_\gamma v(x) \cdot f(x) \sigma_n(x, dx) \tag{38}$$

provides a nondegenerate pairing between the smooth solutions of $\mathcal{L}_a$ and $\mathcal{L}_{-a-n}$.

Example 3. Consider $\mathbb{C}^n$ as a real manifold. Let $\mathcal{L}_{a,b}$ be the following system in $\mathbb{C}^n \backslash 0$:

$$(\sum_{i=1}^n z_i \partial_{z_i} - a) f(z, \bar{z}) = 0, \qquad (\sum_{i=1}^n \bar{z}_i \partial_{\bar{z}_i} - b) f(z, \bar{z}) = 0$$

Then $\tilde{\star}\mathcal{L}_{a,b} = \mathcal{L}_{-a-n,-b-n}$. Then

$$\tilde{G}_{\mathcal{L}_{a,b}}(v(z, \bar{z}), f(z, \bar{z})) = v(z, \bar{z}) \cdot f(z, \bar{z}) \cdot \sigma_n(z, dz) \wedge \sigma_n(\bar{z}, d\bar{z})$$

Let Γ be a chain intersecting any one dimensional subspace in $\mathbb{C}^n$ with multiplicity one. Then

$$\int_\Gamma v(z, \bar{z}) f(z, \bar{z}) \sigma_n(z, dz) \wedge \sigma_n(\bar{z}, d\bar{z}) \tag{39}$$

provides a pairing between the solutions of $\mathcal{L}_{a,b}$ and $\mathcal{L}_{-a-n,-b-n}$. However $H_{2n-2}(\mathbb{C}^n \backslash 0) = H_{2n-2}(S^{2n-1}) = 0$!

A chain Γ can be considered as a discontinuous "section " of the Hopf bundle $\mathbb{C}^n \backslash 0 \longrightarrow \mathbb{C}P^{n-1}$. A better way to think about this integral is the following. The form we integrate can be pushed down to $\mathbb{C}P^{n-1}$, so we integrate over the fundamental cycle.

7. Composition of natural maps between smooth solution spaces. We can not define in general a morphism

$$RHom_{\mathcal{D}}(\mathcal{M}_1, D'(X_1)) \longrightarrow RHom_{\mathcal{D}}(\mathcal{M}_2, D'(X_2))$$

using distributional kernels because of the lack of multiplication of distributions, and a priory there is no way to compose operators

$$RHom_{\mathcal{D}}(\mathcal{M}_1, C^\infty(X_1)) \longrightarrow RHom_{\mathcal{D}}(\mathcal{M}_2, D'(X_2))$$

and

$$RHom_{\mathcal{D}}(\mathcal{M}_2, C^\infty(X_2)) \longrightarrow RHom_{\mathcal{D}}(\mathcal{M}_3, D'(X_3))$$

However the natural linear maps constructed using smooth kernels can be composed. Namely, suppose

$$K_{12} \in RHom_{\mathcal{D}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2, C^\infty(X_1 \times X_2)), \quad \gamma_1 \in H_{l_1}(X_1, \mathbb{R})$$

$$K_{23} \in RHom_{\mathcal{D}}(\star\mathcal{M}_2 \boxtimes \mathcal{M}_3, C^\infty(X_2 \times X_3)), \quad \gamma_2 \in H_{l_2}(X_2, \mathbb{R})$$

They define the corresponding natural maps

$$\bar{K}_{12}^{\gamma_1} : RHom_{\mathcal{D}}(\mathcal{M}_1, C^\infty(X_1)) \longrightarrow RHom_{\mathcal{D}}(\mathcal{M}_2, C^\infty(X_2))$$

$$\bar{K}_{23}^{\gamma_2} : RHom_{\mathcal{D}}(\mathcal{M}_2, C^\infty(X_2)) \longrightarrow RHom_{\mathcal{D}}(\mathcal{M}_3, C^\infty(X_3))$$

Their composition is given by the data

$$K_{23} \circ K_{12} \in RHom_{\mathcal{D}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_3, C^\infty(X_1 \times X_3)), \quad \gamma_1 \in H_{l_1}(X_1, \mathbb{R})$$

where the kernel $K_{23} \circ K_{12}$ is constructed as follows. Let

$$\Delta_2 : X_1 \times X_2 \times X_3 \hookrightarrow X_1 \times X_2 \times X_2 \times X_3$$

be the diagonal imbedding and $\pi_2 : X_1 \times X_2 \times X_3 \longrightarrow X_1 \times X_3$ be the projection. Then

$$\Delta_2{}^! C^\infty(X_1 \times X_2 \times X_2 \times X_3) = C^\infty(X_1 \times X_2 \times X_3)[-d_{X_2}]$$

and

$$H_l(X_2, \mathbb{Z}) \longrightarrow R^{d_{X_2}-l} Hom_{\mathcal{D}}(\pi_{2*} C^\infty(X_1 \times X_2 \times X_3), C^\infty(X_1 \times X_3))$$

Therefore one has canonical morphism

$$H_l(X_2, \mathbb{Z}) \longrightarrow R^{-l} Hom_{\mathcal{D}}(\pi_{2*} \Delta_2{}^! C^\infty(X_1 \times X_2 \times X_2 \times X_3), C^\infty(X_1 \times X_3))$$

According to lemma 3.8 one has $G_{\mathcal{M}} \in p_* \Delta^!(\mathcal{M} \boxtimes \star\mathcal{M})$. Therefore

$$\star\mathcal{M}_1 \boxtimes \mathcal{M}_3 \stackrel{id \boxtimes G_{\mathcal{M}} \boxtimes id}{\longrightarrow} \pi_{2*} \Delta_2{}^! (\star\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \star\mathcal{M}_2 \boxtimes \mathcal{M}_3) \tag{40}$$

There is a canonical map

$$RHom_{\mathcal{D}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2, C^\infty(X_1 \times X_2)) \otimes RHom_{\mathcal{D}}(\star\mathcal{M}_2 \boxtimes \mathcal{M}_3, C^\infty(X_2 \times X_3)) =$$

$$RHom_{\mathcal{D}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \star\mathcal{M}_2 \boxtimes \mathcal{M}_3, C^\infty(X_1 \times X_2 \times X_2 \times X_3)) \stackrel{G_{\mathcal{M}} \times \gamma_2}{\longrightarrow}$$
$$RHom_{\mathcal{D}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_3, C^\infty(X_1 \times X_3))$$

provided by the morphism (40) and morphism

$$\gamma_2 \in R^{-l_2} Hom_{\mathcal{D}}(\pi_{2*} \Delta_2{}^! C^\infty(X_1 \times X_2 \times X_2 \times X_3), C^\infty(X_1 \times X_3))$$

Theorem 5.2 *The kernel $K_{23} \circ K_{12}$ coincides with $(G_{\mathcal{M}} \times \gamma_2)(K_{23} \otimes K_{12})$.*

Proof. Follows immediately from the definitions.

Example. Suppose $\mathcal{M}_i$ are excellent $\mathcal{D}_{X_i}$-modules. In this case usually the natural smooth kernels are just functions

$$K_{12}(x_1, x_2) \in Hom_{\mathcal{D}}(\tilde{\star}\mathcal{M}_1 \boxtimes \mathcal{M}_2, C^\infty(X_1 \times X_2))$$

and

$$K_{23}(x_2, x_3) \in Hom_{\mathcal{D}}(\tilde{\star}\mathcal{M}_2 \boxtimes \mathcal{M}_3, C^\infty(X_1 \times X_2))$$

and the composition is defined by the natural kernel

$$K_{13}(x_1, x_3) = \int_{\gamma_2} G_{\mathcal{M}_2}\Big(K_{12}(x_1, x_2), K_{23}(x_2, x_3)\Big)$$

6 Natural linear maps for excellent $\mathcal{D}$-modules.

1. The general scheme. Let $\mathcal{M}$ and $\mathcal{N}$ be excellent $\mathcal{D}$-modules on manifolds X and Y, i.e. $\tilde{\star}\mathcal{M} := (\star\mathcal{M})[-d_{\mathcal{M}}]$ is a $\mathcal{D}$-module. Let $c_{\mathcal{M}} := codimSS\mathcal{M} = dimX - d_{\mathcal{M}}$. Then solutions

$$f \in Hom_{\mathcal{D}}(\mathcal{M}, C^{\infty}(X)) \quad \text{and} \quad g \in Hom_{\mathcal{D}}(\tilde{\star}\mathcal{M}, D'(X))$$

provide a homomorphism

$$H^{c_{\mathcal{M}}} DR\Big(\tilde{\star}\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M}\Big) \stackrel{\bar{f}\otimes\bar{g}}{\longrightarrow} H^{c_{\mathcal{M}}} DR\Big(C^{\infty}(X) \otimes_{\mathcal{O}} D'(X)\Big) \stackrel{m}{\longrightarrow} H^{d_{\mathcal{M}}} D'^{\bullet}(X)$$

The Green class of $\mathcal{M}$ goes under this map to a cohomology class of degree $d_{\mathcal{M}}$ on X. Recall that we put $DR(\mathcal{M})$ in degrees $[-dimX, 0]$, while the smooth de Rham complex $\mathcal{A}^{\bullet}(X)$ is sitting in degrees $[0, dimX]$.

Let us define a natural linear map

$$I: \quad Sol(\mathcal{M}, C^{\infty}(X)) \longrightarrow Sol(\mathcal{N}, D'(Y))$$

by a kernel

$$K_I(x,y) \in Sol(\tilde{\star}\mathcal{M} \boxtimes \mathcal{N}, D'(X \times Y)) \tag{41}$$

and a cycle γ_X of dimension $d_{\mathcal{M}}$ in X as follows. Let $\tilde{G}_{\mathcal{M}}(\cdot,\cdot)$ be a cocycle in the Cech complex of a covering of of X with coefficients in $DR(\tilde{\star}\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M})$ representing the Green class. (In integral geometry one may usually take a cocycle in the complex $DR(\tilde{\star}\mathcal{M} \otimes_{\mathcal{O}} \mathcal{M})$). Using solutions $K_I(x,y)$ of $\tilde{\star}\mathcal{M}$ (where y is considered as a parameter) and $f(x)$ of $\mathcal{M}$ we get a closed differential form $\tilde{G}_{\mathcal{M}}(K_I(x,y), f(x))$ of degree $d_{\mathcal{M}}$ on X. Set

$$f(x) \longmapsto \int_{\gamma} \tilde{G}_{\mathcal{M}}(K_I(x,y), f(x)) \in Sol(\mathcal{N}, D'(Y)) \tag{42}$$

Under certain assumption on the wave front of the kernel $K_I(x,y)$, which we will assume below, the integral over cycle γ makes sense and the image of (41) lies in $C^{\infty}(Y)$. Then a (natural) inverse for I is an integral transformation

$$J: \quad Sol(\mathcal{N}, C^{\infty}(Y)) \longrightarrow Sol(\mathcal{M}, C^{\infty}(X))$$

$$J: \varphi(x) \longmapsto \int_{\gamma_Y} \tilde{G}_{\mathcal{N}}(K_J(x,y), \varphi(y)) \tag{43}$$

defined via a certain $d_{\mathcal{N}}$-cycle γ_Y in Y and a kernel

$$K_J(x,y) \in Sol(\mathcal{M} \boxtimes \tilde{\star}\mathcal{N}, D'(X \times Y)) \tag{44}$$

This data defines also a transformation

$$J^t: \quad Sol(\tilde{\star}\mathcal{M}, C^{\infty}(X)) \longrightarrow Sol(\tilde{\star}\mathcal{N}, C^{\infty}(Y))$$

$$g(x) \longmapsto \int_{\gamma_X} \tilde{G}_{\mathcal{M}}(g(x), K_J(x,y))$$

There is a canonical map

$$< \cdot,\cdot,\cdot >_{\mathcal{M}} \quad : \quad Sol(\tilde{\star}\mathcal{M}, C^{\infty}(X)) \otimes Sol(\mathcal{M}, C^{\infty}(X)) \otimes H_{d_{\mathcal{M}}}(X, \mathbb{R}) \longrightarrow \mathbb{R} \tag{45}$$

$$< g, f, \gamma_X >_{\mathcal{M}} \quad := \int_{\gamma_X} \tilde{G}_{\mathcal{M}}(g(x), f(x))$$

So if we choose a homology class γ_X we get a pairing

$$< g, f >_{\mathcal{M}} \quad :=< g, f, \gamma_X >_{\mathcal{M}} \tag{46}$$

and a similar one for $\mathcal{N}$.

