Chow polylogarithms and regulators

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Contents

1 Introduction

The classical dilogarithm

$$Li_2(z) := -\int_0^z \log(1-t) d\log t$$

is a multivalued analitic function on $\mathbb{C}P^1\backslash\{0,1,\infty\}$. It has a single-valued version: the Bloch-Wigner function

$$\mathcal{L}_2(z) := ImLi_2(z) + \arg(1-z)\log|z|$$

which satisfies the famous 5-term functional relation. Namely, for any 5 distinct points $z_1, ..., z_5$ on $\mathbb{C}P^1$ one has (r is the cross-ratio).

$$\sum_{i=1}^{5} (-1)^i \mathcal{L}_2(r(z_1, ..., \hat{z}_i, ..., z_5)) = 0$$

In this note we show that the Bloch-Wigner function can be naturally extended to the (infinite dimensional) variety of all algebraic curves in $\mathbb{C}P^3$ which are in sufficiently general position with respect to a given simplex L. (By definition a simplex in $\mathbb{C}P^3$ is a collection of 4 hyperplanes in generic position).

We call the corresponding function the Chow dilogarithm function. When our curve is a straight line we obtain just the Bloch-Wigner function evaluated at the cross-ratio of the 4 intersection points of this line with the faces of the simplex L. It is interesting that even in this case we get a new presentation of $\mathcal{L}_2(z)$.

Any algebraic surface in $\mathbb{C}P^4$ which is in general position with respect to a given simplex produces a 5-term relation for the Chow dilogarithm function. Namely, the intersection of the surface with a codimension 1 face of the simplex provides a curve and a simplex in $\mathbb{C}P^3$. A simplex in $\mathbb{C}P^4$ has 5 codimension 1 faces. The alternating some of the corresponding 5 values of the Chow dilogarithm is zero.

The differential equation for the Chow dilogarithm function reflects the geometry of the intersection points of the corresponding curve with the faces of the simplex.

Finally, one can prove that the Chow dilogarithm function can be expressed by the Bloch-Wigner function.

In general the Chow n-logarithm is a collection of differential forms on Bloch's higher Chow varieties. For example the Chow n-logarithm function lives on the variety of all n-dimensional varieties in $\mathbb{C}P^{2n-1}$ which are in generic position with respect to a given simplex. Each $(n+1)$-dimensional variety in $\mathbb{C}P^{2n}$ generic with respect to a simplex provides a $(2n+1)$-term functional equation for the Chow n-logarithm function.

In particulary we get an explicit definition of the Grassmannian polylogarithms whose existence was conjectured in [BMS] and [HM] (see also [HY]).

The main application is an explicit construction of the Beilinson regulator

$$gr_n^{\gamma} K_{2n-i}(X) \longrightarrow H^i_{\mathcal{D}}(X/\mathbb{R}, \mathbb{R}(n))$$

from an appropriate piece of algebraic K-theory to the Deligne cohomology of an arbitrary regular variety over $\mathbb{R}$. Namely using the (bi)-Grassmannian n-logarithm and the results of [G3] we construct a cocycle in the Deligne cohomology representing the universal Chern class $c_n \in H^{2n}_{\mathcal{D}}(BGL(\mathbb{C})_{\bullet}, R(n))$.

Suppose that X is a projective smooth variety over $\mathbb{Q}$ of dimension $i-1$. Beilinson's conjectures [B] predicts that the image of the regulator map is a lattice whose covolume (with respect to the $\mathbb{Q}$-structure provided by $H^i_{\mathcal{D}}(X/\mathbb{R}, \mathbb{Q}(n))$ coinsides (up to a nonzero rational multiple) with the special value at $s = n$ of the L-function $L(h^{i-1}(X), s)$. Therefore the special values of the L-functions of varieties over number fields should be expressed in terms of the Grassmannian polylogarithms.

In particulary for $X = Spec\mathbb{C}$ we get an explicit construction of the Borel regulator $K_{2n-1}(\mathbb{C}) \to \mathbb{R}$. This togerther with the Borel theorem [Bo2] leads to formulas expressing the special values of ζ-functions of number fields at $s = n$ by means of the Grassmannian n-logarithm function.

In the last section we scetch a construction of the multivalued analitic version of the Chow polylogarithms.

2 Construction of Chow polylogarithms

1 Higher Chow varieties and polylogarithms. A simplex in $\mathbb{C}P^n$ is a collection of hyperplanes $L_0, ..., L_n$ in generic position, i.e. with empty intersection.

Let us choose in $\mathbb{C}P^n$ a simplex L and a generic hyperplane H. We might think about this data as of a simplex in n-dimensional affine space $\mathbb{A}^n := \mathbb{C}P^n \backslash H$

Let $\mathcal{Z}_p^q(L)$ be the variety of all codimension q effective algebraic cycles in $\mathbb{C}P^{p+q}$ which intersect properly (i.e. in right codimension) all faces of the simplex L. It is a union of infinite number of finite dimensional algebraic varieties. (This is the set-up for the definition of Bloch's Higher Chow groups [Bl]).

The intersecion of a cycle with a codimension 1 face L_i of the simplex L provides a map

$$a_i : \mathcal{Z}_p^q(L) \longrightarrow \mathcal{Z}_{p-1}^q(L_i)$$

Further, projection with the center at the vertex l_j of L defines a map

$$b_j : \mathcal{Z}_p^q(L) \longrightarrow Z_p^{q-1}(L_j)$$

Notice that $\mathcal{Z}_0^n(L) = \mathbb{C}P^n \backslash L = (\mathbb{C}^*)^n$. So one can attach to a simplex $L \subset \mathbb{C}P^n$ canonical n-form Ω_L with logarithmic singularities in $\mathbb{C}P^n \backslash L$. Let z_i be a linear homogeneous equation of a hyperplane L_i, then

$$\Omega_L = (2\pi i)^{-n} d\log\frac{z_1}{z_0} \wedge ... \wedge d\log\frac{z_n}{z_0}$$

It is skewsymmetric with respect to reordering of hyperplanes L_i.