Theorem 6.1 *(the Plancherel formula) . Let J be a natural inverse for I: $J \circ I = id_X$. Then for $f \in Sol(\mathcal{M}, C^\infty(X)), g \in Sol(\check{\mathcal{M}}, C^\infty(X))$ one has*

$$< g, f, \gamma_X >_{\mathcal{M}} \quad = \quad < J^t g, If, \gamma_Y >_{\mathcal{N}}$$

Proof. $< g, f, \gamma_X >_{\mathcal{M}} \quad = \quad < g, J \circ If, \gamma_Y >_{\mathcal{N}}$. So the theorem follows from

Lemma 6.2 *Let $\varphi \in Sol(\mathcal{N}, C^\infty(Y))$ and $g \in Sol(\check{\mathcal{M}}, C^\infty(X))$. Then*

$$< g, J\varphi, \gamma_X >_{\mathcal{M}} \quad = \quad < J^t g, \varphi, \gamma_Y >_{\mathcal{N}} \tag{47}$$

Proof. The Green class is multiplicative with respect to the $\boxtimes$ - product. So we can set $\tilde{G}_{\mathcal{M}\boxtimes\mathcal{N}} := \tilde{G}_{\mathcal{M}} \boxtimes \tilde{G}_{\mathcal{N}}$. Consider the following solutions

$$g(x) \boxtimes \varphi(y) \in Sol(\check{\mathcal{M}} \boxtimes \check{\mathcal{N}}, C^\infty(X \times Y))$$

$$K_J(x, y) \in Sol(\mathcal{M} \boxtimes \mathcal{N}, D'(X \times Y))$$

They are solutions to the dual systems. So there is a pairing

$$< g(x) \boxtimes \varphi(y), K_J(x, y), \gamma_X \times \gamma_Y >_{\mathcal{M}\boxtimes\mathcal{N}}$$

We can evaluate it computing first the pairing along X and then along Y. In this case we get the right-hand side of (47). Computing first pairing along Y and then along X we get the left-hand side of (47).

The kernel K_J is a much more simple (and fundamental) object then the actual integral transformation J. The reasons are the following:

1) The kernel K_J is a canonically defined distribution, while the formula for $J\varphi(x)$ depends on a cocycle $\tilde{G}_{\mathcal{N}}$ representing the Green class.

2) Explicit calculation of cocycle $\tilde{G}_{\mathcal{N}}$ can be a nontrivial problem and so the final formula for the right-hand side of (43) could be quite complicated even for a very simple kernel K_J.

So the problem of inversion of the transformation I splits on 3 steps:

Step 1. Find a distribution (44).

Step 2. Compute a cocycle $\tilde{G}_{\mathcal{N}}$ for the Green class.

Step 3. Find a cycle γ_Y.

The distribution (44) should be uniquely defined if exist. However it may not exist. The Green class always exist. Different cocycles representing it together with different choices of cycles γ_Y provides the diversity of concrete inversion formulas. I will demonstrate below how this general scheme works in the simplest concrete problems.

2. The Fourier transform of homogeneous functions and the Radon transform. As everybody knows the Fourier transform in an n-dimensional real vector space V_n is defined by the formula

$$S(V_n) \longrightarrow S(V_n^*); \qquad f(x) \longrightarrow \tilde{f}(\xi) := \int f(x) e^{2\pi i < x, \xi >} d^n x$$

The inverse operator is $f(\xi) \longrightarrow \int f(\xi) e^{-2\pi i < x, \xi >} d^n \xi$. Using the Plancherel formula one can define the Fourier transform of generalized functions.

Let $\Phi_\lambda^+(\mathbb{R}P^{n-1})$ (resp. $\Phi_\lambda^-(\mathbb{R}P^{n-1})$) be space of even (resp. odd) smooth homogeneous functions $f(x)$ on $\mathbb{R}^n \backslash 0$ of degree λ: $f(ax) = |a|^\lambda f(x), a > 0$. Also let $\Psi_\lambda(V_n)$ be the space of homogeneous distribution of degree λ in V_n, and $\Psi_\lambda(V_n) = \Psi_\lambda^+(V_n) \oplus \Psi_\lambda^-(V_n)$ is the decomposition on even and odd parts.

Then $\Phi_\lambda(\mathbb{R}P^{n-1}) \subset \Psi_\lambda(V_n)$. This inclusion is not an isomorphism for integral $\lambda = -k, \quad k \geq n$. One has

$$\Psi_{-k}(V_n)/\Phi_{-k}(\mathbb{R}P^{n-1}) = S^{k-n}(V_n) = \{\delta - \text{ functions of degree } -k \text{ at } 0\}$$

The Fourier transform of generalized functions provides an isomorphism

$$\tilde{\mathcal{F}}_\lambda : \Psi_{-\lambda-n}^\pm(V_n) \longrightarrow \Psi_\lambda^\pm(V_n')$$

Restricting to $\Phi^{\pm}_{-\lambda-n}(\mathbb{R}P^{n-1})$ we get a map

$$\mathcal{F}_\lambda : \Phi^{\pm}_{-\lambda-n}(\mathbb{R}P^{n-1}) \longrightarrow \Phi^{\pm}_{\lambda}((\mathbb{R}P^{n-1})')$$

It is remarkable that there is another way to define the operator $\mathcal{F}_\lambda$. Let me recall that the space of homogeneous degree λ generalized function on $\mathbb{R}$ is 2-dimensional and splits on the even and odd components (with respect to the involution $x \to -x$) generated by

$$\frac{|x|^\lambda}{\Gamma(\frac{\lambda+1}{2})} \quad \text{and} \quad \frac{|x|^\lambda sgn x}{\Gamma(\frac{\lambda+2}{2})}$$

They are both analytic on λ on the whole complex plane. One has

$$\frac{|x|^\lambda}{\Gamma(\frac{\lambda+1}{2})}|_{\lambda=-2k-1} = \frac{(-1)^k k!}{(2k)!} \cdot \delta^{(2k)}(x); \tag{48}$$

$$\frac{|x|^\lambda sgn x}{\Gamma(\frac{\lambda+2}{2})}|_{\lambda=-2k} = \frac{(-1)^k (k-1)!}{(2k-1)!} \cdot \delta^{(2k-1)}(x) \tag{49}$$

Let γ_{n-1} be a cycle generating $H_{n-1}(\mathbb{R}^n\backslash 0;\mathbb{Z})$. The kernel

$$K^+_\lambda(\xi,x) := \frac{|<\xi,x>|^\lambda}{\Gamma(\frac{\lambda+1}{2})}$$

and the cycle γ_{n-1} defines the operator

$$I^+_\lambda : \quad \Phi_{-\lambda-n}(\mathbb{R}P^{n-1}) \longrightarrow \Phi_\lambda((\mathbb{R}P^{n-1})')$$

$$I^+_\lambda : f(x) \longrightarrow \frac{1}{2}\int_{\gamma_{n-1}} f(x)\frac{|<\xi,x>|^\lambda}{\Gamma(\frac{\lambda+1}{2})}\sigma_n(x,dx)$$

The odd kernel

$$K^-_\lambda(\xi,x) := \frac{|<\xi,x>|^\lambda \cdot sgn(<\xi,x>)}{\Gamma(\frac{\lambda+2}{2})}$$

defines an integral transformation

$$(I^-_\lambda f)(\xi) = \int_{\gamma_{n-1}} f(x)K^-_\lambda(\xi,x)\sigma_n(x,dx)$$

Proposition 6.3 $\mathcal{F}^+_\lambda = \pi^{1/2+\lambda}\Gamma(\frac{-\lambda}{2}) \cdot I^+_\lambda, \quad \mathcal{F}^-_\lambda = i \cdot \pi^{1/2+\lambda}\Gamma(\frac{-\lambda}{2}) \cdot I^-_\lambda.$

Set

$$f_\lambda(x) := \pi^{\lambda/2}\frac{|x|^\lambda}{\Gamma(\frac{\lambda+1}{2})}, \qquad g_\lambda(x) := \pi^{\lambda/2}\frac{|x|^\lambda \mathrm{sgn}(x)}{\Gamma(\frac{\lambda+2}{2})}$$

Lemma 6.4

$$\mathcal{F}(f_\lambda(x)) = f_{-1-\lambda}(\xi), \quad \mathcal{F}(g_\lambda(x)) = i \cdot g_{-1-\lambda}(\xi) \tag{50}$$

Proof. See p. 173 in [GS] for an equivalent formula.

Proof of the proposition. Using the polar coordinates $x = r \cdot s$ where $s \in S^{n-1}, |s| = 1$, we have $d^n x = r^{n-1} dr ds_{n-1}$ where ds_{n-1} is the standard volume form on the unit sphere in $\mathbb{R}^n$. Then for $f(x) \in \Phi_{-\lambda-n}(\mathbb{R}P^{n-1})$ we have

$$\int_{\mathbb{R}^n} f(x)e^{2\pi i<\xi,x>}d^n x \quad = \quad \frac{1}{2}\int_{-\infty}^{\infty}|r|^{-\lambda-n}e^{2\pi i r\cdot(\xi,s)}r^{n-1}dr\int_{S^{n-1}}f(e)ds_{n-1} =$$

$$\frac{1}{2}\pi^{\lambda+1/2}\Gamma(\frac{-\lambda}{2})\int_{S^{n-1}}f(e)\frac{|<\xi,s>|^\lambda}{\Gamma(\frac{\lambda+1}{2})}ds_{n-1} \quad = \quad \pi^{-\lambda-1/2}\Gamma(\frac{-\lambda}{2})I^+_\lambda f$$

The proof in the case of odd functions is completely similar.

Corollary 6.5

$$I^+_{-\lambda-n} \circ I^+_\lambda = \frac{\pi^n}{\Gamma(\frac{-\lambda}{2})\Gamma(\frac{\lambda+n}{2})} \cdot Id$$

$$I^-_{-\lambda-n} \circ I^-_\lambda = \frac{\pi^n}{\Gamma(\frac{-\lambda}{2})\Gamma(\frac{\lambda+n}{2})} \cdot Id$$

In particularly using (48) we see that I_{-1} is just a projectively invariant version of the Radon transform:

$$(I_{-1}f)(\xi) = \int_{\gamma_m} f(x)\delta(<\xi, x>)\sigma_{m+1}(x, dx)$$

and the inversion formula looks as follows. When n is even

$$f(y) = \frac{(-1)^{\frac{n-2}{2}}}{2(2\pi)^{n-2}} \int_{\gamma_{n-1}} \hat{f}(\xi)\delta^{(n-2)}(<\xi, y>)\sigma_n(\xi, d\xi)$$

When n is odd

$$f(y) = \frac{(-1)^{n/2-1}(n-2)!}{2(2\pi)^{n-1}} \int_{\gamma_{n-1}} \hat{f}(\xi)(<\xi, y>)^{-n+1}\sigma_n(\xi, d\xi)$$

The operator $I^+_{-\lambda-n}$ is defined on the space of all homogeneous degree λ functions. However it is zero on the subspace of odd functions. The reason is this. A sphere in $\mathbb{R}^n\backslash 0$ representing the generator in $H_{n-1}(\mathbb{R}^n\backslash 0)$ has canonical coorientation "out of the origin". The involution $x \to -x$ preserves it. So it acts on the class γ_{n-1} in the same way as it acts on the orientation class of $\mathbb{R}^n$ and hence on the form $\sigma_{n-1}(x, dx)$: by multiplication by $(-1)^n$. So if $f(x)$ is an odd function the integral $\int_{\gamma_{n-1}} f(x)\sigma_n(\xi, d\xi)$ vanish because the contributions of the opposite parts of the sphere cancel each other.

From our point of view these results looks as follows. Let

$$L_\lambda := \sum_{i=1}^n x_i\partial_{x_i} - \lambda$$

be the Euler operator. Denote the corresponding $\mathcal{D}$-module by $\mathcal{L}_\lambda$. Then $\Phi_\lambda(\mathbb{R}P^{n-1})$ is the space of smooth even solutions of $\mathcal{L}_\lambda$.

It follows from lemma (4.1) that $\star\mathcal{L}_\lambda = \mathcal{L}_{-\lambda-n}[1]$ and the Green class of $\mathcal{L}_\lambda$ is

$$G_{\mathcal{L}_\lambda}(\varphi; v) = \varphi \cdot v \cdot \sigma_n(x, dx)$$

So pairing (46) looks in this case as follows:

$$\Phi_\lambda(\mathbb{R}P^{n-1}) \otimes \Phi_{n-\lambda}(\mathbb{R}P^{n-1}) \longrightarrow \mathbb{R}$$

$$f(x) \otimes g(x) \longmapsto \int_{\gamma_{n-1}} f(x)g(x)\sigma_n(x, dx)$$

Notice that

$$K^\pm_\lambda(x, \xi) \in Sol\Big(\mathcal{L}_\lambda \boxtimes \mathcal{L}_\lambda, \mathcal{D}'(\mathbb{R}^n\backslash 0 \times \mathbb{R}^n\backslash 0)\Big)^\pm \quad (51)$$

One has $\mathcal{L}_\lambda = \tilde{\star}\mathcal{L}_{-\lambda-n}$, so the integral transformation $I^\pm_\lambda$ is just the natural linear map provided by the kernel (51).