Recall that a p-current on a smooth manifold X is a linear continuous functional on the space of $(\dim_{\mathbb{R}} X - p)$-forms with compact support. By a current on a singular variety we will understand a current on its smooth part.

In this paper for each given $q \geq 0$ I will construct explicitly a canonical chain of $(q-p-1)$-currents $\omega_p^q = \omega_p^q(L, H)$ on $\mathcal{Z}_p^q(L)$. The restriction of ω_p^q to the subvariety $\hat{\mathcal{Z}}_p^q(L)$ of smooth cycles in generic position with respect to the simplex L will be a real-analytic differential $(q-p-1)$ form.

Let $H_i := H \cap L_i$. Set $Im(x+iy) := iy$. The currents ω_p^q will satisfy the following conditions:

$$i) \qquad d\omega_0^q(L, H) = Im\Omega_L \tag{1}$$

$$ii) \quad d\omega_p^q(L, H) = \sum_{i=0}^{p+q}(-1)^i a_i^* \omega_{p-1}^q(L, H_i) \tag{2}$$

$$iii) \qquad \sum_{j=0}^{p+q+1}(-1)^j b_j^* \omega_p^q(L, H) = 0 \tag{3}$$

We call the collection ω_p^q the *q-th Chow polylogarithm.* The function $\mathcal{P}_q := \omega_{q-1}^q$ on $\mathcal{Z}_{q-1}^q(L)$ will be called the *Chow polylogarithm function.*

2. The homomorphism r_n. Let X be a variety over $\mathbb{C}$ and $f_1, ..., f_n$ be rational functions on $X(\mathbb{C})$. Set

$$r_n(f_1, ..., f_n) := \tag{4}$$

$$(2\pi i)^{-n}\mathrm{Alt}_n \sum_{j\geq 0} \frac{1}{(2j+1)!(n-2j-1)!} \log|f_1| d\log|f_2| \wedge ... \wedge d\log|f_{2j+1}| \wedge di \arg f_{2j+2} \wedge ... \wedge di \arg f_n$$

Here Alt_n is the operation of alternation:

$$\mathrm{Alt}_n F(x_1, ..., x_n) := \sum_{\sigma \in S_n} (-1)^{|\sigma|} F(x_{\sigma(1)}, ..., x_{\sigma(n)})$$

Let $S^m(\eta_X)$ be the space of smooth n-forms with values at $i\mathbb{R}$ at the generic point of X, i.e. each form is defined on an open subset of $X(\mathbb{C})$. We get a homomorphism

$$r_n : \Lambda^n \mathbb{C}(X)^* \longrightarrow S^{n-1}(\eta_X) \tag{5}$$

such that $dr_n(f_1 \wedge ... \wedge f_n) = Im\Big((2\pi i)^{-n} d\log f_1 \wedge ... \wedge d\log f_n\Big)$

The form (4) is (up to a constant) the product in the real Deligne cohomolgy of 1-cocycles $(\log|f_i|, d\log f_i)$.

3. The primitive $r_n(L; H)$. The form Ω_L has periods in $\mathbb{Z}$. So $Im\Omega_L$ is exact. However there is no canonical choice of a primitive $(n-1)$-form for it. (The group $(\mathbb{C}^*)^n$ acting on $\mathbb{C}P^n \backslash L$ leaves the form invariant and acts non trivially on the primitives). But if we consider a simplex L in the affine complex space $\mathbb{A}^n$ (or, what is the same, choose an additional hyperplane H in $\mathbb{C}P^n$, which should be thought of as the infinite hyperplane) then there is a *canonical* primitive $r_n(L; H)$.

Namely, choose linear homogeneous coordinates $z_0, ..., z_n$ in $\mathbb{C}P^n$ such that L_i is given by equation $z_i = 0$ and H by $\sum z_i = 0$. Then

$$r_n(L; H) := r_n(\frac{z_1}{z_0} \wedge ... \wedge \frac{z_n}{z_0}) \tag{6}$$

is a primitive of $Im\Omega_L$ in $\mathbb{C}P^n \backslash L$. The element $\frac{z_1}{z_0} \wedge ... \wedge \frac{z_n}{z_0}$ is skew-symmetric with respect to reordering of coordinates, so the same is true for the form $r_n(L; H)$.

4. The main construction. The form $r_n(L; H)$ defines an $(n-1)$-current on $\mathbb{C}P^n$.

Definition 2.1. *ω_p^q is the Radon transform of the current $r_{p+q}(L; H)$ in $\mathbb{C}P^{p+q}$ over the family of cycles Y_ξ parametrized by $\mathcal{Z}_p^q(L)$.*

More precisely, this means the following. Consider the incidence variety:

$$\Gamma_p := \{(x, \xi) \in \mathbb{C}P^{p+q} \times \mathcal{Z}_p^q(L) \quad \text{such that} \quad x \in Y_\xi\}$$

where Y_ξ is the cycle in $\mathbb{C}P^{p+q}$ corresponding to $\xi \in \mathcal{Z}_p^q(L)$. We get a double bundle

$$\begin{array}{ccc} & \Gamma_p & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{C}P^{p+q} & & \mathcal{Z}_p^q(L) \end{array}$$

Then

$$\omega_p^q := \pi_{2*}\pi_1^* r_{p+q}(L;H)$$

The fibers of π_2 are compact, so the push forward of currents is well defined. It commute with the De Rham differential d:

$$\pi_{2*}d = d\pi_{2*} \tag{7}$$

Theorem 2.2. *The currents $\{\omega_p^q\}$ satisfy the condition (2).*

Proof. Let $\hat{L}_i$ be the simplex cut by L in the hyperplane L_i. Consider the $(n-2)$-form $r_{n-1}(\hat{L}_i; H_i)$ in L_i as $(n-2)$-current in $\mathbb{C}P^n$.

Lemma 2.3. $dr_n(L;H) = Im(\Omega_L) + \sum_{i=0}^{n}(-1)^i r_{n-1}(\hat{L}_i;H_i)$

Proof . Use the formula $d(di\arg z) = 2\pi i\delta(z)dzd\overline{z}$.