3. The complex space. Let λ and μ be complex numbers such that $n := \lambda - \mu$ is an integer. Let

$$\Phi_{\lambda,\mu}(\mathbb{C}P^m) := \{f \quad | \quad f(az, \bar{a}\bar{z}) = a^\lambda\bar{a}^\mu f(z, \bar{z})\}$$

be the space of smooth homogeneous function in $\mathbb{C}^{m+1}\backslash 0$ of the bidegree λ, μ. Consider the kernel

$$K^{\mathbb{C}}_{\lambda}(x,\xi) = \frac{<\xi, x>^{\lambda} \cdot <\bar{\xi}, \bar{x}>^{\mu}}{\Gamma(\frac{s+|n|+2}{2})}$$

where $s = \lambda + \mu$. It is a homogeneous generalized function. It defines an integral transformation

$$I^{\mathbb{C}}_{\lambda,\mu} : \Phi_{-\lambda-m-1,-\mu-m-1}(\mathbb{C}P^m) \longrightarrow \Phi_{\lambda,\mu}((\mathbb{C}P^m)')$$

$$f(z,\bar{z}) \longrightarrow \int_{\mathbb{C}P^m} f(z,\bar{z}) K^{\mathbb{C}}_{\lambda}(z,\xi)\sigma_m(z,dz) \wedge \sigma_m(\bar{z}, d\bar{z})$$

Here the integral has the following meaning. The form we integrate can be pushed down to $\mathbb{C}P^m$, so we integrate over the fundamental cycle. One has (see [GGV]).

$$K^{\mathbb{C}}_{\lambda}(z,\xi)|_{\lambda=-k-1,\mu=-l-1} = \frac{\pi(-1)^{k+l+1} j!}{k! l!} \delta^{k,l}(z,\bar{z})$$

where $j = min(k,l)$. In particularly applying the above results to the case $k = 0, l = 0$ we come to the Radon transform of smooth homogeneous functions of degree $(-m,-m)$ in $\mathbb{C}P^m$:

$$I^{\mathbb{C}}_m : \Phi_{-m,-m}(\mathbb{C}P^m) \longrightarrow \Phi_{-1,-1}((\mathbb{C}P^m)')$$

$$(I^{\mathbb{C}}_m f)(\xi) = \hat{f}(\xi) = (i/2)^m \int_{\mathbb{C}P^m} f(x)\delta(<\xi, x>)\sigma_m(x,dx) \wedge \sigma_m(\bar{x}, d\bar{x})$$

The projectively invariant inversion formula is

$$(J^{\mathbb{C}}_m f)(y) = c^{\mathbb{C}}_m \int_{\mathbb{C}P^m} \hat{f}(\xi)\delta^{(m-1,m-1)}(<\xi, y>)\sigma_m(\xi, d\xi) \wedge \sigma_m(\bar{\xi}, d\bar{\xi})$$

where $c^{\mathbb{C}}_m = (-1)^{m-1}(m-1)!(\pi)^{-2m+2}(i/2)^m$.

7 Integral geometry on the family of spheres

1. The integral transformation. Let

$$S^m = \{x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2 = 0\}/\mathbb{R}^*$$

be a sphere in $\mathbb{R}P^{m+1}$. The stereographic projection identifies the family of its hyperplane sections with the family of all spheres in $\mathbb{R}^m$.

Let $Q_{m+1} := \{x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2 = 0\}$ be a cone in $\mathbb{R}^{m+2}\backslash 0$. It has two connected components: Q^+_{m+1} in the half space $x_{m+2} > 0$ and its opposite Q^-_{m+1}.

Denote by $\Phi_\lambda(S^m)$ the space of all smooth homogeneous functions of degree λ on the cone Q^+_{m+1}. Let $SO(m+1,1)_0$ be the connected component of unity of the group $O(m+1,1)$. It acts on the cone Q^+_{m+1}.

Let β_m be a hyperplane section of Q^+_{m+1} which is isomorphic to a sphere. The orientation of $\mathbb{R}^{m+2}$ provides an orientation of β_m: the cycle β_m is cooriented out of the origin in the cone, and the cone itself coorientated outside of the convex component in $\mathbb{R}^{m+2}$. Let β^+_m be an oriented this way cycle. Its homology class is a generator of $H_m(Q^+_{m+1}, \mathbb{Z})$.

Lemma 7.1 *There is a nondegenerate $SO(m+1,1)_0$ - invariant pairing*

$$<\cdot,\cdot>_{S^m}: \quad \Phi_{-\lambda-m}(S^m) \otimes \Phi_\lambda(S^m) \longrightarrow \mathbb{R}$$

defined by the formula

$$<f, g>_{S^m} := \int_{\beta^+_m} \delta(x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2) f(x) g(x) \sigma_{m+2}(x, dx)$$

Here we integrate the closed m-form on Q^+_{m+1}. By definition it is the restriction to Q^+_{m+1} of any form α_m satisfying the condition

$$d(x_1^2+...+x_{m+1}^2-x_{m+2}^2)\wedge\alpha_m = f(x)g(x)\sigma_{m+2}(x,dx)$$

The restriction is well defined on Q^+_{m+1}.

Proof. The $SO(m+1,1)_0$ - invariance is obvious.

Let $\xi_1,...,\xi_{m+2}$ be coordinates in $(\mathbb{R}^{m+2})'$ dual to x_i and $<\xi,x>=\sum\xi_i x_i$. Consider the kernel

$$K^+_\lambda(\xi,x) := \frac{|<\xi,x>|^\lambda}{\Gamma(\frac{\lambda+1}{2})} \tag{52}$$

Set

$$\Delta := \partial^2_{\xi_1}+...+\partial^2_{\xi_{m+1}}-\partial^2_{\xi_{m+2}}; \qquad L_\lambda := \sum_{i=1}^{m+2}\xi_i\partial_{\xi_i}-\lambda$$

Let us denote by $\mathcal{M}_\lambda$ the $\mathcal{D}$-module on $\mathbb{R}^{m+2}$ corresponding to the system

$$\mathcal{M}_\lambda : \qquad L_\lambda f=0, \quad (x_1^2+...+x_{m+1}^2-x_{m+2}^2)f=0$$

and by $\mathcal{N}_\lambda$ the $\mathcal{D}$-module on $(\mathbb{R}^{m+2})'$corresponding to the system of differential equations

$$\mathcal{N}_\lambda : \qquad L_\lambda\varphi=0, \quad \Delta\varphi=0$$

Then

$$K^+_\lambda(\xi,x)\in Sol\Big(\mathcal{M}_\lambda\boxtimes\mathcal{N}_\lambda, D'(\mathbb{R}^{m+2}\times(\mathbb{R}^{m+2})')\Big)^+$$

is an even solution of this system. Notice that $\mathcal{M}_\lambda=\tilde{\star}\mathcal{M}_{-\lambda-m}$. So the kernel $K^+_\lambda(\xi,x)$ defines an operator

$$I^+_\lambda : \Phi_{-\lambda-m}(S^m)\longrightarrow Sol(\mathcal{N}_\lambda)^+; \tag{53}$$

$$(I^+_\lambda f)(\xi)=\int_{\beta^+_m}\delta(x_1^2+...+x_{m+1}^2-x_{m+2}^2)f(x)K^+_\lambda(\xi,x)\sigma_{m+2}(x,dx)$$

Consider the following domain:

$$\tilde\Gamma_0 := \{\xi|\xi_1^2+...+\xi_{m+1}^2-\xi_{m+2}^2=0\} \qquad \tilde\Gamma_1 := \{\xi|\xi_1^2+...+\xi_{m+1}^2>\xi_{m+2}^2\}$$

Remark. The functions $I^\pm_\lambda f(\xi)$ are a priory smooth only in the complement to the cone $\tilde\Gamma_0$. Indeed, the integral transform I^+_λ, for instance, is written in affine coordinates as

$$I^+_\lambda(f)(\xi)=\int f(x_1,...,x_{m+1})\delta(x_1^2+...+x_{m+1}^2-1)\frac{|<\xi',x>+s|^\lambda}{\Gamma(\frac{\lambda+1}{2})}d^{m+1}x$$

where $\xi=(\xi',s)$ and $<\xi',x>=\sum\xi_i x_i$. Set $\xi_1=1,\xi_i=0$ for $i>1$. Then

$$I^+_\lambda(f)(1,0,...,0;s)=\int\tilde f(x_1)\frac{|x_1+s|^\lambda}{\Gamma(\frac{\lambda+1}{2})}dx_1$$

where $\tilde f(x_1):=\int f(x)\delta(x_1^2+...+x_{m+1}^2-1)dx_2...dx_{m+1}$. The function $\tilde f(x_1)$ vanishes outside of the segment $[-1,1]$, smooth inside of it but not smooth near $x_1=\pm1$. The integral $\int|x|^\lambda f(x)dx$ is regularized near $x=0$ in assumption that the function $f(x)$ is smooth near zero.

Similarly the kernel

$$K^-_\lambda(\xi,x) := \frac{|<\xi,x>|^\lambda\cdot sgn(<\xi,x>)}{\Gamma(\frac{\lambda+2}{2})}$$

is an odd solution of the system $\mathcal{M}_\lambda \boxtimes \mathcal{N}_\lambda$. It defines an intertwiner operator for the group $O(m+1,1)$:

$$I_\lambda^- : \Phi_{-\lambda-m}(S^m) \longrightarrow Sol(\mathcal{N}_\lambda)^-;$$

Notice that

$$\Phi_\lambda^\pm(S^m) = Sol\Big(\mathcal{M}_\lambda|_{\mathbb{R}^{m+2}\backslash 0}, C^\infty(\mathbb{R}^{m+2}\backslash 0)\Big)^\pm$$

So the operators $I_\lambda^\pm$ are natural linear operators between smooth solution spaces.

In this chapter we will work with the restriction of the functions $I_\lambda^\pm$ *to the domain* $\tilde{\Gamma}_1$. Our first goal is to invert the operator

$$I_\lambda^\pm : \Phi_{-\lambda-m}(S^m) \longrightarrow Sol(\mathcal{N}_\lambda|_{\tilde{\Gamma}_1})^\pm;$$

2. The Green class. Now we make the crucial step. Consider the following m-form:

$$\omega_m(\varphi; v) := \tag{54}$$

$$\sum_{1\le i<j\le m+2} (-1)^{i+j-1}\Big(\xi_i \cdot \varepsilon_j(v\cdot \varphi'_{\xi_j} - v'_{\xi_j}\cdot \varphi) - \xi_j \cdot \varepsilon_i(v \cdot \varphi'_{\xi_i} - v'_{\xi_i}\cdot\varphi)\Big) d\xi_1 \wedge ...\hat{d\xi_i}...\hat{d\xi_j}... \wedge d\xi_{m+2}$$

Here $\varepsilon_{m+2} = -1$ and $\varepsilon_j = 1$ if $j \neq m+2$. Let $\omega_{m+1}(\varphi;\Delta;v)$ be the Green form for the Laplacian Δ:

$$\omega_{m+1}(\varphi;\Delta;v) = \sum_{1\le j\le m+2} (-1)^{j-1}\varepsilon_j(\varphi_{\xi_j}\cdot v - \varphi\cdot v_{\xi_j}) d\xi_1 \wedge ...\hat{d\xi_j}... \wedge d\xi_{m+2} \tag{55}$$

Then (54) is the contraction of the Green form (55) with the Euler vector field L:

$$\omega_m(\varphi; v) = -\frac{1}{2} i_L \omega_{m+1}(\varphi;\Delta;v)$$

Remark. More generally, for any homogeneous differential operator P with constant coefficients in $\mathbb{R}^n$ the Green form for the system $Pf = 0, L_a f = 0$ is equal to $-\frac{1}{2} i_L \omega_{n-1}(\varphi;\Delta;v)$.

It can be also written as follows:

$$\omega_m(\varphi;v) = [\xi v, \varepsilon \cdot \frac{\partial}{\partial \xi}\varphi, d\xi, ..., d\xi] - [\xi\varphi, \varepsilon\cdot\frac{\partial}{\partial\xi} v, d\xi, ..., d\xi]$$

Here $[\xi v, \varepsilon\cdot\frac{\partial}{\partial\xi}\varphi, d\xi, ..., d\xi]$ means the determinant of the following matrix:

$$\begin{pmatrix} \xi_1 v & \varepsilon_1 \cdot \varphi'_{\xi_1} & d\xi_1 & ... & d\xi_1 \\ \xi_2 v & \varepsilon_2\cdot\varphi'_{\xi_2} & d\xi_2 & ... & d\xi_2 \\ ... & ... & ... & ... & ... \\ \xi_{m+2} v & \varepsilon_{m+2}\cdot\varphi'_{\xi_{m+2}} & d\xi_{m+2} & ... & d\xi_{m+2} \end{pmatrix}$$

Lemma 7.2 *The form* $\omega_m(\varphi;v)$ *can be pushed down to* Γ

Proof. An easy calculation.