Lemma 2.4. $\pi_{2*}Im(\Omega_L) = 0$

Proof. $Im(\Omega_L)$ is a sum of forms of type $(p+q,0)$ and $(0,p+q)$. So its integral over any family of complex varieties is zero.

Theorem follows immediately from (7), and this lemmas.

Notice that (1) is true just by the definition. Finally, (3) is provided by

Lemma 2.5.

$$\sum_{j=0}^{n+1}(-1)^j b_j^*\omega_0^n = 0.$$

Proof. Let $s(z_0, ..., z_n) := z_1/z_0 \wedge ... \wedge z_n/z_0$. Lemma follows from the identity

$$\sum_{j=0}^{n+1}(-1)^j s(z_0, ..., \hat{z}_j, ..., z_{n+1}) = 0$$

5. An interpretation on the language of the real Deligne cohomology. Let $\mathcal{A}_X^p$ be the space of all p-currents on X. The De Rham complex of currents $(\mathcal{A}_X^\bullet, d)$ is a resolution of the constant sheaf $\mathbb{R}$.

The n-th Deligne complex $\mathbb{R}(n)_\mathcal{D}$ can be defined as the total complex associated with the following bicomplex:

$$\begin{array}{ccccccccccc}
\mathcal{A}_X^0(n-1) & \stackrel{d}{\longrightarrow} & \mathcal{A}_X^1(n-1) & \stackrel{d}{\longrightarrow} & \dots & \stackrel{d}{\longrightarrow} & \mathcal{A}_X^n(n-1) & \stackrel{d}{\longrightarrow} & \mathcal{A}_X^{n+1}(n-1) & \stackrel{d}{\longrightarrow} & \dots \\
 & & & & & & \uparrow \pi_n & & \uparrow \pi_n & & \\
 & & & & & & \Omega_X^n & \stackrel{\partial}{\longrightarrow} & \Omega_X^{n+1} & \stackrel{\partial}{\longrightarrow} &
\end{array} \tag{8}$$

Here $\pi_n : \mathcal{A}_X^m \otimes \mathbb{C} \longrightarrow \mathcal{A}_X^m(n-1) := \mathcal{A}_X^m \otimes \mathbb{R}(n-1)$ is the projection induced by $\mathbb{C} = \mathbb{R}(n-1) \oplus \mathbb{R}(n) \longrightarrow \mathbb{R}(n-1)$. Further, $\mathcal{A}_X^0(n-1)$ placed in degree 1 and $(\Omega_X^\bullet, \partial)$ is the de Rham complex of holomorphic forms with logarithmic singularities at infinity.

$\hat{\mathcal{Z}}_p^q(L)$ for $p \geq 0$ form a truncated simplicial variety $\hat{\mathcal{Z}}_\bullet^q$. The conditions i) and ii) just mean that the sequence of forms ω_p^q is a $2q$-cocycle in the bicomplex computing the Deligne cohomology $H^{2q}(\hat{Z}_\bullet^q, \mathbb{R}(q)_{\mathcal{D}})$.

6. First application: the bi-Grassmannian polylogarithms. Let us denote by $\hat{G}_p^q$ the Grassmannian of p-planes in $\mathbb{A}^{p+q}$ in generic position with respect to a given simplex L.

The operations a_i and b_j transforms planes to planes. So we get the following diagram of manifolds which is called the bi-Grassmannian $\hat{G}(q)$:

$$
\begin{array}{lccccccc}
 & & & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow \\
\hat{G}(q) := & & & \stackrel{\rightarrow}{\stackrel{\dots}{\rightarrow}} & \hat{G}_1^{q+1} & \stackrel{\rightarrow}{\stackrel{\dots}{\rightarrow}} & \hat{G}_0^{q+1} \\
 & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow & & \downarrow \dots \downarrow \\
 & \dots \stackrel{\rightarrow}{\stackrel{\dots}{\rightarrow}} & \hat{G}_2^q & \stackrel{\rightarrow}{\stackrel{\dots}{\rightarrow}} & \hat{G}_1^q & \stackrel{\rightarrow}{\stackrel{\dots}{\rightarrow}} & \hat{G}_0^q
\end{array}
$$

Here the horisontal arrows are the arrows a_i and the vertical ones are b_j. The bi-Grassmannian $\hat{G}(n)$ is a truncated semisimplicial scheme: $\hat{G}(n)_{(k)} := \coprod_{p+q=k} \hat{G}_p^q$.

Let $\psi_p^q(q)$ be the restriction of the differential form ω_p^q to $\hat{G}_p^q$. The conditions i), ii) they satisfy are exactly the defining conditions for the single-valued Grassmannian polylogarithm whose existence was conjectured by Hain-MacPherson and Beilinson-Schechtman ([HM], [BMS], see also the pioneering work [GGL]). So we proved this conjecture.

R.Hain and J.Yang [HY] proved that for a certain semisimplicial Zariski open subset $U_\bullet \subset \hat{G}_\bullet^n$ there exists a class in $H^{2n}(U_\bullet, \mathbb{R}(n)_{\mathcal{D}})$ whose $\Omega^n_{\hat{G}_0^n}$-component is $d\log z_1 \wedge \dots \wedge d\log z_n$. The other components can be represented by iterated integrals.

After this the property iii) plays the key role. Namely, set $\psi_p^{q+i}(q) = 0$ if $i > 0$. Then the condition iii) guarantee that the forms $\psi_p^{q+i}(q) = 0$ where $i \geq 0$ is a $2q$-cocycle in the bicomplex computing the Deligne cohomology $H^{2q}(\hat{G}(q)_\bullet, \mathbb{R}(q)_{\mathcal{D}})$. We will call it *the bi-Grassmannian n-logarithm*. ([G3]).