Theorem 7.3 *a)* $\mathcal{N}_a = \tilde{\star}\mathcal{N}_b$ *where* $a+b+m=0$.

b) The form $\omega_m(\varphi;v)$ *represents the Green class* $G_{\mathcal{N}_a}(\varphi;v)$ *of the system* $\mathcal{N}_a$.

Corollary 7.4 *The form* $\omega_m(\varphi;v)$ *is closed if the functions* φ *and* v *satisfy the following systems of differential equations:*

$$L_a\varphi = 0, \quad \Delta\varphi = 0 \qquad and \quad L_b v = 0, \quad \Delta v = 0 \quad where \quad a+b+m=0.$$

In the rest of this chapter we will use extensively this corollary (but not the fact that $\omega_m(\varphi; v)$ represents the Green class). So let me first give a straightforward proof independent of the proof of theorem (7.3).

Proof. One has

$$d\omega_m(v;\varphi) = \sum_{1\leq j\leq m+2} (-1)^{j-1}\Big(\xi_j(v\cdot\Delta\varphi - \Delta v\cdot\varphi) -$$

$$\varepsilon_j \cdot L_{-m-1}(v\cdot\varphi^{'}_{\xi_j} - v^{'}_{\xi_j}\cdot\varphi)\Big)d\xi_1\wedge ...\wedge \hat{d\xi_j}\wedge ...\wedge d\xi_{m+2}$$

Indeed, applying $\frac{\partial}{\partial\xi_i}d\xi_j$ to $\omega_{n-2}(v;\varphi)$ we get

$$\sum_{1\leq i<j\leq m+2} (-1)^{j-1}\Big(\xi_j(v\cdot\varphi^{''}_{\xi_i\xi_i} - v^{''}_{\xi_i\xi_i}\cdot\varphi) - \xi_i\frac{\partial}{\partial\xi_i}\varepsilon_j\cdot(v\cdot\varphi^{'}_{\xi_j} - \varphi\cdot v^{'}_{\xi_j}) -$$

$$-(j-1)\varepsilon_j\cdot(v\cdot\varphi^{'}_{\xi_j} - \varphi\cdot v^{'}_{\xi_j})\Big)d\xi_1\wedge ...\wedge \hat{d\xi_j}\wedge ...\wedge d\xi_{m+2}$$

Similarly we compute the contribution of $\frac{\partial}{\partial\xi_j}d\xi_j$ and take the sum.

Proof of theorem (7.3). Consider a complex of $\mathcal{D}$ - modules $\mathcal{D} \stackrel{d}{\longrightarrow} \mathcal{D}^2 \stackrel{d}{\longrightarrow} \mathcal{D}$ sitting in degrees [-2,0] (d has degree +1) which we visualize as:

$$\begin{array}{ccccc} & & \mathcal{D} & & \\ & \Delta\nearrow & & \searrow L_a & \\ \mathcal{D} & & & & \mathcal{D} \\ & -L_{a-2}\searrow & & \nearrow\Delta & \\ & & \mathcal{D} & & \end{array}$$

One has $[L_a, \Delta] = -2\Delta$, so $\Delta L_{-1} + (-L_{-3})\Delta = 0$, i.e. we get a complex.

This is a resolution of the $\mathcal{D}$- module $\mathcal{N}_a$. Indeed, consider a filtration on $\mathcal{D}$ such that the degree of x and $\frac{\partial}{\partial x}$ is +1. Then both L_a and Δ have degree +2. Shifting the filtration in the second term of the resolution down by 2 and in the third down by 4 we get a filtered complex. The associated graded quotient complex is a Koszul resolution. So our complex is also a resolution. The part a) follows easyly from this..

To calculate the Green class we use theorem (**??**) for this resolution. The complex $*\mathcal{P}^\bullet := Hom_{\mathcal{D}}(\mathcal{P}^\bullet, \mathcal{D}^\Omega)[m+2]$ is concentrated in degrees $[-(m+2), -m]$ and looks as follows:

$$\begin{array}{ccccc} & & \mathcal{D}^\Omega & & \\ & \Delta\swarrow & & \nwarrow L_a^* & \\ \mathcal{D}^\Omega & & & & \mathcal{D}^\Omega \\ & -L_{a-2}^*\nwarrow & & \swarrow\Delta & \\ & & \mathcal{D}^\Omega & & \end{array}$$

A homomorphism of $\mathcal{D}$-modules $\mathcal{D}_X \longrightarrow C^\infty(X)$ is determined by its value at $1\in\mathcal{D}_X$. So one can represent the complex $Hom_{\mathcal{D}}(\mathcal{P}^\bullet, C^\infty(\mathbb{R}^{m+2}))$ by the following picture where $\varphi_0, \varphi_1, \varphi_1', \varphi_2$ are the values of the corresponding homomorphisms at 1:

$$\begin{array}{ccccc} & & \varphi_1 & & \\ & \Delta\swarrow & & \nwarrow L_a & \\ \varphi_2 & & & & \varphi_0 \\ & -L_{a-2}\nwarrow & & \swarrow\Delta & \\ & & \varphi_1^{'} & & \end{array}$$

Similary one can make a picture for $Hom_{\mathcal{D}}(\star\mathcal{P}^{\bullet}, C^{\infty}(\mathbb{R}^{m+2}))$:

$$\begin{array}{ccccc} & & v_{m+1} & & \\ \Delta \nearrow & & & & \searrow L_a^t \\ v_m & & & & v_{m+2} \\ -L_{a-2}^t \searrow & & & & \nearrow \Delta \\ & & v'_{m+1} & & \end{array}$$

Recall that $Hom_{\mathcal{D}}(\mathcal{D}^{\Omega}, C^{\infty}(X)) = \mathcal{A}^m(X)$. So $v's$ are forms of top degree. Then

$$(L_a\varphi_0 \cdot v_{m+1} - \varphi_0 \cdot L_a^t v_{m+1}) + (\Delta\varphi_0 \cdot v'_{m+1} - \varphi_0 \cdot \Delta v'_{m+1}) =$$

$$d\omega_{m+1}(\varphi_0; v_{m+1}, v'_{m+1})$$

and

$$(\Delta\varphi_1 \cdot v_m - \varphi_1 \cdot \Delta v_m) + ((-L_{a-2})\varphi'_1 \cdot v_m - \varphi'_1 \cdot (-L_{a-2})^t v_m) =$$
$$d\omega_{m+1}(\varphi_1, \varphi'_1; v_m)$$

where

$$\omega_{m+1}(\varphi_0; v_{m+1}, v'_{m+1}) := \varphi_0 \cdot v_{m+1}\sigma_{m+2}(\xi, d\xi) +$$
$$\sum_{i=1}^{n} \varepsilon_i \Big((\frac{\partial\varphi_0}{\partial\xi_i} \cdot v'_{m+1} - \varphi_0 \cdot \frac{\partial v'_{m+1}}{\partial\xi_i} \Big) (-1)^{i-1} d\xi_1 \wedge ... \wedge \hat{d\xi_i} \wedge ... \wedge d\xi_{m+2}$$

and

$$\omega_{m+1}(\varphi_1, \varphi'_1; v_m) := -\varphi'_1 \cdot v_m \sigma_{m+2}(\xi, d\xi) +$$
$$\sum_{i=1}^{m+2} \varepsilon_i \Big(\frac{\partial\varphi_1}{\partial\xi_i} v_m - \varphi_1 \frac{\partial v_m}{\partial\xi_i} \Big) (-1)^{i-1} d\xi_1 \wedge ... \wedge \hat{d\xi_i} \wedge ... \wedge d\xi_{m+2}$$

Further,

$$\omega_{m+1}(\varphi_0; \Delta v_m, -L_{a-2}^t v_m) + \omega_{m+1}(L_a\varphi_0, \Delta\varphi_0; v_m) = d\omega_m(\varphi_0; v_m)$$

where $\omega_m(\varphi; v)$ is the Green form (54).

3. Construction of the inverse operator. We have defined in s. 7.1 the domain $\tilde{\Gamma}_1 = \{\xi_1^2 + ... + \xi_{m+1}^2 > \xi_{m+2}^2\}$. Let $\Gamma_1 = \tilde{\Gamma}_1/\mathbb{R}_+^*$ be the manifold of all oriented rays inside $\tilde{\Gamma}_1$. Its closure $\Gamma = \Gamma_0 \cup \Gamma_1$ parametrizes *oriented* hyperplane sections of the sphere S^m (here $\Gamma_0 = \tilde{\Gamma}_0/\mathbb{R}_+^*$).

$\Gamma = S^{m+1}\backslash\mathcal{D}_+ \cup \mathcal{D}_-$ where $\mathcal{D}_+$ is a ball $\{\xi_1^2 + ... + \xi_{m+1}^2 < \xi_{m+2}^2\}/(\mathbb{R}^*)^+$ and $\mathcal{D}_- = -\mathcal{D}_+$. Therefore $H_m(\Gamma, \mathbb{Z}) = \mathbb{Z}$. Consider the cycle γ_m of rays in the hyperplane $\xi_{m+2} = 0$. It is cooriented by the function ξ_{m+2} (or, more invariantly, by the choice of one of the balls $\mathcal{D}_+$). So orientation of $\mathbb{R}^{m+2}$ provides an orientation of the cycle. Denote by γ_m^+ the oriented this way cycle. Its homology class is a generator of $H_m(\Gamma, \mathbb{Z})$.

There is a nondegenerate pairing

$$< \cdot, \cdot >_{\mathcal{N}_\lambda}: \quad Sol(\mathcal{N}_\lambda)^+ \otimes Sol(\mathcal{N}_{-\lambda-m})^- \longrightarrow \mathbb{R}; \quad < \varphi, v >_{\mathcal{N}_\lambda} := \int_{\gamma_m^+} \omega_m(\varphi, v)$$

Remark. This pairing would have being zero if φ and v have the same parity. Indeed, in this case the involution $\xi \longmapsto -\xi$ multiplies the form $\omega_m(\varphi, v)$ by $(-1)^{m+2}$ and the cycle γ_m by $(-1)^{m+1}$, so the contributions to the integral coming from the antipodal parts of the cycle are canceled.

Let K be a compact hypersurface in Γ. Its homology class $[K] \in H_n(\Gamma)$ is equal to $d(K) \cdot [\gamma_M^+]$. The integer $d(K)$ is the intersection number of the class $[K]$ with the cycle consisting of all oriented spheres passing through a given point $x \in S^m$ and tangent to a given hyperplane in T_xS^m.

According to part a) of theorem (7.3) $\check{\star}\mathcal{N}_\lambda = \mathcal{N}_{-m-\lambda}$. So by the general philosophy the kernel $K^-_\lambda(\xi, x)$ define integral operators

$$J^+_\lambda : Sol_{C^\infty}(\mathcal{N}_{-\lambda-m})^+ \longrightarrow \Phi_\lambda(S^m) \tag{56}$$

$$(J^+_\lambda \varphi)(\xi) = \frac{1}{2}\int_K \omega_m(\varphi; K^-_\lambda(\xi, x))$$

and similarly the even kernel $K^+_\lambda(\xi, x)$ provides operators

$$J^-_\lambda : Sol_{C^\infty}(\mathcal{N}_{-\lambda-m})^- \longrightarrow \Phi_\lambda(S^m)$$

Notice that J^+_λ is defined by an odd kernel, and J^-_λ by an even kernel.

Further, there are operators

$$(J^\pm_{-\lambda-m})^t : \Phi_\lambda(S^m) \longrightarrow Sol_{C^\infty}(\mathcal{N}_{-\lambda-m})^\mp$$

$$(J^\pm_{-\lambda-m})^t(g) = \int_{\beta^\pm_m} \delta(x_1^2 + ... + x_{m+1}^2 - x_{m+2}^2) g(x) K^\mp_\lambda(x, \xi)\sigma_{m+2}(x, dx)$$

Theorem 7.5 *a) These operators are intertwiners for the group $SO(m+1,1)_0$.*

b) For any m-cycle $K \in \Gamma$ one has

$$d(K)\cdot < f, g >_{S^m} = c(\lambda) \cdot \int_K \omega_m \Big(I^\pm_\lambda f; (J^\pm_{-\lambda-m})^t g \Big)$$

where

$$c(\lambda) = \frac{\pi^{m+1}}{\Gamma(\frac{-\lambda}{2})\Gamma(\frac{\lambda+m+2}{2})}$$

c) In particular $d(K) \cdot J^\pm_{-\lambda-m} \circ I^\pm_\lambda = c(\lambda) \cdot Id$.

The part b) can be viewed as the universal form of the Plancherel theorem for the integral transformation $I^\pm_\lambda$.

Proof. a) The operator $J^\pm_\lambda$ is an intertwiner for the following three reasons.

1. A group element $g \in SO(m+1,1)_0$ sends form $\omega_m(\varphi, v)$ to the form $\omega_m(g\cdot\varphi, g\cdot v)$. Indeed, the form ω_m is a cocycle representing the Green class for the system $\mathcal{N}_\lambda$. This system as well as the volume form in $\mathbb{R}^{m+2}$ is invariant under the action of the group $SO(m+1,1)_0$.

2. A *connected* Lie group acts trivially on the homology.

In the definition of the inverse operator J^-_λ we can integrate over an m-cycle $\tilde{K} \subset (\mathbb{R}^{m+2})'$ projecting to K. So $J^\pm_\lambda$ apriory defined for any smooth function $\varphi(\xi)$. However it commutes with the group action only on the subspace $Sol(\mathcal{N}_\lambda, C^\infty(\mathbb{R}^{m+2}))$. Indeed, g moves the cycle $\tilde{K}$ to a different cycle $g\tilde{K}$ *homologous to the initial one.* To compare the integrals we use the Stokes formula for the form $\omega_m(\varphi; K_\lambda(\xi, x))$. The integrals will be the same only if the form is closed. This happened only if $\varphi(\xi) \in Sol_{C^\infty}(\mathcal{N}_\lambda)$.

b). Let $n = (0 : .. : 0 : 1 : 1)$ be the "North pole"in S^m. The variety Γ_n parametrizing the hyperplane sections of the sphere S^m passing through the point n is a hyperplane given by equation $\xi_{m+1} + \xi_{m+2} = 0$.

It is sufficient to prove these formulas for one cycle K. Let $\pi_n : (x_1, .., .x_{m+2}) \to (x_1, .., .x_m, x_{m+1} - x_{m+2})$ be the projection along the line n. Set $\tilde{x} := (x_1, .., .x_m)$, $v := x_{m+1} - x_{m+2}$. Assume that $f \in \Phi_\lambda(S^m)$ vanishes near the line n. Then π_n identifies $f|_{Q^+_{m+2}}$ with a function $\varphi(\tilde{x}, v) := f(\tilde{x}, v, -(x_1^2 + ... + x_m^2)/v)$ on the hyperplane $x_{m+2} = 0$, which vanish at $v \le \varepsilon$, $\varepsilon > 0$. Let $\alpha^+_m := \pi_n(\gamma^+_m)$. The restriction of $I^+_\lambda f(\xi)$ to $\xi_{m+1} + \xi_{m+2} = 0$ can be written as

$$(-1)^m \int_{\alpha^+_m} \frac{\varphi(\tilde{x}, v)}{v} \frac{|\tilde{\xi}\tilde{x} + v\cdot(\xi_{m+1} - \xi_{m+2})/2|^\lambda}{\Gamma(\frac{\lambda+1}{2})} \sigma_{m+1}(\tilde{x}, v)$$

This and the following lemma shows that for $K = \Gamma_n$ part b) reduces to the Plancherel theorem and the inversion formula for the generalized Radon transform in the projective space (see s. 6.2-6.3). Set $\xi' = (\xi_1, ..., \xi_{m+1})$.

Lemma 7.6 . *The restriction of the form $\omega_m(\varphi; v)$ to Γ_n is equal to*

$$\omega_m(\varphi;v)|_{\Gamma_n} = (-1)^{m+1}\Big(v\cdot(\partial_{\xi_{m+1}}-\partial_{\xi_{m+2}})\varphi-\varphi\cdot(\partial_{\xi_{m+1}}-\partial_{\xi_{m+2}})v\Big)\sigma_{m+1}(\xi',d\xi')$$

Integrating by parts we get $2\cdot(-1)^m\varphi\cdot(\partial_{\xi_{m+1}}-\partial_{\xi_{m+2}})v\sigma_{m+1}(\xi',d\xi')$

4. An example: the Radon transform along the hyperplane sections of a sphere . The generalized functions (52) has no poles on λ. One has

$$K^+_{-(2k+1)}(\xi,x) = \frac{(-1)^k k!}{(2k)!}\cdot\delta^{(2k)}(<\xi,x>);$$

$$K^-_{-2k}(\xi,x) = \frac{(-1)^k (k-1)!}{(2k-1)!}\cdot\delta^{(2k-1)}(<\xi,x>)$$

$$K^+_{-2k}(\xi,x) = \frac{(-1)^k 2^k\sqrt{\pi}}{(2k-1)!!} <\xi,x>^{-2k}$$

$$K^-_{-(2k+1)}(\xi,x) = \frac{(-1)^k 2^k\sqrt{\pi}}{(2k-1)!!} <\xi,x>^{-2k-1}$$

So we get the following integral transformation. For $f\in\Phi_{1-m}(S^m)$ set

$$If(\xi) = \int_{\beta^+_m}\delta(x_1^2+...+x_{m+1}^2-x_{m+2}^2)f(x)\delta(<\xi,x>)\sigma_{m+2}(x,dx)$$

The function $If(\xi)$ is zero outside Γ. Consider the following kernel:

$$K_{-(m-1)}(\xi,x) := \delta^{(m-2)}(<\xi,x>)\quad \textit{for odd } \mathrm{m} \textit{ and } \quad <\xi,x>^{-(m-1)}\quad \textit{for even } \mathrm{m}$$

It defines an integral transformation acting on $g\in\Phi_{-1}(S^m)$:

$$(J^t)g(\xi) = \int_{\beta^+_m}\delta(x_1^2+...+x_{m+1}^2-x_{m+2}^2)g(x)K_{-(m-1)}(<\xi,x>)\sigma_{m+2}(x,dx)$$

Theorem 7.7 *a) For any m-cycle $K\in\Gamma$ one has*

$$d(K)\cdot <f,g>_{S^m} = c_m\cdot\int_K\omega_m\Big(If;J^tg\Big)$$

where $-c_m = \frac{(-1)^{(m-1)/2}}{(2\pi)^{m-1}}$ *for odd m and* $\frac{(-1)^{m/2}(m-1)!}{(2\pi)^m}$ *for even m.*

b) In particular

$$d(K)\cdot f(x) = c_m\cdot\int_K\omega_m\Big(If;K_{-(m-1)}(\xi,x)\Big) \tag{57}$$

So the inversion formula is local for odd m and nonlocal for even m.

Theorem (7.7) is a special case of theorem (7.5).

The inverse operator J provided by the kernel $K_{-(m-1)}(\xi,x)$ looks as follows:

$$(J\varphi)(x) := \int_K\sum_{1\le i<j\le n}(-1)^{i+j-1}\Big(\varphi(\xi)(\xi_ix_j-\xi_jx_i)\delta^{(m-1)}(<\xi,x>)-$$

$$(\xi_i\varphi'_{\xi_j}-\xi_j\varphi'_{\xi_i})\delta^{(m-2)}(<\xi,x>)\Big)d\xi_1\wedge...\wedge\hat{d}\xi_i\wedge...\wedge\hat{d}\xi_j\wedge...\wedge d\xi_n$$

for odd m and

$$(J\varphi)(x) := \int_K\sum_{1\le i<j\le n}(-1)^{i+j-1}\Big(\varphi(\xi)(\xi_ix_j-\xi_jx_i)<\xi,x>^{-m}-$$

$$(\xi_i\varphi_{\xi_j}-\xi_j\varphi_{\xi_i}) < \xi, x >^{-(m-1)}\Big) d\xi_1\wedge ...\wedge \hat{d\xi_i}\wedge ...\wedge \hat{d\xi_j}\wedge ...\wedge d\xi_n$$

for even m.

5. Admissible families of spheres. Restricting the integral operator $I^{\pm}_{\lambda}$ to a family K of oriented spheres we get an integral transformation

$$I^{\pm}_{\lambda,K}: \Phi_\lambda(S^m)\longrightarrow \Psi^{\pm}_{-\lambda-m}(K)$$

A priory the restriction of the form $\omega_m(\varphi;v)$ to a hypersurface K depends not only on the restriction of the functions φ and v on K, but on their first derivatives in the normal direction to K. Therefore for general K the right hand side of (57) can not be computed if know only $I^{\pm}_{\lambda,K}(f)$. So it does not give an inversion formula for the integral transformation $I^{\pm}_{\lambda,K}$.

Definition 7.8 *A hypersurfaces $K\subset\Gamma$ is called admissible if the restriction of the form $\omega_m(\varphi;v)$ to K depends only on the restrictions of smooth solutions $\varphi\in Sol_{C^\infty}(\mathcal{N}_\lambda)$, $v\in Sol_{C^\infty}(\mathcal{N}_\lambda)$ to K.*

This means that there exists a bidifferential operator $\nu: C^\infty(K)^{\otimes 2}\longrightarrow \mathcal{A}^m(K)$ such that for any φ, v as above $\omega_m(\varphi;v)|_K=\nu(\varphi|_K, v|_K)$.

It is worth to compare this definition of admissibility with the one usually used in integral geometry, see [G3].

Let C be a submanifold in S^m. Consider the family Γ_C of oriented hyperplane sections of the sphere S^m tangent to C. For example if C is a point then $d(\Gamma_C)=1$.

Lemma 7.9 *For any $C\subset S^m$ the hypersurface Γ_C is admissible.*

Proof. For $C=n$ this follows from the lemma (7.6). Indeed, the vector field $(\partial_{\xi_{m+1}}-\partial_{\xi_{m+2}})$ is tangent to the hyperplane Γ_n.

In general we proceed as follows. The form $\omega_m(\varphi;v)$ is given by a bidifferential operator of first order (see (54)), so its restriction to K is determined by the restriction of the functions φ and v to the 1-st infinitesimal neighborhood of K. Let $\eta\in\Gamma_C$ and $t(\eta)$ be the tangency point of the hyperplane $<\eta,x>=0$ with C. Then the tangent space to Γ_K at a point $t(\eta)$ coincides with $\Gamma_{t(\eta)}$.

6. Inversion of the integral transform related to an admissible family. The restriction of the form $\omega_m(I^{\pm}_\lambda(f);K^{\mp}_\lambda(\xi,x))$ to Γ_C depends only $I^{\pm}_{\lambda,\Gamma_C}f$. So one can expect the inversion formula

$$d(\Gamma_C)f(x)=c(\lambda)\cdot\int_{\Gamma_C}\omega_m\Big(I^{\pm}_\lambda(f);K^{\mp}_\lambda(\xi,x)\Big)\Big) \tag{58}$$

similar to (57). However the cycle Γ_C lies in the closure Γ of Γ_0, while the function $I^{\pm}_\lambda(f)$ was well defined only inside of Γ_0. For the same reason the form $\omega_m(I^{\pm}_\lambda(f);K^{\mp}_\lambda(\xi,x))$ is closed only inside of Γ_0 (and outside of Γ). So it is a priory unclear whether the formula makes sense and is it possible to use the Stokes theorem.

To avoid this trouble we consider the integral transformation $I^{\pm}_{\lambda,\Gamma_C}$ only on the subspace $C^\infty(S^m,C)$ of the functions vanishing in a very small neighborhood of the subvariety C in S^m.

Let $\hat{C}\in\Gamma$ be the subvariety of spheres of radius zero with center at points of C. Let $\Psi^{\pm}_\lambda(\Gamma_C;\hat{C})$ be the subspace of $\Psi^{\pm}_\lambda(\Gamma_C)$ consisting of functions smooth near $\hat{C}$. Then $I^{\pm}_\lambda f$ is smooth in a neighborhood of $\hat{C}$. So we get an integral transformation

$$I^{\pm}_{\lambda,\Gamma_C}:\Phi_{-\lambda-m}(S^m,C)\longrightarrow\Psi^{\pm}_\lambda(\Gamma_C;\hat{C})$$

Now we may apply the Stokes formula near $\hat{C}$. Assuming this let us perturbate the cycle Γ_C near the boundary of Γ by moving it a little bit inside of Γ. Geometrically this means that we replace small spheres tangent to C by close to them small spheres which are not tangent to C.

Remarks. 1. The cycle K becomes homologous to 0 in the sphere S^{m+1} parametrizing *all* oriented hyperplanes.

2. One can deform smoothly the cycle K out of the domain Γ. However doing this we must cross *all* the points of the boundary Γ_1 of Γ. Therefore we *can not* use the Stokes formula to compare

$$\int_K \omega_m\Big(I_\lambda^{\pm}(f); K_\lambda^{\mp}(\xi,x)\Big) \quad \text{and} \quad \int_{K'} \omega_m\Big(I_\lambda^{\pm}(f); K_\lambda^{\mp}(\xi,x)\Big)$$

where K is inside Γ_0 and K' outside Γ. This is very natural: otherwise we would prove that they are equal, and so equal to zero since the cycle K' is homologous to zero in the complement to Γ.

So we can reduce the investigation of the integral to the study of a similar integral over a cycle K inside Γ_0, which was done above. Therefore we come to the following conclusion:

Theorem 7.10 *For an admissible family Γ_C the operator $J_\lambda^{\pm}$ provides an operator*

$$J^{\pm}_{\lambda,\Gamma_C} : \Psi_\lambda^{\pm}(\Gamma_C, \hat{C}) \longrightarrow \Phi_{-\lambda-m}(S^m, C)$$

such that

$$c(\lambda) \cdot I^{\pm}_{\lambda,\Gamma_C} \circ J^{\pm}_{\lambda,\Gamma_C} = d(K) \cdot Id$$

7. Geometry of the family of spheres. The group $SO(m+1,1)$ acts on the family of all spheres in S^m. A remarkable fact is that a bigger symmetry group, $SO(m+1,2)$, acts as a group of contact transformations on the family of all spheres (including the points, which are spheres of zero radius!).