3 Properties of Chow polylogarithm functions

1. Basic integral and its general properties

Lemma 3.1. *For any rational functions $f_1, \dots, f_n$ on X the $(n-1)$-form $r_n(f_1 \wedge \dots \wedge f_n)$ defines a current on $X(\mathbb{C})$.*

Proof. We may suppose that the divisors of functions f_i are disjoint because (5) is a homomorphism. Resolving singularities we reduce lemma to the case when these divisors have normal crossing. Our statement is local, so we can assume that in local coordinates $z_1, \dots, z_m$ one has $f_1 = z_1, \dots, f_k = z_k$ and $div f_j$ for $j > k$ does not intersect the origin. After this the statement of lemma is obvious.

In particulary if $dim X = n$ the integral

$$\int_{X(\mathbb{C})} r_{2n+1}(f_1, ..., f_{2n+1}) \tag{9}$$

is convergent.

Proposition 3.2. *Integral (??) is a rational multiple of*

$$\int_{X(\mathbb{C})} \sum_{j=1}^{2n+1} (-1)^j \log|f_j| d\log|f_1| \wedge ... \wedge d\hat{\log}|f_j| \wedge ... \wedge d\log|f_{2n+1}| \tag{10}$$

Theorem 3.3. *Suppose that $dim X = n$. Then integral (9) does not change if we multiply one of the functions f_i by a non zero constant.*

A simplex L in $\mathbb{A}^n$ determines n coordinate functions z_i such that $< z_i, l_j >= \delta_{i,j}$. So on a subvariety $X \subset \mathbb{A}^n$ n rational functions appear.

Theorem (3.3) just means that the function $\mathcal{P}_n$ is invariant under the natural action of the torus $(\mathbb{C}^*)^n$ on $\mathcal{Z}^n_{n-1}(L)$ and so does not depend on the choice of the hyperplane H. The forms ω^n_p for $p < n-1$ depend on H.

$(\mathbb{C}^*)^n$-orbits on $\hat{G}^n_{n-1}$ are paprametrized by *configurations* of $2n$ generic hyperplanes in $\mathbb{C}P^{n-1}$. Namely, the orbit of a plane $h \in \hat{G}^n_{n-1}$ is determined by the $2n$-tuple of hyperplanes $\{h \cap L_i\}$ in h considered modulo PGL_n-equivalence. In the next section we describe the Grassmannian n-logarithm using this language.

2. The Grassmannian n-logarithm function. Let $h_1, ..., h_{2n}$ be *arbitrary* $2n$ hyperplanes in $\mathbb{C}P^{n-1}$. Choose an additional hyperplane h_0. Let f_i be a rational function on $\mathbb{C}P^{n-1}$ with divisor $h_i - h_0$. It is defined up to a scalar factor. Set

$$\mathcal{P}_n(h_1, ..., h_{2n}) := \int_{\mathbb{C}P^{n-1}} r_{2n-1}(\sum_{j=1}^{2n} (-1)^j f_1 \wedge ... \wedge \hat{f_j} \wedge ... \wedge f_{2n})$$

It is skewsymmetric by the definition. Notice that

$$\sum_{j=1}^{2n} (-1)^j f_1 \wedge ... \wedge \hat{f_j} \wedge ... \wedge f_{2n} = \frac{f_1}{f_{2n}} \wedge \frac{f_2}{f_{2n}} \wedge ... \wedge \frac{f_{2n-1}}{f_{2n}}$$

So if $g_1, ..., g_{2n-1}$ are rational functions such that $div g_i = h_i - h_{2n}$ then

$$\mathcal{P}_n(h_1, ..., h_{2n}) = \int_{\mathbb{C}P^{n-1}} r_{2n-1}(g_1, ..., g_{2n-1})$$

Theorem 3.4. *The function $\mathcal{P}_n$ has the following properties:*

a) For any $2n+1$ hyperplanes in $\mathbb{C}P^n$ one has

$$\sum_{j=1}^{2n+1} (-1)^j \mathcal{P}_n(h_j \cap h_1, ..., h_j \cap h_{2n+1}) = 0$$

b) For any $2n+1$ hyperplanes in $\mathbb{C}P^{n-1}$ one has

$$\sum_{j=1}^{2n+1}(-1)^j\mathcal{P}_n(h_1,...,\hat{h}_j,...,h_{2n+1})=0$$

Proof.
a) Let $g_1,...,g_{2n+1}$ be rational functions on $\mathbb{C}P^n$ with $divg_i=h_i-h_0$. Then

$$dr_{2n}\Big(\sum_{j=1}^{2n+1}(-1)^j g_1\wedge...\wedge\hat{g}_j\wedge...g_{2n+1}\Big)= \tag{11}$$

$$2\pi i\delta(f_j)df_j\wedge\bar{d}f_j\wedge r_{2n-1}\Big(\sum_{j\neq i}(-1)^j g_1\wedge...\hat{g}_i\wedge...\wedge\hat{g}_j\wedge...g_{2n+1}\Big)$$

(Notice that $d\log g_1\wedge...\wedge d\,\widehat{\log}\,g_j\wedge...\wedge d\log g_{2n+1}=0$ on $\mathbb{C}P^n$). Integrating (11) over $\mathbb{C}P^n$ we get a).

b) is obvious: we apply r_{2n-1} to zero element. Theorem is proved.

Conjecture 3.5. *The Chow n-logarithm function can be expressed by the Grassmannian n-logarithm function.*

4. Relation with classical polylogarithms. The classical polylogarithms ([Lei]) are defined by the following absolutely convergent series

$$Li_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^n}\qquad |z|<1$$

They are continued analytically to a covering of $\mathbb{C}P^1\backslash\{0,1,\infty\}$ by induction

$$Li_n(z):=\int_0^z Li_{n-1}(t)\frac{dt}{t},\qquad Li_1(z)=-\log(1-z)$$

The classical n-logarithm $Li_n(z)$ has a remarkable single-valued version (Zagier's function, see [Z1] and [BD]):

$$\mathcal{L}_n(z)\quad:=\quad\begin{matrix}\mathrm{Re} & (n:\ \text{odd})\\ \mathrm{Im} & (n:\ \text{even})\end{matrix}\ \left(\sum_{k=0}^{n-1}\beta_k\log^k|z|\cdot Li_{n-k}(z)\right)\ ,\quad n\geq 2$$

Here $\frac{2x}{e^{2x}-1}=\sum_{k=0}^{\infty}\beta_k x^k$, so $\beta_k=\frac{2^kB_k}{k!}$ where B_k are Bernoulli numbers.