Namely, let

$$X_{m+1} := \{\eta_1^2 + ... + \eta_{m+1}^2 - \eta_{m+2}^2 - \eta_{m+3}^2 = 0\}/\mathbb{R}_+^*$$

be the $m+1$-dimensional quadric of signature $(m+1,2)$. Its affine part $\eta_{m+3} \neq 0$ is isomorphic to the hyperboloid $\Gamma_0 = \{\xi_1^2 + ... + \xi_{m+1}^2 - \xi_{m+2}^2 = 1\}$. The complement to the affine part is the projectivization of the cone $\{\xi_1^2 + ... + \xi_{m+1}^2 - \xi_{m+2}^2 = 0\}$, i.e. it is a sphere $\{\xi_1^2 + ... + \xi_{m+1}^2 = 1\}$. The quadric X_{m+1} parametrizes all oriented hyperplane sections of the sphere S^m. The hyperboloid Γ_0 parametrizes all oriented spheres of non zero radius.

Let $A \subset S^m \times X_{m+1}$ be the incidence subvariety. Consider the double bundle corresponding to this family and its symplectization:

$$\begin{array}{ccc} & A & \\ p_1 \swarrow & & \searrow p_2 \\ S^m & & X_{m+1} \end{array} \qquad \begin{array}{ccc} & N_A^*(S^m \times X_{m+1}) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ T^*S^m & & T^*X_{m+1} \end{array}$$

Let

$$\Sigma := \pi_2(N_A^*(S^m \times X_{m+1})) \subset T^*X_{m+1}$$

Then $\Sigma_\xi := T_\xi^* X_{m+1} \cap \Sigma$ is a nondegenerate quadratic cone in the cotangent bundle to ξ. This cone is dual to the cone in the tangent space to the quadric at the point ξ given by intersection of the quadric with the hyperplane in the projective space tangent to the quadric at ξ.

The hypersurface Σ is foliated on curves: bicharacteristics. This foliation is invariant under the action of the multiplicative group $\mathbb{R}^*$ on T^*X_{m+1}.

Lemma 7.11 *a) Projection along the bicharaceristics gives the $\mathbb{R}^*$ equivariant fibration*

$$\pi_\Sigma : \Big(\Sigma\backslash \quad \{\text{zero section}\}\Big) \longrightarrow \Big(T^*S^m\backslash \quad \{\text{zero section}\}\Big)$$

b) The projection of a bicharacteristic to X_{m+1} consists of all spheres tangent to a given hyperplane at a given point.

So the manifold of all bicharacteristics is identified with the projectivization of the cotangent bunlde to S^m.

Geometrically $P(\Sigma\setminus$ {zero section}) is the set of all pairs

{ a contact element h at a point $x \in S^m$, a sphere tangent to h at x }

The group $SO(m+1,2)$ acts on X_{m+1} and hence on Σ. Thanks to the lemma the group $SO(m+1,2)$ acts as a group of homogeneous symplectomorphisms on T^*S^m. It preserves the family of homogeneous Lagrangian subvarieties given by the conormal bundles to spheres (including the spheres of zero radius).

8. The Hamilton-Jacoby method for description of admissible families of spheres. A hypersurface $K' \subset X_{m+1}$ is characteristic if its conormal bundle in X_{m+1} is contained in Σ, i.e. for any nonsingular $\xi \in K'$ the tangent plane $T_\xi K'$ is tangent to the "light cone" $\Sigma^*_\xi \subset T_\xi X_{m+1}$.

Proposition 7.12 *An irreducible hypersurface $K' \subset X_{m+1}$ is admissible if and only if it is characteristic.*

Proof. We already proved in lemma (7.9) that if K' is characteristic then it is admissible. Let us prove the converse statement. Since $\omega_m(\varphi; v)$ is given by a bidifferential operator of order $(1,1)$ it is enough to check that the restriction of the differential form $\omega_m(\varphi; v)$ to any noncharacteristic hyperplane does depend on the derivatives of φ and v in the direction transversal to this hyperplane. The group $SO(m+1,1)$ acts transitively on the variety of noncharacteristic hyperplanes in the tangent spaces $T_\xi X_{m+1}$. So it is sufficient to check the statement above for the hyperplane $\xi_{m+2} = 0$. One has

$$\omega_m(\varphi; v)|_{\xi_{m+2}=0} = \sum_{1\le i\le m+1} (-1)^{i+m-1}\xi_i \cdot (v \cdot \varphi'_{\xi_{m+2}} - v'_{\xi_{m+2}} \cdot \varphi) d\xi_1 \wedge ... \hat{d\xi_i} ... \wedge d\xi_{m+1}$$

The proposition follows.

The following lemma is well known

Lemma 7.13 *Any algebraic irreducible homogeneous Lagrangian subvariety in T^*X is isomorphis to the conormal bundle to an algebraic irreducible subvariety $Y \subset X$*

Theorem 7.14 *Any admissible hypersurface in Γ is a piece of a hypersurface Γ_C for a certain $C \subset S^m$.*

Proof. We may assume that K' is irreducible. According to proposition (7.12) $N^*_{K'}X_{m+1}$ is a Lagrangian subvariety in Σ, so π_Σ it projects it down to a Lagrangian subvariety in T^*S^m, which by the above lemma must have form $N^*_C S^m$.

8 Holonomic kernels and their composition: the bicategory of D-modules

1. Motivations. As we emphasized before the composition of natural linear maps defined by distributional kernels not always exists. However when it is defined we come to the problem of computation of the composition. Many important problems of analysis can be considered as special cases of this one. For instance in integral geometry both the integral transformation and its inverse should be treated as natural linear maps between solution spaces of $\mathcal{D}$-modules, so to invert an integral transformation we should be able to compute the composition of natural linear maps.

Let us assume for a moment that $\mathcal{M}_i$ are excellent $\mathcal{D}$-modules. Then usually the natural kernels are *distributions satisfying holonomic system of differential equations.* This means that the image of homomorphism

$$\tilde{\star}\mathcal{M}_1 \boxtimes \mathcal{M}_2 \longrightarrow D'(X_1 \times X_2) \tag{59}$$

provided by the kernel

$$K_{12}(x_1, x_2) \in Hom_{\mathcal{D}}\Big(\tilde{\star}\mathcal{M}_1 \boxtimes \mathcal{M}_2, D'(X_1 \times X_2)\Big)$$

is a holonomic $\mathcal{D}$-module. Let us denote it by $\mathcal{K}_{12}$ and by $\tilde{\star}\mathcal{M}_1 \boxtimes \mathcal{M}_2 \xrightarrow{\alpha_{12}} \mathcal{K}_{12}$ the corresponding morphism of $\mathcal{D}$-modules. So (59) is a composition

$$\tilde{\star}\mathcal{M}_1 \boxtimes \mathcal{M}_2 \xrightarrow{\alpha_{12}} \mathcal{K}_{12} \hookrightarrow D'(X_1 \times X_2)$$

The idea to keep only the first arrow suggests the following definition

Definition 8.1 *A holonomic kernel on $X_1 \times X_2$ is a collection $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{K}_{12}; \alpha)$ where*

$$\mathcal{M}_1 \in D^b_{coh}(\mathcal{D}_{X_1}), \quad \mathcal{M}_2 \in D^b_{coh}(\mathcal{D}_{X_2}), \quad \mathcal{K}_{12} \in D^b_{hol}(\mathcal{D}_{X_1 \times X_2})$$

and

$$\alpha \in RHom_{\mathcal{D}_{X_1 \times X_2}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2, \mathcal{K}_{12})$$

A holonomic kernel is a finer algebraic version of a holonomic distribution on $X_1 \times X_2$ then the $\mathcal{D}$-module which this distribution satisfies.

Example. Suppose that $\mathcal{M}_i = \mathcal{D}_{X_i}$ for $i = 1, 2$. Then $\tilde{\star}\mathcal{D}_{X_1} = \mathcal{D}_{X_1}$ and $\mathcal{D}_{X_1} \times \mathcal{D}_{X_2} = \mathcal{D}_{X_1 \times X_2}$. Morphisms of $\mathcal{D}$-modules $\mathcal{D}_{X_1 \times X_2} \longrightarrow \mathcal{K}$ are defined by their value on the generating section 1 and correspond just to the *sections* of $\mathcal{K}_{12}$.

For instance, if $X_1 = X_2 = \mathbb{A}^1$ and $\mathcal{K}_{12}$ is the $\mathcal{D}$-module of delta functions on the diagonal the morphisms above correspond to sections $f(x)\delta^{(k)}(x - y)$.

It seems that the notion of a bicategory is the appropriate language to discuss the holonomic kernels and their composition.

2. Bicategories. A complete definition of (lax) bicategory see in [Be] or p.200 [KV]. In particular a notion of bicategory $\mathcal{C}$ includes the following data:

a set Ob$\mathcal{C}$ of objects;

for any 2 objects a set of 1-morphisms from A to B;

for any two 1-morphisms α_1, α_2 between A and B a set of 2-morphisms between α_1 and α_2.

For any 2 objects A_1 and A_2 of a bicategory there is a category $Mor_1(A_1, A_2)$ of 1-morphisms from A_1 to A_2. The objects in this category are 1-morphisms from A_1 to A_2; the morphisms between given two 1-morphisms from A_1 to A_2 are given by the 2-morphisms between these 1-morphisms.

The composition of 1-morphisms provides a bifunctor

$$Mor_1(A_1, A_2) \times Mor_1(A_2, A_3) \longrightarrow Mor_1(A_1, A_3)$$

The archetypal example is the bicategory of all categories. Its objects are categories and for any two categories $\mathcal{A}$ and $\mathcal{B}$ the category $Mor_1(\mathcal{A}, \mathcal{B})$ is the category of functors from $\mathcal{A}$ to $\mathcal{B}$.

3. A bicategory of $\mathcal{D}$-modules. Below we work in the derived category. In particular all morphisms are morphisms in the derived category.

The objects of the bicategory are pairs $(X, \mathcal{M})$ where X is an algebraic variety over a field k (char $k = 0$) and $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$.

By definition 1-morphisms between the 2 objects $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are holonomic kernels

$$\star\mathcal{M} \boxtimes \mathcal{N} \xrightarrow{\alpha} K$$

It is the composition of 1-morphisms which makes the whole story relevant to integral geometry. Roughly speaking it answers to the question "what system of differential equations satisfies the kernel of the composition of 2 natural maps ?" and motivated by s. 5.7 above.

Let $\Delta_2 : X_1 \times X_2 \times X_3 \hookrightarrow X_1 \times X_2 \times X_2 \times X_3$ be the diagonal embedding of X_2 and $\pi_2 : X_1 \times X_2 \times X_3 \to X_1 \times X_3$ be the projection. Consider the objects $(X_i, \mathcal{M}_i)$ where $i = 1, 2, 3$.