Consider the following special family of $(n-1)$-planes h_a in $\mathbb{A}^{2n-1}$ defined in coordinates $z_1,...,z_{2n-1}$ by equations

$$z_n=1-z_1,\quad z_{n+i}=z_i-z_{i+1},\quad (i=1,...,n-2),\qquad z_{2n-1}=z_{n-1}-a \tag{12}$$

If $n>2$ h_a even does not intersect properly the codimension 2 faces of L. However the function $\mathcal{P}_n(h_a)$ was defined in s.3.

Theorem 3.6. *The value of the function $\mathcal{P}_n$ on the plane (12) is equal to* $(2\pi i)^{-n}\mathcal{L}_n(a)$

To visualize the configuration of hyperplanes $\{h \cap L_i\}$ in h consider the projectively dual configuration of points in $\mathbb{C}P^{n-1}$. For the plane h_a it looks as follows. Consider the set of vertices $l_1, ..., l_n$ of a simplex and points $m_1 \in l_1l_2, m_2 \in l_2l_3, ..., m_n \in l_nl_1$ on its sides (see the picture). Then the configuration of points $(l_1, ..., l_n, m_1, ..., m_n)$ corresponds to a certain plane h_a and any plane h_a produces such a configuration.

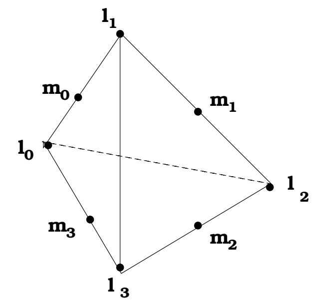

Figure 1:

The constant a has the following geometric interpretation. Let $\hat{m}_i$ be the point of intersection of the line l_il_{i+1} with the hyperplane passing through all the points m_j exept m_i. Then $a = r(l_i, l_{i+1}, m_i, \hat{m}_{i+1})$ (here r is the cross-ratio).

Theorem (3.6), and proposition (3.2) imply that

$$\mathcal{L}_n(a) = c_n \cdot (2\pi)^n \int_{\mathbb{C}P^{n-1}} \log|1 - z_1| \prod_{i=1}^{n-1} d\log|z_i| \wedge \prod_{i=1}^{n-2} d\log|z_i - z_{i+1}| \wedge d\log|z_{n-1} - a|$$

(c_n is a rational constant). This presentation seems new even for the dilogarithm.

4 Cocycles for all continuous cohomology classes of $GL_N(\mathbb{C})$

Let $H_c^{2n-1}(GL_n(\mathbb{C}), \mathbb{R})$ be the space of continuous cohomology of the Lie group $GL_n(\mathbb{C})$. It is known that

$$H_c^*(GL_n(\mathbb{C}), \mathbb{R}) = \Lambda_{\mathbb{R}}^*(c_1, c_3, ..., c_{2n-1})$$

where $c_{2k-1} \in H_c^{2n-1}(GL_k(\mathbb{C}), \mathbb{R})$ are certain canonical classes called the Borel classes ([Bo1]). In this section we will construct measurable cocycles for all Borel classes. Measurable and continuous cohomology of a Lie group G are isomorphic.

1. A $(2n-1)$-cocycle of the Lie group $GL_n(\mathbb{C})$. Choose a hyperplane h in $\mathbb{C}P^{n-1}$. Then the function

$$c_{2n-1}^n(g_1, ..., g_{2n}) := \mathcal{P}_n(g_1h, ..., g_{2n}h) \tag{13}$$

is a (2n-1)-cocycle of $GL_n(\mathbb{C})$. The cocycle condition is just the property b) in theorem (3.4). The cocycle $c_{2n-1}^n(g_1, ..., g_{2n})$ is defined everywhere on $GL_n(\mathbb{C})^{2n}$ but discontinuous near the identity.

2. A $(2n-1)$**-cocycle for the Borel class of** $GL_{n+m}(\mathbb{C})$. Theorem (3.4a) guarantee that this cocycle can be extended to the group $GL(\mathbb{C})$. Namely, to do this for the group $GL_{n+m}(\mathbb{C})$ we should proceed as follows.

It is convinient to consider the dual function $\tilde{\mathcal{P}}_n$ on configurations of $2n$ points in $\mathbb{C}P^{n-1}$. By definition its value value at a configuration of points is equal to the value of $\mathcal{P}_n$ on the projectively dual configuration of hyperplanes in $\mathbb{C}P^{n-1}$

Let us call by m-flag in $\mathbb{C}P^k$ a sequence of subspaces $L_\bullet := L_0 \subset L_1 \subset ... \subset L_{m-1}$ where $dim L_i = i$.

Let $H_1 * H_2$ be the joining of planes H_1, H_2. In general $dim(H_1 * H_2) = dim H_1 + dim H_2 - 1$.

For $2n$ generic $(m+1)$-flags $L_\bullet^{(1)}, ..., L_\bullet^{(2n)}$ in $\mathbb{C}P^{n+m-1}$ set

$$\tilde{\mathcal{P}}_n^{(m)}(L_\bullet^{(1)}, ..., L_\bullet^{(2n)}) := \sum_{j_1+...+j_{2n}=m} \tilde{\mathcal{P}}_n\Big((L_{j_1-1}^{(1)} * ... * L_{j_{2n}-1}^{(2n)} | L_{j_1}^{(1)}, ..., L_{j_{2n}}^{(2n)})\Big)$$

Here $(L_{j_1-1}^{(1)} * ... * L_{j_{2n}-1}^{(2n)} | L_{j_1}^{(1)}, ..., L_{j_{2n}}^{(2n)})$ is the configuration of $2n$ points in $\mathbb{C}P^{n-1}$ obtained by the projection of $L_{j_k}^{(k)}$ with the center at $L_{j_1-1}^{(1)} * ... * L_{j_{2n}-1}^{(2n)}$. More precisely, the set of all planes of dimension $j_1 + ... + j_{2n}$ containing $L_{j_1-1}^{(1)} * ... * L_{j_{2n}-1}^{(2n)}$ is a projective space of dimension $n-1$. Each $L_{j_k}^{(k)}$ provides a point in this space.