Definition 8.2 *The composition of 1-morphisms*

$$\star\mathcal{M}_1 \boxtimes \mathcal{M}_2 \xrightarrow{\alpha_{12}} \mathcal{K}_{12} \quad \textit{and} \quad \star\, \mathcal{M}_2 \boxtimes \mathcal{M}_3 \xrightarrow{\alpha_{23}} \mathcal{K}_{23}$$

is the 1-morphism

$$\star\mathcal{M}_1 \boxtimes \mathcal{M}_3 \xrightarrow{\alpha_{13}} \mathcal{K}_{13}$$

where

$$\mathcal{K}_{13} = \mathcal{K}_{12} \circ \mathcal{K}_{23} := \pi_{2*}\Delta_2^!\Big(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23}\Big)$$

and the morphism α_{13} *is the composition of the morphisms* $id \boxtimes G \boxtimes id$ *and* $\alpha_{12} \boxtimes \alpha_{23}$*:*

$$\star\mathcal{M}_1 \boxtimes \mathcal{M}_3 \overset{id\boxtimes G\boxtimes id}{\longrightarrow} \pi_{2*}\Delta_2^!\Big(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \star\mathcal{M}_2 \boxtimes \mathcal{M}_3\Big) \overset{\alpha_{12}\boxtimes\alpha_{23}}{\longrightarrow} \pi_{2*}\Delta_2^!\Big(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23}\Big)$$

A 2-morphism between 1-morphisms

$$\star\mathcal{M} \boxtimes \mathcal{N} \xrightarrow{\alpha_1} \mathcal{K}_1 \quad \text{and} \quad \star\, \mathcal{M} \boxtimes \mathcal{N} \xrightarrow{\alpha_2} \mathcal{K}_2$$

is a morphism $\varphi_{12} : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$ making the following diagram commutative:

$$\begin{array}{ccc} & \star\mathcal{M}_1 \boxtimes \mathcal{M}_3 & \\ \alpha_1 \swarrow & & \searrow \alpha_2 \\ \mathcal{K}_1 & \overset{\varphi_{12}}{\longrightarrow} & \mathcal{K}_2 \end{array}$$

A 2-morphism between a holonomic kernel $\alpha'_{13} : \star\mathcal{M}_1 \boxtimes \mathcal{M}_3 \to \mathcal{K}'_{13}$ and the composition $\mathcal{K}_{12} \circ \mathcal{K}_{23}$ of holonomic kernels $\alpha_{12} : \star\mathcal{M}_1 \boxtimes \mathcal{M}_3 \to \mathcal{K}_{12}$ and $\alpha_{23} : \star\mathcal{M}_2 \boxtimes \mathcal{M}_3 \to \mathcal{K}_{23}$ is provided by the following commutative diagram

$$\begin{array}{ccc} \star\mathcal{M}_1 \boxtimes \mathcal{M}_3 & \overset{id\boxtimes G\boxtimes id}{\longrightarrow} & R\pi_{2*}\Delta_2^!\Big(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \star\mathcal{M}_2 \boxtimes \mathcal{M}_3\Big) \\ \Big\downarrow \alpha'_{13} & & \Big\downarrow R\pi_{2*}\Delta_2^!(\alpha_{12} \boxtimes \alpha_{23}) \\ \mathcal{K}'_{13} & \longrightarrow & R\pi_{2*}\Delta_2^!\Big(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23}\Big) \end{array}$$

The composition of 2-morphisms is defined in an obvious way.
The identity 1-morphism $Id_{\mathcal{M}}$. For any $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$ there is a canonical morphism

$$I_{\mathcal{M}} : \star\mathcal{M} \boxtimes \mathcal{M} \longrightarrow \delta_\Delta[d_X]$$

corresponding via (**??**) to the identity map $Id \in Hom_{\mathcal{D}_X}(\star\mathcal{M}, \star\mathcal{M})$.

We will say that the 1-morphism α_{23} is weakly inverse to the 1-morphism α_{12} (see (**??**)) if there is a 2-morphism from the identity 1-morphism $Id_{\mathcal{M}}$ to the composition of 1-morphisms $\alpha_{23} \circ \alpha_{12}$. This means that the following diagram is commutative:

$$\begin{array}{ccc} \star\mathcal{M}_1 \boxtimes \mathcal{M}_1 & \overset{id\boxtimes G\boxtimes id}{\longrightarrow} & R\pi_{2*}\Delta_2^!\Big(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2 \boxtimes \star\mathcal{M}_2 \boxtimes \mathcal{M}_1\Big) \\ \Big\downarrow I_{\mathcal{M}_1} & & \Big\downarrow R\pi_{2*}\Delta_2^!(\alpha_{12} \boxtimes \alpha_{23}) \\ \delta_\Delta & \overset{\varphi}{\longrightarrow} & R\pi_{2*}\Delta_2^!\Big(\mathcal{K}_{12} \boxtimes \mathcal{K}_{23}\Big) \end{array}$$

Remark. These definitions make sense for any (not necessarily holonomic) $\mathcal{K}_{ij} \in D^b_{coh}(\mathcal{D}_X)$.
3. On composition of holonomic kernels. Recall that for $\mathcal{M}, \mathcal{N} \in D^b_{coh}(\mathcal{D}_X)$ one has

$$\mathcal{M} \overset{!}{\otimes} \mathcal{N} := \Delta^!(\mathcal{M} \boxtimes \mathcal{N}) = \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}[-d_X]$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding.

Definition 8.3 *Let* $\mathcal{K} \in D^b_{coh}(\mathcal{D}_{X_1 \times X_2})$*. Then it defines a functor*

$$\bar{\mathcal{K}} : D^b_{coh}(\mathcal{D}_{X_1}) \longrightarrow D^b_{coh}(\mathcal{D}_{X_2}) \qquad \bar{\mathcal{K}}(\mathcal{M}) := p_{2*}(\mathcal{K} \overset{!}{\otimes} p_1^! \mathcal{M})$$

This is motivated by the following proposition (compare with s. 8.2):

Proposition 8.4

$$RHom_{\mathcal{D}_{X_1\times X_2}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2, \mathcal{K}_{12}) = RHom_{\mathcal{D}_{X_2}}(\mathcal{M}_2, \mathcal{K}_{12}(\mathcal{M}_1))$$

Proof. Let $p_i : X_1 \times X_2 \longrightarrow X_i$ be natural projections. We have

$$RHom_{\mathcal{D}}(\star\mathcal{M}_1 \boxtimes \mathcal{M}_2, \mathcal{K}_{12}) = RHom_{\mathcal{D}}(p_1^!(\star\mathcal{M}_1)[-d_{X_2}] \otimes_{\mathcal{O}} p_2^!(\mathcal{M}_2)[-d_{X_1}], \mathcal{K}_{12}) =$$

$$RHom_{\mathcal{D}}(p_2^!(\mathcal{M}_2)[-d_{X_1}], \star p_1^! \star (\mathcal{M}_1)[d_{X_2}] \otimes_{\mathcal{O}} \mathcal{K}_{12}) =$$

$$RHom_{\mathcal{D}}(p_2^*(\mathcal{M}_2)[d_{X_1}], \mathcal{K}_{12} \otimes_{\mathcal{O}} p_1^!(\mathcal{M}_1)[-d_{X_2}]) =$$

$$RHom_{\mathcal{D}}(\mathcal{M}_2, p_{2_*}(\mathcal{K}_{12} \otimes_{\mathcal{O}} p_1^!(\mathcal{M}_1)[-d_{X_1} - d_{X_2}]) = RHom_{\mathcal{D}}(\mathcal{M}_2, \mathcal{K}_{12}(\mathcal{M}_1))$$

Proposition 8.5 *There is natural isomorphism of functors:*

$$\mathcal{K}_{23} \mathbin{\bar{\circ}} \mathcal{K}_{12} = \bar{\mathcal{K}}_{23} \circ \bar{\mathcal{K}}_{13}$$

Proof.
Consider the following diagrams:

$$\begin{array}{ccccccc} & & & X_1\times X_2\times X_3 & & & \\ & & \pi_3 \swarrow & \downarrow \pi_2 & \searrow \pi_1 & & \\ & X_1\times X_2 & & X_1\times X_3 & & X_2\times X_3 & \\ p_1\swarrow & & \searrow p_2 & & q_1 \swarrow & & \searrow q_2 \\ X_1 & & & X_2 & & & X_3 \end{array}$$

and

$$\begin{array}{ccc} & X_1\times X_3 & \\ r_1\swarrow & & \searrow r_2 \\ X_1 & & X_3 \end{array}$$

Let $\Delta_2 : X_1 \times X_2 \times X_3 \hookrightarrow X_1 \times X_2 \times X_2 \times X_3$ is the diagonal imbedding of X_2.

Lemma 8.6 *Let* $\mathcal{K}_{12} \in D^b_{coh}(\mathcal{D}_{X_1\times X_2})$ *and* $\mathcal{K}_{23} \in D^b_{coh}(\mathcal{D}_{X_2\times X_3})$. *Then*

$$\mathcal{K}_{12} \circ \mathcal{K}_{23} = \pi_{2*}(\pi_3^!\mathcal{K}_{12} \otimes_{\mathcal{O}} \pi_1^!\mathcal{K}_{23})$$

Proof. Follows immediately from $\pi_3 = \tau_{12} \circ \Delta_2$ and $\pi_1 = \tau_{23} \circ \Delta_2$. One has

$$K_{23}(K_{12}(\mathcal{M})) = q_{3*}(q_2^! p_{2*}(p_1^!\mathcal{M}_1 \overset{!}{\otimes} K_{12}) \overset{!}{\otimes} K_{23}) =$$

$$q_{3*}(\pi_{1*}\pi_3^!(p_1^!\mathcal{M}_1 \overset{!}{\otimes} K_{12}) \overset{!}{\otimes} K_{23}) = q_{3*}(\pi_{1*}(\pi_1^!\mathcal{M}_1 \overset{!}{\otimes} \pi_{12}^! K_{12}) \overset{!}{\otimes} K_{23}) \overset{1}{=}$$

$$q_{3*}\pi_{1*}(\pi_1^!\mathcal{M}_1 \overset{!}{\otimes} \pi_3^! K_{12} \overset{!}{\otimes} \pi_1^! K_{23}) = \pi_{3*}(\pi_1^!\mathcal{M}_1 \overset{!}{\otimes} \pi_3^! K_{12} \overset{!}{\otimes} \pi_1^! K_{23}) =$$

$$r_{3*}\pi_{2*}(\pi_1^!\mathcal{M}_1 \overset{!}{\otimes} \pi_3^! K_{12} \overset{!}{\otimes} \pi_1^! K_{23}) =$$

$$r_{3*}\pi_{13*}(\pi_{2!} r_1^!\mathcal{M}_1 \overset{!}{\otimes} \pi_3^! K_{12} \overset{!}{\otimes} \pi_1^! K_{23}) \overset{2}{=} r_{3*}(r_1^!\mathcal{M}_1 \overset{!}{\otimes} K_{12} \circ K_{23})$$

Here (1) is by the base change for the diagram

$$\begin{array}{ccccc} & & X_1 \times X_2 \times X_3 & & \\ & \pi_3 \swarrow & & \searrow \pi_1 & \\ X_1 \times X_2 & & & & X_2 \times X_3 \\ & p_2 \searrow & & \swarrow q_2 & \\ & & X_2 & & \end{array}$$

and (2) is by projection formula $f_*(A \overset{!}{\otimes} f^! B) = f_* A \overset{!}{\otimes} B$.

Lemma 8.7 *The composition of 1-morphisms*

$$(\mathcal{M}_1, \mathcal{M}_2, \mathcal{K}_{12}; \alpha_{12}) \quad on \quad X_1 \times X_2 \qquad and \qquad (\mathcal{M}_2, \mathcal{M}_3, \mathcal{K}_{23}; \alpha_{23}) \quad on \quad X_2 \times X_3$$

is a 1-morphisms

$$(\mathcal{M}_1, \mathcal{M}_3, \mathcal{K}_{13}; \alpha_{13}) \quad on \quad X_1 \times X_3$$

where $\mathcal{K}_{13} := \mathcal{K}_{23} \circ \mathcal{K}_{12}$ *and* α_{23} *is the composition:*

$$\mathcal{M}_3 \longrightarrow \bar{\mathcal{K}}_{23}(\mathcal{M}_2) \overset{\bar{\mathcal{K}}_{23}(\tau_{12})}{\longrightarrow} \bar{\mathcal{K}}_{23}(\bar{\mathcal{K}}_{12}(\mathcal{M}_1)) = \bar{\mathcal{K}}_{13}(\mathcal{M}_1)$$

4. Natural linear maps provided by algebraic kernels. Let $\mathcal{M}_i \in D^b_{coh}(\mathcal{D}_{X_i})$, $i = 1, 2$, and $\mathcal{K}_{12} \in D^b_{coh}(\mathcal{D}_{X_1 \times X_2})$. Suppose we are given the following data:

1) an algebraic kernel

$$\alpha_{12} \in RHom_{\mathcal{D}_{X_2}}(\mathcal{M}_2, \mathcal{K}_{12}(\mathcal{M}_1)); \qquad K_{12} \in RHom_{\mathcal{D}_{X_1 \times X_2}}(\mathcal{K}_{12}, \mathcal{O}_{X_1 \times X_2})$$

2) an element

$$\gamma \in RHom_{\mathcal{D}}(p_{2*}\mathcal{O}(X_1 \times X_2), \mathcal{O}(X_2))$$

We will construct a linear map

$$RHom_{\mathcal{D}}(\mathcal{M}_1, C^\infty(X_1)) \longrightarrow RHom_{\mathcal{D}}(\mathcal{M}_2, C^\infty(X_2))$$

related to this data. Namely, by the functoriality

$$RHom_{\mathcal{D}}(\mathcal{M}_1, C^\infty(X_1)) \longrightarrow RHom_{\mathcal{D}}(p_1^! \mathcal{M}_1, p_1^! C^\infty(X_1)) \longrightarrow$$
$$RHom_{\mathcal{D}}(p_1^! \mathcal{M}_1, C^\infty(X_1 \times X_2)[d_{X_2}]) \overset{K_{12} \otimes}{\longrightarrow}$$
$$RHom_{\mathcal{D}}(\mathcal{K}_{12} \otimes_{\mathcal{O}} p_1^! \mathcal{M}_1, C^\infty(X_1 \times X_2)[d_{X_2}]) \longrightarrow$$
$$RHom_{\mathcal{D}}(p_{2*}(\mathcal{K}_{12} \otimes_{\mathcal{O}} p_1^! \mathcal{M}_1), p_{2*} C^\infty(X_1 \times X_2)[d_{X_1}]) \overset{\gamma}{\longrightarrow}$$
$$RHom_{\mathcal{D}}(\mathcal{K}_{12}(\mathcal{M}_1), C^\infty(X_2)[-d_{X_2}])$$

The morphism α_{12} provides the last arrow

$$RHom_{\mathcal{D}}(\mathcal{K}_{12}(\mathcal{M}_1), C^\infty(X_2)[-d_{X_2}]) \longrightarrow RHom_{\mathcal{D}}(\mathcal{M}_2, C^\infty(X_2)[-d_{X_2}])$$

If

$$\alpha_{12} \in R^f Hom_{\mathcal{D}}(\mathcal{M}_2, \mathcal{K}_{12}(\mathcal{M}_1)) \quad , K_{12} \in R^k Hom_{\mathcal{D}}(\mathcal{K}_{12}, \mathcal{O})$$

and

$$\gamma \in R^{d_{X_1}-l} Hom_{\mathcal{D}}(p_{2*}C^\infty(X_1\times X_2), C^\infty(X_2))$$

then we get a linear map

$$R^j Hom_{\mathcal{D}}(\mathcal{M}_1, C^\infty(X_1)) \longrightarrow R^{j+k+l-d_{X_1}} Hom_{\mathcal{D}}(\mathcal{M}_2, C^\infty(X_2))$$

5. Algebraic version of the Radon transform of (holonomic) functions. Any 1-morphism $\gamma : A_2 \longrightarrow A_3$ provides a functor

$$F_\beta : Mor_1(A_1, A_2) \longrightarrow Mor_1(A_1, A_3) \qquad \alpha \longmapsto \beta \circ \alpha$$

There is an object $\star$ corresponding to the one-dimensional vector space considered as a $\mathcal{D}$-module over a point.