Choose an $(m+1)$-flag $L_\bullet$ in $\mathbb{C}P^{n+m-1}$. Set

$$c_{2n-1}^{n+m}(g_1, ..., g_{2n}) := \tilde{\mathcal{P}}_n^{(m)}(g_1 L_\bullet^{(1)}, ..., g_{2n} L_\bullet^{(2n)}) \tag{14}$$

This function is defined on a Zariski open subset of $GL_{n+m}(\mathbb{C})^{2n}$ for which the flags in the right hand side of (14) are in generic position. The cocycles c_3^k where also considered by K.Igusa (unpublished).

Theorem 4.1. *a)* $c_{2n-1}^{n+m}(g_1, ..., g_{2n})$ *is a measurable cocycle of the Lie group* $GL_{n+m}(\mathbb{C})$.

b) The cohomology class of $(2\pi i)^n c_{2n-1}^{n+m}$ *in* $H_c^{2n-1}(GL_n(\mathbb{C}), \mathbb{R})$ *is a non zero rational multiple of the Borel class.*

Proof. a) follows from the Key lemma in s.2.1 in[G3] and b) from theorem 5.12 in [G3]. The existence of the bi-Grassmannian n-logarithm is the main ingredient of the proof.

This leads to an explicit computation of the Borel regulator $K_{2n-1}(\mathbb{C}) \longrightarrow \mathbb{R}$ and so, thanks to the Borel theorem [Bo2], to formulas for special values of Dedekind zeta-functions at $s = n$. (These results should not be confused with Zagier's conjecture, which remaines unproved for $n > 3$).

5 Explicit construction of Beilinson's regulator

1. Affine flags. An affine p-flag is a p-flag $L^\bullet$ together with a choice of vectors $l_i \in L_i/L_{i-1}$ for all $1 \leq i \leq p$. We will denote affine p-flags as $(l_1, ..., l_p)$. Several affine p-flags are in general position if all the corresponding subspaces L_i are in generic position.

Let X be a G-variety. There is a simplicial variety $BX_\bullet$ where $BX_{(i)} := G\backslash X^{i+1}$. Let $\tau_{\geq n}BX_\bullet$ be the n-truncated simplicial variety, where $\tau_{\geq n}BX_{(i)} = 0$ for $i < n$ and $BX_{(i)}$ otherwise.

Denote by $A^p(k)$ the manifold of all affine p-flags in an k-dimensional vector space V^k over a field F. The group $GL(V^k)$ acts on it. Let $\hat{B}A^p(k)_\bullet \subset BA^p(k)_\bullet$ be the semisimplicial subvariety consisting of *configurations* of affine p-flags in generic position in V^k.

2. A correspondence between the affine flags and the bi-Grassmannian ([G3], §3). A finite correspondence between (semi)simplicial varieties $X_\bullet$ and $Y_\bullet$ is a (semi)simplicial subvariety $Z_\bullet \subset X_\bullet \times Y_\bullet$ finite over $X_\bullet$.

There is the following finite correspondence T between the truncated semisimplicial varieties $\tau_{\geq n}\hat{B}A^{m+1}(n+m)_\bullet$ and $\hat{G}(n)_\bullet$. For a point

$$a = (v_1^{(0)}, ..., v_{m+1}^{(0)}; ...; v_1^{(k)}, ..., v_{m+1}^{(k)}) \in \tau_{\geq n}\hat{B}A^{m+1}(n+m)_{(k)}$$

representing a configuration of $(k+1)$ affine $(m+1)$-flags in a vector space of dimension $n+m$ set

$$T(a) := \cup_{q=0}^{k-n} \cup_{i_0+...+i_k=m-q} \alpha^{-1}(L_{i_0}^{(0)} \oplus ... \oplus L_{i_k}^{(k)} | v_{i_0+1}^{(0)}, ..., v_{i_k+1}^{(k)})$$

Here $(L_{i_0}^{(0)} \oplus ... \oplus L_{i_k}^{(k)} | v^{(0)} i_0 + 1, ..., v_{i_k+1}^{(0)})$ is the configuration of vectors in the space $V^m / \oplus_{s=0}^{k}$ $L_{i_s}^{(s)}$ obtained by the projections of vectors $v_{i_0+1}^{(0)}, ..., v_{i_k+1}^{(k)}$ and $\alpha^{-1}(...)$ is the corresponding point of the appropriate Grassmannian, see (1.4) in [G3].

Theorem 5.1. *T is a correspondence between $\tau_{\geq n}\hat{B}A^{m+1}(n+m)_\bullet$ and $\hat{G}(n)_\bullet$*

Proof. This is the Key lemma in s.2.1 of [G3] translated to the language of semisimplicial varieties.

3. A construction of a cocycle representing Beilinson's class in $H^{2n}(BGL(\mathbb{C})_\bullet, \mathbb{R}(n)_\mathcal{D})$. Set $G := GL(V^{n+m})$. Recall that $BG_\bullet$ is the following simplicial variety:

$$BG_\bullet := pt \underset{s_1}{\overset{s_0}{\Leftarrow}} G \underset{s_2}{\overset{s_0}{\Lleftarrow}} G^2 \underset{s_3}{\overset{s_0}{\Lleftarrow\!\!\leftarrow}} G^3 \underset{s_4}{\overset{s_0}{\leftarrow \cdots \leftarrow}}$$

Denote by $\mathcal{D}^{p,q}(X)$ the space of (p,q) currents on X. To compute the hypercohomology with coefficients in the complex of sheaves $\mathbb{R}_\mathcal{D}(n)$ (see (8) we will replace $\Omega^{n+\bullet}$ by its Dolbeaux resolution $\mathcal{D}^{n+\bullet,\bullet}$.

Then a cocycle representing a class in $H^{2n}(BG(\mathbb{C})_\bullet, \mathbb{R}(n)_\mathcal{D})$ is a sequence of currents

$$\alpha_i \in \mathcal{A}^i(G(\mathbb{C})^{2n-1-i}), \quad \text{and} \quad \beta_j \in \mathcal{D}^{n,j}(G(\mathbb{C})^{n-j}), \qquad 0 \leq i,j \leq n-1$$

satisfying the following conditions ($s^* = \sum(-1)^k s_k^*$):

$$d\alpha_i = s^*\alpha_{i+1}, \quad d\alpha_{n-1} = Im(2\pi i)^{-n}\beta_n, \quad d\beta_j = s^*\beta_{j+1} \qquad (15)$$

Choose an affine $(m+1)$-flag $L_\bullet$ in V^{n+m}. It defines canonical map

$$BG(\mathbb{C})_\bullet \xrightarrow{p_L} BA^{m+1}(n+m)$$

Let $\hat{B}G(\mathbb{C})_\bullet := p_L^* \hat{B}A^{m+1}(n+m)$. One has

$$\hat{B}G(\mathbb{C})_\bullet \xrightarrow{p_L} \hat{B}A^{m+1}(n+m) \xrightarrow{T} \hat{G}(n)(\mathbb{C})$$

Set $\hat{\alpha}_i := p_L^* T^*(\psi^n_{n-i-1}(n))$. The Dolbeaux currents α_{n-1} and β_j satisfying these properties where constructed in [G3] (see s.4.2 and s. 5.5 there). The currents $\hat{\alpha}_i$ satisfy the conditions (15) because the bi-Grassmannian polylogarithms satisfy similar ones.

The currents $\hat{\alpha}_i$ can be canonically extended to certain currents α_i on $BG(\mathbb{C})_\bullet$ satisfying the same conditions. Details will be published elsewhere.

Using this Beilinson's regulator (γ is the Adams filtration)

$$gr^\gamma_n K_{2n-j}(X) \longrightarrow H^j(X, \mathbb{R}(n)_\mathcal{D})$$

for any algebraic variety X over $\mathbb{C}$ is obtained by the standard procedure ([B]).

6 The Abel-Jacobi map for Higher Chow groups

1. Regulator to the real Deligne cohomology. Let $z^r(X,i)$ be the free abelian group generated by irreducible codimension r algebraic cycles in $X \times \mathbb{A}^i$ which intersect properly all faces $X \times L_I$.

The intersection with codimension 1 faces provides $z^r(X,\bullet)$ with a structure of simplicial abelian group. Let $z^r_\bullet(X)$ be the corresponding (homological) complex. Its homology are Bloch's Higher Chow groups: $CH^r(X,i) = H_i z^r_\bullet(X)$

Now let X be a smooth variety. Let us construct homomorphisms

$$A_i : z^r(X,i) \longrightarrow \mathcal{A}^{2r-i-1}(X(\mathbb{C})) \qquad D_i : z^r(X,i) \longrightarrow \mathcal{D}^{r,r-i}(X(\mathbb{C}))$$

Denote by π_A (resp π_X) projection of $X \times \mathbb{A}^i$ to $\mathbb{A}^i$ (resp X). If $\omega \in S^{2r-i-1}(X(\mathbb{C}))$ is a smooth test form and $Y \in z^r(X,i)$ then

$$< A_i(Y), \omega > := \int_{Y(\mathbb{C})} \pi_X^*\omega \cdot \pi_A^* r_i(L; H) \qquad < D_i(Y), \omega > := \int_{Y(\mathbb{C})} \pi_X^*\omega \cdot \pi_A^* \Omega_L$$

If $Y \to \pi_X(Y)$ is not a finite map the second integral vanish thanks to the type considerations. This is always the case if $i > r$.

Let us cook up from $z^r_\bullet(X)$ a cohomological complex setting $z^{r,\bullet}(X) := z^r(X, 2r-\bullet)$. Set $A^\bullet := A_{2r-\bullet}$, $D^\bullet := D_{2r-\bullet}$.

Theorem 6.1. *$Y \in z^{r,i}(X) \longmapsto (A^i(Y), D^i(Y))$ provides a homomorphism from the complex $z^{r,\bullet}(X)$ to the Dolbeaux resolution of the Deligne complex (8)*

In particulary we get the regulator map $CH^r(X,i) \longrightarrow H^{2r-i}(X(\mathbb{C}), \mathbb{R}(r)_\mathcal{D})$ (compare with [Bl2]). It is quite remarkable that it is defined explicitely on the level of complexes.

2. The Abel-Jacobi map. One has exact sequence

$$0 \longrightarrow \frac{H^{2r-i-1}(X,\mathbb{C})}{H^{2r-i-1}(X,\mathbb{Z}(r)) + F^r H^{2r-i-1}(X,\mathbb{C})} \longrightarrow H^{2r-i}(X, \mathbb{Z}(r)_\mathcal{D}) \longrightarrow$$

$$H^{2r-i}(X,\mathbb{Z}(r))\cap F^rH^{2r-i}(X,\mathbb{C})\longrightarrow 0$$

We will construct first the map

$$CH^r(X,i)\longrightarrow H^{2r-i}(X,\mathbb{Z}(r))\cap F^rH^{2r-i}(X,\mathbb{C}) \tag{16}$$

Namely, let us think of $\mathbb{A}^m$ as of hyperplane $\sum_{i=0}^m x_m = 1$. Consider canonical chains $\Delta_m := \{x_i \in \mathbb{R}, x_i \geq 0\}$. Let $Y \in z^r(X,i)$, so Y is a cycle in $X(\mathbb{C})\times\mathbb{A}^i(\mathbb{C})$. Set $r(Y) := \pi_X(Y \cap X(\mathbb{C}))\times\Delta_i$. This is a chain of codimension $2r-i$ in $X(\mathbb{C})$. If Y is cycle in the complex $z^r(X,\bullet)$ then $r(Y)$ is a topological cycle. Its homology class is the image of Y under the map (16).