The category $Mor_1(\star, (X, \mathcal{M}))$ looks as follows. Its objects are pairs: a holonomic complex of $\mathcal{D}$-modules $\mathcal{L}$ on X and a morphism $\alpha : \mathcal{M} \to \mathcal{L}$. The morphisms are provided by $\varphi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ making the corresponding diagram commutative. We will call it the category of $\mathcal{D}$-modules under $\mathcal{M}$ on X.

Therefore the 1-morphisms $(\alpha, \mathcal{K}) : (X, \mathcal{M}) \to (X, \mathcal{N})$ provide functors from the category of $\mathcal{D}$-modules under $\mathcal{M}$ on X to the category of $\mathcal{D}$-modules under $\mathcal{N}$ on Y.

6. Examples. Let me first discuss the analytic properties of the Radon transform in $\mathbb{R}^2$.

$$\varphi(x,y) \longmapsto \hat\varphi(\xi_1,\xi_2,s) := \int \varphi(x,y)\delta(\xi_1 x + \xi_2 y - s)dxdy$$

the 1-form

$$\kappa\hat\varphi(\xi_1,\xi_2) := \hat\varphi'_s(\xi_1 d\xi_2 - \xi_2 d\xi_1)$$

is closed on the subvariey $\xi_1 x + \xi_2 y - s = 0$. Here (x,y) is a given point. Integral of this 1-form over any cycle in (ξ_1,ξ_2) plane is zero.

Consider the line through the point (x,y) corresponding to $\xi = (\xi_1,\xi_2)$. On a line minus a point ((x,y) in our case),there is canonical multiplicatively invariant measure ($\frac{dt}{t}$). Let $L(\xi) := \int \varphi(x-\xi_2 t, y+\xi_1 t)\frac{dt}{t}$ be the integral over this measure. Then

$$\int_\xi^\eta (\kappa\hat\varphi)(\xi_1,\xi_2) = L(\eta) - L(\xi) \tag{60}$$

where we integrate over any path connecting points ξ and η.

In particular in the affine picture

$$\varphi(x,y) \longmapsto \int_\infty^\infty \varphi(x, ax+b)dx, \quad \kappa\hat\varphi = \hat\varphi'_b da \qquad \int \hat\varphi'_b(a, y-ax)da = \int_{-\infty}^\infty \varphi(x,y)\frac{dy}{y} \tag{61}$$

The 1-form $\kappa\hat\varphi$ is exact on the image of functions vanishing at the point (x,y). For example

$$\kappa(I(x\varphi)) = (I(x\varphi))'_b da = (I\varphi)'_a da = d(I\varphi)$$

Formula (60) follows immediately from this.

Now let us turn to the $\mathcal{D}$-module picture. Set

$$X_1 = \{(x,y)\} = \mathbb{R}^2, \quad X_2 = \{(a,b)\} = \mathbb{R}^2, \quad X_3 = \{(x',y')\} = \mathbb{R}^2$$

and $\mathcal{M}_{X_i} = \mathcal{D}_{X_i}$. Notice that $\star\mathcal{D}_{X_i} \boxtimes \mathcal{D}_{X_{i+1}} = \mathcal{D}_{X_i\times X_{i+1}}[d_{X_i}]$. Set

$$\delta(A) = \delta(y - ax - b) \quad \text{and} \quad \delta(A') = \delta(y' - ax' - b)$$

$$\mathcal{K}_{12} := \mathcal{D}_{X_1\times X_2}\cdot\delta(A), \quad \mathcal{K}_{23} := \mathcal{D}_{X_2\times X_3}\cdot\delta(A')$$

$$\alpha_{12}[-2] : 1_{X_1\times X_2} \longmapsto \delta(A), \quad \alpha_{12}[-2] : 1_{X_2\times X_3} \longmapsto \delta^{(1)}(A')$$

The $\mathcal{D}$-module $\mathcal{K}_{13}$ has a more complicated structure which can be described as follows.

$$0 \longrightarrow \mathcal{O}_{X_1\times X_3} \oplus \delta_{\Delta_{13}} \longrightarrow \mathcal{K}_{13} \longrightarrow \delta_V \longrightarrow 0 \qquad (62)$$

Here $\Delta_{13} \subset X_1 \times X_2$ is the diagonal and V is the divisor of pairs of points (p, p') with $x = x'$ (I.e. the vertical line through p contain the point p'). Let

$$j : X_1 \times X_3 \backslash V \hookrightarrow X_1 \times X_3 \quad i : V \hookrightarrow X_1 \times X_3 \quad f : V\backslash\Delta_{13} \hookrightarrow V$$

Then (62) is the Bauer sum of the following two standard extensions:

$$0 \longrightarrow \mathcal{O}_{X_1\times X_3} \longrightarrow j_*j^*\mathcal{O}_{X_1\times X_3} \longrightarrow \delta_V \longrightarrow 0$$

and

$$i_*\Big(0 \longrightarrow \delta_{\Delta_{13}} \longrightarrow f_!f^*\mathcal{O}_V \longrightarrow \mathcal{O}_V \longrightarrow 0\Big)$$

To see this consider the variety $\mathcal{A} := \{p, l, p'\} \subset X_1 \times X_2 \times X_3$ such that $p, p' \in l$ and its closure $\bar{\mathcal{A}}$ in $X_1 \times \bar{X}_2 \times X_3$. Notice that $\bar{\mathcal{A}}$ is the blow up of the diagonal Δ_{13} in $X_1 \times X_3$.

Then $\mathcal{K}_{13} = \pi_{2*}\mathcal{O}_{\mathcal{A}}$. One has $\bar{\pi}_{2*}\mathcal{O}_{\bar{\mathcal{A}}} = \mathcal{O}_{X_1\times X_3} \oplus \delta_{\Delta_{13}}$. Further, notice that $\bar{\mathcal{A}}\backslash\mathcal{A}$ projects isomorphically to V. So one has

$$0 \longrightarrow \mathcal{O}_{\bar{\mathcal{A}}} \longrightarrow g_*\mathcal{O}_{\mathcal{A}} \longrightarrow \delta_V \longrightarrow 0$$

Taking direct image of this extension to $X_1 \times X_3$ we get (62)).

Proposition 8.8 *The formula*

$$\delta(x - x')\delta(y - y') \longmapsto \delta(y - ax - b) \otimes \delta^{(1)}(y' - ax' - b)dadb \qquad (63)$$

defines a homomorphism of $\mathcal{D}$-modules $\delta_\Delta \longrightarrow \mathcal{K}_{13}$ and hence a 2-morphism $Id_{\mathcal{D}_{X_1}} => (\alpha_{13}, \mathcal{K}_{13})$.

Proof. We have to show that applying to the right hand side of (63) any differential equations which the left hand side satisfy, we will get exact 2-form in the de Rham complex with respect to (a, b) variables. This follows from the formulas

$$(x - x') \cdot \delta(A) \otimes \delta^{(1)}(A')dadb = d\Big(\delta(A) \otimes \delta(A')(xda + db)\Big)$$

$$(y - y') \cdot \delta(A) \otimes \delta^{(1)}(A')dadb = d\Big(\delta(A) \otimes \delta(A')a(xda + db)\Big)$$

$$(\partial_x + \partial_{x'})\delta(A) \otimes \delta^{(1)}(A')dadb = d\Big(\delta(A) \otimes \delta^{(1)}(A')ada\Big)$$

$$(\partial_y + \partial_{y'})\delta(A) \otimes \delta^{(1)}(A')dadb = d\Big(\delta(A) \otimes \delta^{(1)}(A)da\Big)$$

It is amazing to see the structure of the extension (62) from this point of view. Namely, $\delta(A) \otimes \delta(A')dadb$ is a generator of $\mathcal{K}_{13}$, so $(x - x')\delta(A) \otimes \delta(A')dadb$ is the generator of the submodule $\mathcal{O}_{X_1\times X_3}$ and $\delta(A) \otimes \delta^{(1)}(A')dadb$ generates the submodule $\delta_{\Delta_{13}}$.

The Radon transform over the lines in the space.

$$X_1 = \{(x, y, z)\} = \mathbb{A}^3, \quad X_2 = \{(a_1, a_2, b_1, b_2)\} = \mathbb{A}^4, \quad X_3 = \{(x', y', z')\} = \mathbb{A}^3$$

and

$$\mathcal{M}_{X_i} = \mathcal{D}_{X_i} \quad \text{for} \quad i = 1, 3; \qquad \mathcal{M}_{X_2} = \mathcal{D}_{X_2} \cdot (\frac{\partial^2}{\partial a_1 \partial b_2} - \frac{\partial^2}{\partial a_2 \partial b_1})$$

Notice that $\star\mathcal{M}_{X_2} = \mathcal{M}_{X_2}[3]$.

Let $A \subset X_1 \times X_2$ be the correspondence $\{y - a_1x - b_1 = 0, z - a_2x - b_2 = 0\}$ defining our family of lines. Set

$$\delta(A) = \delta(y - a_1x - b_1) \cdot \delta(z - a_2x - b_2), \quad \delta(A') = \delta(y' - a_1x' - b_1) \cdot \delta(z' - a_2x' - b_2)$$

Then

$$\mathcal{K}_{12} := \mathcal{D}_{X_1\times X_2} \cdot \delta(A) \qquad \mathcal{K}_{23} := \mathcal{D}_{X_2\times X_3} \cdot \delta(A')$$

$$\alpha_{12} : 1_{X_1\times X_2} \longmapsto \delta(A)[-3] \qquad \alpha_{23} : 1_{X_2\times X_3} \longmapsto \delta(A')[-3]$$

Proposition 8.9 *The formula*

$$\delta(x-x')\delta(y-y')\delta(z-z') \longmapsto G_{\mathcal{M}_2}\Big(\delta(A'),\delta(A)\Big)$$

defines a homomorphism of $\mathcal{D}$-modules $\delta_\Delta \longrightarrow \mathcal{K}_{23}\circ\mathcal{K}_{12}$ providing a 2-morphism $Id_{\mathcal{D}_{X_1}} => (\alpha_{23},\mathcal{K}_{23})\circ(\alpha_{12},\mathcal{K}_{12})$

Here

$$G_{\mathcal{M}_2}(g,f) = \frac{1}{2}\Big((g'_{a_1}\cdot f - g\cdot f'_{a_1})da_1\wedge da_2\wedge db_1 - (g'_{a_2}\cdot f - g\cdot f'_{a_2})da_1\wedge da_2\wedge db_2$$

$$(g'_{b_1}\cdot f - g\cdot f'_{b_1})da_1\wedge db_1\wedge db_2 - (g'_{b_2}\cdot f - g\cdot f'_{b_2})da_2\wedge db_1\wedge db_2\Big)$$

Relation with the "form κ" of [GGrS]. The Green class can be represented by another cocycle

$$\kappa_{\mathcal{M}} := \delta(A)\otimes\Big(\frac{\partial}{\partial b_1}\delta(A')da_1 + \frac{\partial}{\partial b_2}\delta(A')da_2\Big)\wedge(xda_1+db_1)\wedge(xda_2+db_2)$$

Its great advantage is "locality": it is a cocycle in the de Rham complex with support in the incidence subvariety A.

The expression $\frac{\partial f}{\partial b_1}da_1 + \frac{\partial f}{\partial b_2}da_2$ is the "1-form κ". It is a 1-form on the incidence subvariety A.

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