If it is zero one can define the Abel-Jacobi map

$$CH^r(X,i)\longrightarrow \frac{H^{2r-i-1}(X,\mathbb{C})}{H^{2r-i-1}(X,\mathbb{Z}(r))+F^rH^{2r-i-1}(X,\mathbb{C})}$$

as follows. Choose a chain b_Y which bounds the cycle $r(Y)$. Then

$$\omega\longmapsto\int_{b(Y(\mathbb{C}))}\omega-\int_{Y(\mathbb{C})}\pi_X^*\omega\cdot\pi_A^*r_i(L;H)$$

is the current representing the cohomology class of the Abel-Jacobi map.

7 The multivalued analytic version of Chow polylogarithms

1. A definition. Let $\tilde{\mathcal{Z}}_m^n$ be the variety parametrizing collections $(X_m; f_1, ..., f_{n+m})$ where f_i are rational functions with normal crossing divisors on an m-dimensional complex variety X_m. (In particulary we require that all irreducible components of $div f_i$ has multiplicity ± 1). Choose a coordinate z on P^1. The function f_i defines a rational map $X\longrightarrow (P^1)^{n+m}$. So $\tilde{\mathcal{Z}}_m^n$ is the higher cubical Chow variety ([BK]).

Let me scetch a construction of multivalued analytic $(n-m-1)$-forms $L_m^n(X_m; f_1, ..., f_{n+m})$ on these varieties satisfying the conditions similar to (1),(2).

There is a canonical m-chain $[\infty,0]^m\subset(\mathbb{C}P^1)^m$. Let

$$(f_1, ..., f_m) : X(\mathbb{C})\longrightarrow(\mathbb{C}P^1)^m$$

Set $\gamma^0_{f_1,...,f_m} := (f_1, ..., f_m)^{-1}[\infty,0]^m$. Let X_{f_i} be the divisor of the function f_i. Then this chain defines a relative homology class in $H_m(X(\mathbb{C}), X_{f_1}\cup...\cup X_{f_m})$. Let $\gamma_{f_1,...,f_m}$ be a chain relatively homotopic to $\gamma^0_{f_1,...,f_m}$. Suppose that it is in generic position with respect to the divisors of the functions $f_{m+1}, ..., f_n$. Set

$$L_m^n(X; f_1, ..., f_{m+n}) := \frac{1}{m!(n-1)!}\text{Alt}_{m+n}\int_{\gamma_{f_1,...,f_m}}\log f_{m+1}d\log f_{m+2}\wedge...\wedge d\log f_{m+n} \tag{17}$$

We call this collection of forms the multivalued Chow n-logarithm.

There are the "boundary maps"

$$a_i : (X; f_1, ..., f_{m+n}) \longrightarrow (X_{f_i}; f_1, ..., \hat{f}_i, ..., f_{m+n})$$

(the restriction of the functions f_j, $j \neq i$, to the divisor X_{f_i}). The main properties are

$$dL^n_m = (n+1) \cdot \sum_{i=1}^{m+n} (-1)^i a_i^* L^n_{m-1} \qquad \text{and} \qquad \sum_{i=1}^{2n} (-1)^i a_i^* L^n_{n-1} \in (2\pi i)^n \mathbb{Q}$$

for appropriately choosen branches of these multivalued forms.

In [HaM1], [HaM2] M.Hanamura and R.MacPherson suggested an interesting geometrical construction of a sequence of multivalued analytic forms on Grassmannians satisfying conditions similar to (1),(2), see also [H] for some existence results in this direction.

2. An example: the multivalued Chow dilogarithm function. It is defined as follows:

$$L_2(X; f_1, f_2, f_3) := \frac{1}{3} \text{Alt}_3 \int_{\gamma_{f_1}} \log f_2 d \log f_3$$

Here γ_f is a path on $X(\mathbb{C})$ relatively homotopic to $f^{-1}[\infty, 0]$: its boundary is the divisor $(f^{-1}(0)) - (f^{-1}(\infty))$.

It satisfies the differential equation

$$dL_2(X; f_1, f_2, f_3) = \text{Alt}_3 \sum_{x \in X(\mathbb{C})} v_x(f_1) \log f_2(x) d \log f_3(x)$$

where $v_x(f_1)$ is the order of zero of function f_1 at the point x.

3. Motivic interpretation. The integral

$$\int_{\gamma_{f_1, ..., f_m}} \log f_{m+1} d \log f_{m+2} \wedge ... \wedge d \log f_{m+n}$$

is a period of the following mixed motive:

$$M(X; f_1, ..., f_{2n-1}) := H^n\Big((\mathbb{G}_m, 1)^n \times (P^1)^{n-1}, X \cup B_X)\Big)$$

Here $\mathbb{G}_m := P^1 \backslash \{0, \infty\}$, and B_X is constructed as follows. Let $K_{n-1} \subset (P^1)^n$ be "the algebraic cube". It is union of the hyperplanes where one of the coordinates 0 or ∞. For each $x \in X \cap K_{n-1} \times (P^1)^n$ consider the line through x in the direction of f_{m+1}-axis. B_x is the union of all these lines.

This mixed motive has canonical n-framing. Namely we have a distinguished vector in $gr^W_{2n} M(X; f_1, ..., f_{2n-1})$ and a functional on $gr^W_0 M(X; f_1, ..., f_{2n-1})$. The vector is given by the form $d \log f_n \wedge ... \wedge d \log f_{2n-1}$. The functional corresponds to a relative $2n$-cycle whose boundary component on X is $\gamma_{f_1, ..., f_{n-1}}$. Our function is the integral of the form $d \log f_n \wedge ... \wedge d \log f_{2n-1}$ over this $2n$-chain.

A more detailed exposition [G5] of these constructions will appear later.

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